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CERTAIN CLASSES OF INFINITE SERIES DEDUCIBLE FROM MELLIN-BARNES TYPE OF CONTOUR INTEGRALS

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Abstract. Certain interesting single (or double) infinite series associated with hypergeometric functions have been expressed in terms of Psi (or Digamma) function $\psi(z)$, for example, see Nishimoto and Srivastava [8], Srivastava and Nishimoto [13], Saxena [10], and Chen and Srivastava [5], and so on. In this sequel, with a view to unifying and extending those earlier results, we *first* establish two relations which some double infinite series involving hypergeometric functions are expressed in a single infinite series involving $\psi(z)$. With the help of those series relations we derived, we *next* present two functional relations which some double infinite series involving $\overline{H}$-functions, which are defined by a generalized Mellin-Barnes type of contour integral, are expressed in a single infinite series involving $\psi(z)$. The results obtained here are of general character and only two of their special cases, among numerous ones, are pointed out to reduce to some known results.

1. Introduction and Preliminaries

Certain interesting single (or double) infinite series associated with hypergeometric functions (1.4) have recently been expressed in terms of Psi (or Digamma) function $\psi(z)$ in (1.1), for example, see Nishimoto and Srivastava [8], Srivastava and Nishimoto [13], Saxena [10], Chen and Srivastava [5] and Srivastava and Choi [15], and so on. In this connection, with a view to unifying and extending those earlier results, we *first* establish two relations which some double infinite series involving hypergeometric functions are expressed in a single infinite series involving $\psi(z)$. With the help of those series relations we derived, we *next* present two functional relations which some double infinite series involving $\overline{H}$-functions in (3.1), which are defined by a generalized Mellin-Barnes type of contour integral, are expressed in a single

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infinite series involving $\psi(z)$. The results obtained here are of general character and only two of their special cases, among numerous ones, are pointed out to reduce to some known results.

To do this, we begin by recalling the Psi (or Digamma) function $\psi(z)$ (cf. [15, Section 1.2] and [16, p. 24]) defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.1}$$

and the following well-known (rather classical) result (see, for example, [16, p. 352]):

$$\sum_{n=1}^{\infty} \frac{(\nu)_n}{n(\lambda)_n} = \psi(\lambda) - \psi(\lambda-\nu) \quad \left(\Re(\lambda-\nu) > 0;\ \lambda \notin \mathbb{Z}_0^-\right), \tag{1.2}$$

where Γ is the familiar Gamma function, $(\lambda)_n$ denotes the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,\, 2,\, 3,\, \cdots\}), \end{cases} \tag{1.3}$$

and $\mathbb{C}$ and $\mathbb{Z}_0^-$ are the sets of complex numbers and nonpositive integers, respectively.

A natural generalization of the hypergeometric functions ${}_2F_1$, ${}_1F_1$, *et cetera* (considered in the vast literature; see, for example, [16, p. 71]) is accomplished by the introduction of an arbitrary number of numerator and denominator parameters. The resulting series:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1,\, \cdots,\, \alpha_p; \\ \beta_1,\, \cdots,\, \beta_q; \end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\, \cdots,\, \alpha_p;\, \beta_1,\, \cdots,\, \beta_q;\, z), \end{aligned} \tag{1.4}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by (1.3), is known as the *generalized Gauss* (*and Kummer*) *series*, or simply, the *generalized hypergeometric series.*

The summation formula (1.2) and its obvious special cases were revived, in recent years, as illustrations emphasizing the usefulness of fractional calculus in evaluating infinite sums. For a detailed historical account of (1.2), and of its various consequences and generalizations have been presented by Nishimoto and Srivastava [8]. A systematic account of certain family of infinite series which can be expressed in terms of Digamma functions together with their relevant unification and generalization has been given by Srivastava [14], Al-Saqabi *et al.* [1] and Aular de Duran *et al.* [2].

From the aforementioned work of Nishimoto and Srivastava [8], we choose to recall here two interesting consequences of the summation formula (1.2), which are contained in Theorem 1 below.

