J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2013.20.4.277
Volume 20, Number 4 (November 2013), Pages 277–286

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

SOME MULTI-STEP ITERATIVE SCHEMES FOR SOLVING NONLINEAR EQUATIONS

Arif Rafiq [a], Ayesha Inam Pasha [b] and Byung-Soo Lee [c,*]

Abstract. In this paper, we suggest and analyze a family of multi-step iterative methods which do not involve the high-order differentials of the function for solving nonlinear equations using a different type of decomposition (mainly due to Noor and Noor [15]). We also discuss the convergence of the new proposed methods. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative method. Our results can be considered as an improvement and refinement of the previous results.

1. Introduction

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations (see for example [1, 6-16]). These methods can be classified as one-step and two-step methods.

Abbasbandy [1] and Chun [6] have proposed and studied several one-step and two-step iterative methods with higher order convergence by using the decomposition technique of Adomian [2-5].

In their methods, they have used the higher order differential derivatives which is a serious drawback. To overcome this drawback, Noor and Noor [14-15] developed two-step and three step iterative methods by combining the well-known Newton method with other one-step and two-step methods.

Following the lines of [14-15], we suggest and analyze a family of multi-step iterative methods which do not involve the high-order differentials of the function for solving nonlinear equations using a different type of decomposition (mainly due to Noor and Noor [15]). We also discuss the convergence of the new proposed methods.

Received by the editors August 29, 2013. Revised November 5, 2013. Accepted Nov. 7, 2013.
2010 *Mathematics Subject Classification.* 47J05, 47J25.
Key words and phrases. nonlinear equations, iterative methods.
*Corresponding author.

Several numerical examples are given to illustrate the efficiency and the performance of the new iterative method. Our results can be considered as an improvement and refinement of the previous results.

2. Iterative Methods

Consider the nonlinear equation

$$f(x) = 0. \tag{1}$$

We assume that α is a simple root of (1) and β is an initial guess sufficiently close to α. We can rewrite (1) as a coupled system using the Taylor series

$$f(\beta) + (x-\beta)f'(\beta) + g(x) = 0, \tag{2}$$

$$g(x) = f(x) - f(\beta) - (x-\beta)f'(\beta). \tag{3}$$

We can rewrite (3) in the following form

$$x = \beta - \frac{f(\beta)}{f'(\beta)} - \frac{g(x)}{f'(\beta)} \tag{4}$$

$$= c + N(x), \tag{5}$$

where

$$c = \beta - \frac{f(\beta)}{f'(\beta)} \tag{6}$$

and

$$N(x) = -\frac{g(x)}{f'(\beta)}. \tag{7}$$

In order to prove the multi-step iterative methods, He [9] and Lao [11] have considered the case with the definition that

$$g(x_0) = 0, \tag{8}$$

and Noor and Noor [15] have considered the case

$$f(x_0) = g(x_0). \tag{9}$$

For the derivation of multi-step iterative methods for solving nonlinear equations, the condition (9) introduced by Noor and Noor [15] which is actually

$$f(\sum_{i=0}^{\infty} x_i) = g(\sum_{i=0}^{\infty} x_i) \tag{10}$$

is the stronger one. We rectify this error and also remove such kind of conditions. For this purpose, we substitute (3) into (7) to obtain

$$N(x) = x - \beta - \frac{f(x)}{f'(\beta)} + \frac{f(\beta)}{f'(\beta)}. \tag{11}$$

We now construct a sequence of higher order iterative methods by using the following decomposition method which is mainly due to Noor and Noor [15]. This decomposition of the nonlinear operator $N(x)$ is quite different than that of Adomian decomposition. The main idea of this technique is to look for a solution of (4) having the series form

$$x = \sum_{i=0}^{\infty} x_i. \tag{12}$$

The nonlinear operator N can be decomposed as

$$N(x) = N(\sum_{i=0}^{\infty} x_i) = N(x_0) + \sum_{i=1}^{\infty} \left[N\Big(\sum_{j=0}^{i} x_j \Big) \right]. \tag{13}$$

Combining (4), (12) and (13), we have

$$x = c + N(x_0) + \sum_{i=1}^{\infty} \left[N\Big(\sum_{j=0}^{i} x_j \Big) \right]. \tag{14}$$

Thus we have the following iterative scheme

$$\begin{aligned} x_0 =& c, \\ x_1 =& N(x_0), \\ x_2 =& N(x_0 + x_1), \\ x_3 =& N(x_0 + x_1 + x_2), \\ &\vdots \\ x_{n+1} =& N(x_0 + x_1 + \cdots + x_n) \, ; n = 1, 2, \cdots . \end{aligned} \tag{15}$$