Theorem 1 ([8]). *Let $\{R_k\}_{k=0}^{\infty}$ be an arbitrary bounded sequence of complex numbers. Then we have*

$$\sum_{n=1}^{\infty}\frac{(\nu)_n}{n(\lambda)_n}\sum_{k=0}^{\infty}\frac{R_k}{(\lambda+n)_k}\frac{z^k}{k!}=\sum_{k=0}^{\infty}\frac{R_k}{(\lambda)_k}\left[\psi(\lambda+k)-\psi(\lambda-\nu+k)\right]\frac{z^k}{k!} \tag{1.5}$$
$$\left(\Re(\lambda-\nu)>0;\ \lambda\notin\mathbb{Z}_0^-\right)$$

and

$$\sum_{n=1}^{\infty}\frac{(\nu)_n}{n(\lambda)_n}\sum_{k=0}^{\infty}\frac{(\nu+n)_k}{(\lambda+n)_k}R_k\frac{z^k}{k!}=\sum_{k=0}^{\infty}\frac{(\nu)_k}{(\lambda)_k}\left[\psi(\lambda+k)-\psi(\lambda-\nu)\right]R_k\frac{z^k}{k!} \tag{1.6}$$
$$\left(\Re(\lambda-\nu)>0;\ \lambda\notin\mathbb{Z}_0^-\right),$$

provided that each of the series involved converges absolutely.

2. Generalizations of the Results in Theorem 1

In this section, we establish certain generalizations of the formulas (1.5) and (1.6).

Theorem 2. *Let $\{R_k\}_{k=0}^{\infty}$ be an arbitrary bounded sequence of complex numbers and set*

$$\begin{aligned}U_n(\alpha,\beta,\mu,\eta,\rho(k);z):=&\sum_{k=0}^{\infty}R_k\frac{\Gamma(\mu+\rho(k))\Gamma(\mu+\alpha+\beta+\eta+\rho(k))}{\Gamma(\mu+\eta+\rho(k))\Gamma(\mu+\alpha+\beta+n+\rho(k))}\\&\cdot{}_3F_2\left[\begin{matrix}\alpha+n,\ \alpha+\beta,\ -\eta;\\ \mu+\alpha+\beta+n+\rho(k),\ \alpha;\end{matrix}\ 1\right]\frac{z^k}{k!}.\end{aligned} \tag{2.1}$$

Then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\alpha)_n}{n}U_n(\alpha,\beta,\mu,\eta,\rho(k);z)=&\sum_{k=0}^{\infty}R_k\left[\psi(\mu+\rho(k))+\psi(\mu+\alpha+\beta+\eta+\rho(k))\right.\\&\left.-\psi(\mu+\beta+\rho(k))-\psi(\mu+\eta+\rho(k))\right]\frac{z^k}{k!}\end{aligned} \tag{2.2}$$
$$\left(\Re(\mu+\beta)>0;\ \Re(\mu+\eta)>0;\ \mu\notin\mathbb{Z}_0^-;\ \Re(\rho(k))\geq 0\ \textit{for}\ k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right),$$

provided that each of the series involved converges absolutely.

Theorem 3. *Let $\{R_k\}_{k=0}^{\infty}$ be an arbitrary bounded sequence of complex numbers and set*

$$V_n(\alpha,\beta,\mu,\eta,\rho(k);z) := \sum_{k=0}^{\infty} R_k \frac{\Gamma(\mu+\rho(k))\Gamma(\alpha+n+\rho(k))}{\Gamma(\mu+\alpha+\beta+n+\rho(k))} \cdot {}_3F_2\left[\begin{matrix} \alpha+n+\rho(k),\ \alpha+\beta,\ -\eta; \\ \mu+\alpha+\beta+n+\rho(k),\ \alpha+\rho(k); \end{matrix}\ 1\right] \frac{z^k}{k!}.$$

Then we get

$$\tag{2.3} \begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} V_n(\alpha,\beta,\mu,\eta,\rho(k);z) &= \sum_{k=0}^{\infty} R_k \frac{\Gamma(\alpha+\rho(k))\Gamma(\mu+\eta+\rho(k))}{\Gamma(\mu+\alpha+\beta+\eta+\rho(k))} \\ &\cdot [\psi(\mu+\rho(k)) + \psi(\mu+\alpha+\beta+\eta+\rho(k)) - \psi(\mu+\beta) - \psi(\mu+\eta+\rho(k))] \frac{z^k}{k!} \end{aligned}$$

$$\left(\Re(\mu+\beta) > 0;\ \Re(\mu+\eta) > 0;\ \mu \notin \mathbb{Z}_0^-;\ \Re(\rho(k)) \geq 0\ for\ k \in \mathbb{N}_0\right),$$

provided that each of the series involved converges absolutely.