Then

$$\begin{aligned} x_1 + x_2 + \cdots + x_{n+1} = & N(x_0) + N(x_0 + x_1) + N(x_0 + x_1 + x_2) \\ & + \cdots + N(x_0 + x_1 + x + \cdots + x_n) \, ; n = 1, 2, \cdots \end{aligned} \tag{16}$$

and

$$x = c + \sum_{i=1}^{\infty} x_i. \tag{17}$$

It follows from (6), (11) and (15), that

$$x_0 = c = \beta - \frac{f(\beta)}{f'(\beta)} \tag{18}$$

and

$$x_1 = N(x_0) = -\frac{f(x_0)}{f'(\beta)}. \tag{19}$$

From (14), (16) and (17) we have

$$x \approx c = x_0 = \beta - \frac{f(\beta)}{f'(\beta)}.$$

This allows us to suggest the following one-step iterative method for solving (1).

Algorithm 1 For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ f'(x_n) \neq 0,\ n = 0, 1, 2, \cdots$$

which is known as *"Newton's Method"* and it has the *second order convergence.* Again using (15), (17)-(19), we conclude that

$$\begin{aligned} x &\approx c + x_1 = x_0 + N(x_0) \\ &= \beta - \frac{f(\beta)}{f'(\beta)} - \frac{f(x_0)}{f'(\beta)}. \end{aligned}$$

Using this relation, we can suggest the following two-step iterative methods for solving (1).

Algorithm 2 For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

Predictor-Step

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},\ f'(x_n) \neq 0,$$

Corrector-Step

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}.$$

This Algorithm is commonly known as *"Double-Newton Method"* with the *third order convergence.*

Again using (11) and (15), we can calculate

$$\begin{aligned} x_2 &= N(x_0+x_1) \\ &= -\frac{f(x_0)}{f'(\beta)} - \frac{f(x_0+x_1)}{f'(\beta)}. \end{aligned} \tag{20}$$

From (11), (15)-(20), we conclude that

$$\begin{aligned} x &\approx c + x_1 + x_2. \\ &= x_0 + N(x_0) + X(x_0+x_1). \\ &= \beta - \frac{f(\beta)}{f'(\beta)} - 2\frac{f(x_0)}{f'(\beta)} - \frac{f(x_0+x_1)}{f'(\beta)}. \end{aligned}$$

Using this, we can suggest and analyze the following two-step iterative method for solving (1).

Algorithm 3 (AA) For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

Predictor-Step

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\ , f'(x_n) \neq 0,\ n = 0, 1, 2, \cdots . \tag{21}$$

$$z_n = -\frac{f(y_n)}{f'(x_n)}, \tag{22}$$

Corrector-Step

$$x_{n+1} = y_n + 2z_n - \frac{f(y_n+z_n)}{f'(x_n)}. \tag{23}$$

Again using (11) and (15), we have

$$\begin{aligned} x_3 &= N(x_0+x_1+x_2) \\ &= -\frac{f(x_0)}{f'(\beta)} - \frac{f(x_0+x_1)}{f'(\beta)} - \frac{f(x_0+x_1+x_2)}{f'(\beta)}. \end{aligned}$$

From (11), (17)-(20), we have

$$\begin{aligned} x &\approx c + x_1 + x_2 + x_3 \\ &= \beta - \frac{f(\beta)}{f'(\beta)} - 2\frac{f(x_0)}{f'(\beta)} - 2\frac{f(x_0+x_1)}{f'(\beta)} - \frac{f(x_0+x_1+x_2)}{f'(\beta)}. \end{aligned}$$

Using this, we can suggest and analyze the following iterative method for solving (1).

Algorithm 4 For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

Predictor-Step

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},\ f'(x_n) \neq 0,\ n = 0, 1, 2, \cdots . \tag{24}$$

$$z_n = -\frac{f(y_n)}{f'(x_n)}, \tag{25}$$

$$w_n = -\frac{f(y_n + z_n)}{f'(x_n)}, \tag{26}$$

Corrector-Step

$$x_{n+1} = y_n + 2z_n + 2w_n - \frac{f(y_n + z_n + w_n)}{f'(x_n)}. \tag{27}$$

3. Convergence Analysis

Theorem 1. *Let $\beta \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to β, then the three-step iterative method defined by Algorithm 3 has the second-order convergence.*