Proof of Theorems 2 and 3. For sake of convenience, let the left-hand side of the (2.2) be denoted by $\mathcal{I}$. Then, substituting for U_n from (2.1) and applying the definitions (1.3) and (1.4), we have

$$\mathcal{I} = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)}{n\,\Gamma(\alpha)} \sum_{k=0}^{\infty} R_k \frac{\Gamma(\mu+\rho(k))\Gamma(\mu+\alpha+\beta+\eta+\rho(k))}{\Gamma(\mu+\eta+\rho(k))\Gamma(\mu+\alpha+\beta+n+\rho(k))} \cdot \left\{ \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+n+l)\,\Gamma(\alpha)\Gamma(\mu+\alpha+\beta+n+\rho(k))\,(\alpha+\beta)_l\,(-\eta)_l}{\Gamma(\alpha+n)\,\Gamma(\alpha+l)\,\Gamma(\mu+\alpha+\beta+n+\rho(k)+l)\,l!} \right\} \frac{z^k}{k!}.$$

$$\tag{2.4} \begin{aligned} &= \sum_{k=0}^{\infty} R_k \frac{\Gamma(\mu+\rho(k))\Gamma(\mu+\alpha+\beta+\eta+\rho(k))}{\Gamma(\mu+\eta+\rho(k))\Gamma(\mu+\alpha+\beta+\rho(k))} \sum_{l=0}^{\infty} \frac{(\alpha+\beta)_l\,(-\eta)_l}{(\mu+\alpha+\beta+\rho(k))_l\,l!} \\ &\cdot \left\{ \sum_{n=1}^{\infty} \frac{(\alpha+l)_n}{n\,(\mu+\alpha+\beta+\rho(k)+l)_n} \right\} \frac{z^k}{k!}, \end{aligned}$$

where the inversion of the order of summation can be justified by the absolute convergence of the series involved. The innermost series in (2.4) is summable by means of the well-known result (1.2). We thus have

$$\mathcal{I} = \sum_{k=0}^{\infty} R_k \frac{\Gamma(\mu+\rho(k))\Gamma(\mu+\alpha+\beta+\eta+\rho(k))}{\Gamma(\mu+\eta+\rho(k))\Gamma(\mu+\alpha+\beta+\rho(k))} \cdot \left\{ \sum_{l=0}^{\infty} \frac{(\alpha+\beta)_l\,(-\eta)_l}{(\mu+\alpha+\beta+\rho(k))_l\,l!} \Psi(\mu+\alpha+\beta+\rho(k)+l) - \Psi(\mu+\beta+\rho(k)) \right\} \frac{z^k}{k!},$$

provided that $\Re(\mu+\beta)>0$, $\mu\notin\mathbb{Z}_0^-$, $\Re(\rho(k))\geq 0$ for all $k\in\mathbb{N}_0$.
Now we have

(2.5)
$$\begin{aligned}\mathcal{I}=\sum_{k=0}^{\infty}\Bigg[R_k\,&\frac{\Gamma(\mu+\rho(k))\Gamma(\mu+\alpha+\beta+\eta+\rho(k))}{\Gamma(\mu+\eta+\rho(k))\Gamma(\mu+\alpha+\beta+\rho(k))}\\ &\cdot\left\{\sum_{l=0}^{\infty}\frac{(\alpha+\beta)_l\,(-\eta)_l}{(\mu+\alpha+\beta+\rho(k))_l\,l!}\;\left[\Psi(\mu+\alpha+\beta+\rho(k)+l)-\Psi(\mu+\alpha+\beta+\rho(k))\right]\right.\\ &\left.+\left[\Psi(\mu+\alpha+\beta+\rho(k))-\Psi(\mu+\beta+\rho(k))\right]\,{}_2F_1\left[\begin{matrix}\alpha+\beta,-\eta;\\ \mu+\alpha+\beta+\rho(k);\end{matrix}\;1\right]\right\}\Bigg]\,\frac{z^k}{k!}.\end{aligned}$$