Proof. Let $\beta \in I$ be a simple zero of f. Since f is sufficiently differentiable function, by expanding $f(x_n)$ and $f'(x_n)$ about β, we get

$$f(x_n) = f'(\beta)[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots], \tag{28}$$

$$f'(x_n) = f'(\beta)[12c_2 e_n + 3c_3 e_n^2 + \cdots], \tag{29}$$

where $c_k = \frac{f_k(\beta)}{k! f'(\beta)}$, $k = 1, 2, 3, \cdots$ and $e_n = x_n - \beta$.
Now from (28) and (29), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 - 2(c_3 + c_2^2)e_n^3 - (3c_4 + 5c_2 c_3)e_n^4 + \cdots . \tag{30}$$

From (18) and (30), we have

$$y_n = \beta + c_2 e_n^2 + 2(c_3 + c_2^2)e_n^3 + (3c_4 + 5c_2 c_3)e_n^4 + \cdots . \tag{31}$$

Now expanding $f(y_n)$ about β and using (31), we get

$$f(y_n) = f'(\beta)[c_2 e_n^2 + 2(c_3 + c_2^2)e_n^3 + (3c_4 + 5c_2 c_3)e_n^4 + \cdots]. \tag{32}$$

Now from (29) and (32), we have

$$z_n = -[c_2 e_n^2 + (2c_3 + 4c_2^2)e_n^3 + (3c_4 + 12c_2 c_3 + 4c_2^3)e_n^4 + \cdots]. \tag{33}$$

Now again expanding $f(y_n + z_n)$ about β and using (31) and (33), we have

$$f(y_n + z_n) = f'(\beta)[-2c_2^2 e_n^3 + (-4c_2^3 - 7c_2c_3)e_n^4 + \cdots]. \tag{34}$$

From (29) and (34), we get

$$\frac{f(y_n + z_n)}{f'(x_n)} = 2c_2^2 e_n^3 + 7c_2c_3 e_n^4 + \cdots . \tag{35}$$

From (23), (33) and (35), one obtains

$$e_{n+1} = c_2 e_n^2 + 2(c_3 + 4c_2^2)e_n^3 + \cdots . \tag{36}$$

Hence it is proved. □

Theorem 2. *Let $\beta \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to β, then the four-step iterative method defined by Algorithm 4 has the second-order convergence.*

Proof. From (26) and (35), we have

$$w_n = 2c_2^2 e_n^3 + 7c_2c_3 e_n^4 + \cdots . \tag{37}$$

Now again expanding $f(y_n + z_n + w_n)$ about β and using (23), (25) and (37), we get

$$f(y_n + z_n + w_n) = f'(\beta)[-4c_2^3 e_n^4 + (-17c_2^2 c_3 - 8c_2^4)e_n^5 + \cdots]. \tag{38}$$

From (29) and (38), we have

$$\frac{f(y_n + z_n + w_n)}{f'(x_n)} = -4c_2^3 e_n^4 - 17c_2^2 c_3 e_n^5 + \cdots . \tag{39}$$

From (27), (31), (33), (37) and (39), one obtains

$$e_{n+1} = 4c_2 e_n^2 + (8c_3 + 24c_2^2)e_n^3 + \cdots . \tag{40}$$

Hence it is proved. □

4. Numerical Examples

We present some examples to illustrate the efficiency of the new developed three-step iterative methods. We compare the Newton method (NM), the method (NR1) [14], the method (NR2) [15] and the method (AA). Put $\epsilon = 10^{-15}$.

The following stopping criteria is used for computer programs

(1) $|x_{n+1} - x_n| < \epsilon$,
(2) $|f(x_{n+1})| < \epsilon$.

As for the convergence criteria, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations (IT) to approximate the zero, the approximate zero x_0 and the value $f(x_0)$ and δ (see Table 1).

The examples are the same as in Chun [6]:

$$F_1(x) = \sin^2 x - x^2 + 1,$$
$$F_2(x) = x^2 - e^x - 3x + 2,$$
$$F_3(x) = \cos x - x,$$
$$F_4(x) = (x-1)^3 - 1,$$
$$F_5(x) = x^3 - 10,$$
$$F_6(x) = x \cdot e^{x^2} - \sin^2 x + 3\cos x + 5,$$
$$F_7(x) = e^{x^2+7x-30} - 1.$$

Table 1

	IT	x_n	$f(x_n)$	δ
$F_1 : x_0 = 1$				
NM	7	1.404491648215341	3.331e-16	3.059e-13
NR1	6	1.404491648215341	-8.882e-16	2.22e-16
NR2	8	1.404491648215341	-4.441e-16	7.349e-14
AA	8	1.404491648215341	3.331e-16	4.244e-11
$F_2 : x_0 = 2$				
NM	6	0.257530285439861	0	9.864e-14
NR1	5	0.257530285439861	0	2.209e-12
NR2	7	0.257530285439861	4.4409e-16	5.884e-15
AA	6	0.257530285439861	-4.441e-16	8.109e-08
$F_3 : x_0 = 1.7$				
NM	5	0.739085133215161	-4.441e-16	3.259e-08
NR1	4	0.739085133215161	0	1.849e-08
NR2	6	0.739085133215161	0	7.627e-14
AA	5	0.739085133215161	1.1102e-16	1.827e-08
$F_4 : x_0 = 3.5$				
NM	8	2	0	2.878e-11
NR1	7	2	0	2.2204e-15
NR2	6	2	0	6.702e-11
AA	7	2	0	2.22e-15

$F_5 : x_0 = 1.5$				
NM	7	2.1544346900318837217592935665	4.83e-13	2.741e-07
NR1	6	2.1544346900318837217592935665	1.776e-15	1.761e-08
NR2	7	2.1544346900318837217592935665	1.776e-15	1.770e-10
AA	7	2.1544346900318837217592935665	-1.776e-14	5.087e-08
$F_6 : x_0 = -2$				
NM	9	-1.2076478271309189270094165484	-2.664e-15	4.26e-11
NR1	8	-1.2076478271309189270094165484	-6.217e-15	3.002e-09
NR2	7	-1.2076478271309189270094165484	-2.664e-15	2.273e-10
AA	6	-1.2076478271309189270094165484	1.687e-14	1.980e-08
$F_7 : x_0 = 3.5$				
NM	13	3	0	2.53e-13
NR1	11	3	0	1.079e-09
NR2	8	3	-7.105e-15	1.057e-10
AA	9	3	0	3.553e-15

5. Conclusions

We have suggested a family of one-step, two-step, three-step and four-step iterative methods for solving nonlinear equations. It is important to note that the implementation of these multi-step methods does not require the computation of higher order derivatives compared to most other methods of the same order.

References

1. S. Abbasbandy: Improving Newton Raphson method for nonlinear equations by modified Adomian decomposition method. *Appl. Math. Comput.* **145** (2003) 887-893.
2. G. Adomian: Nonlinear Stochastic Systems and Applications to Physics. Kluwer Academic Publishers, Dordrecht, 1989.
3. ______: A review of the decomposition method and some recent results for nonlinear equations. *Math. Comput. Model.* **13** (1992), no. 7, 17-43.
4. ______: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer, Boston, MA, 1994.
5. G. Adomian & R. Rach: Noise terms in decomposition series solution *Comput. Math. Appl.* **24** (1992), no. 11, 61-64.
6. C. Chun: Iterative methods improving Newton method by the decomposition method. *Comput. Math. Appl.* **50** (2005) 1559-1568.

7. V. Daftardar-Geiji & H. Jafari: An iterative method for solving nonlinear functional equations. *J. Math. Anal. Appl. 316* (2006) 753-763.
8. M. Frontini & E. Sormani: Some variants of Newton method with third order convergence. *J. Comput. Appl. Math.* **140** (2003) 419-426.
9. J.H. He: A new iteration method for solving algebraic equations. *Appl. Math. Comput.* **135** (2003) 81-84.
10. H.H. Homeier: On Newton-type methods with cubic convergence. *J. Comput. Appl. Math.* **176** (2005) 425-432.
11. X. Lao: A note on the new iteration for solving algebraic equations. *Appl. Math. Comput.* **171** (2005) 1177-1183.
12. M.A. Noor: Numerical Analysis and Optimization, Lecture Notes. COMSATS Institute of Information Technology, Islamabad, Pakistan, 2006.
13. ______: New family of iterative methods for nonlinear equations. *Appl. Math. Comput.* **190** (2007) 553-558.
14. M.A. Noor & K.I. Noor: Three-step iterative methods for nonlinear equations. *Appl. Math. Comput.* **183** (2006) 322-327.
15. ______: Some iterative schemes for nonlinear equations. *Appl. Math. Comput.* **183** (2006) 774-779.
16. S. Weerakoon & G.I. Fernando: A variant of Newton method with accelerated third order convergence. *Appl. Math. Lett.* **17** (2000) 87-93.

[a]DEPARTMENT OF MATHEMATICS, LAHORE LEADS UNIVERSITY, LAHORE, PAKISTAN
Email address: aarafiq@gmail.com

[b]CIIT, LAHORE, PAKISTAN
Email address: ayeshams2@gmail.com

[c]DEPARTMENT OF MATHEMATICS, KYUNGSUNG UNIVERSITY, BUSAN 608-736, KOREA
Email address: bslee@ks.ac.kr