Upon using the following known summation formula [5, p. 380, Eq. (2.5)]:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\;[\psi(c+n)-\psi(c)]\\ &\quad=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,[\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b)]\\ &\qquad\qquad\left(\Re(c-a-b)>0;\;c\notin\mathbb{Z}_0^-\right),\end{aligned}\tag{2.6}$$

and Gauss's well-known summation theorem for ${}_2F_1(a,\,b;c;1)$ (see, *e.g.*, [16, p. 64, Eq. (7)]; see also [12]), after a little simplification, we are easily led to the desired result (2.2). □

The equality (2.3) in Theorem 3 will be established in a similar way as in the proof of equality (2.2).

Remark. The results [14, Theorem 3] look very similar to those in Theorems 2 and 3 here. Yet, it is easy to see that the results in Theorems 2 and 3 here are neither special nor general cases of those in [14, Theorem 3] and vice versa.

3. Definition and Existence Conditions of $\overline{H}$-function

A lot of research work has recently come up on the study and development of a function that is more general than the Fox H-function (see, *e.g.*, [10, 11]), popularly known as $\overline{\text{H}}$-function. It was introduced by Inayat-Hussain [6, 7] and now stands on a fairly firm footing through the following contributions of various authors [3, 4, 6, 7, 9, 10].

The $\overline{H}$-function is defined and represented in the following manner [6]:

$$\overline{H}_{p,q}^{m,n}[z]=\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j,A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ (b_j,\beta_j)_{1,m},(b_j,\beta_j,B_j)_{m+1,q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L z^{\xi}\;\overline{\phi}(\xi)\;d\xi,\tag{3.1}$$

where $z \neq 0$ and

$$\overline{\phi}(\xi) := \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \xi)}. \tag{3.2}$$

It may be noted that the $\overline{\phi}(\xi)$ contains fractional powers of some of the Gamma functions. Here z may be real or complex but is not equal to zero, and an empty product is interpreted as unity; m, n, p, and q are integers such that $1 \leq m \leq q$, $0 \leq n \leq p$; $\alpha_j > 0$ $(j = 1, \ldots, p)$, $\beta_j > 0$ $(j = 1, \ldots, q)$ and a_j $(j = 1, \ldots, p)$ and b_j $(j = 1, \ldots, q)$ are complex numbers. The exponents A_j $(j = 1, \ldots, n)$ and B_j $(j = m+1, \ldots, q)$ take on non-integer values.

The nature of the contour L, sufficient conditions of convergence of defining integral (3.1) and other details about the $\overline{\mathrm{H}}$-function can be seen in [4, 6, 7].

The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result given by Rathie [9]:

$$\overline{H}_{p,q}^{m,n}[z] = o\left(|z|^{\alpha}\right) \quad \text{as} \quad |z| \to 0,$$

where

$$\alpha = \min_{1 \leq j \leq m} \Re\left(\frac{b_j}{\beta_j}\right).$$

The following series representation for the $\overline{H}$-function given by Saxena *et al.* [11] will be required later on:

$$\overline{H}_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j,A_j)_{1,n},(a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j,B_j)_{m+1,q} \end{matrix}\right] = \sum_{k=0}^{\infty} \sum_{h=1}^{m} \overline{f}(\zeta)\, z^{\zeta}, \tag{3.3}$$

where

$$\overline{f}(\zeta) = \frac{\prod_{\substack{j=1 \\ j \neq h}}^{m} \Gamma(b_j - \beta_j \zeta) \prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j \zeta)\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j \zeta)\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \zeta)} \frac{(-1)^k}{k! \beta_h} \tag{3.4}$$

and

$$\zeta = \zeta(h,k) = \frac{b_h + k}{\beta_h}. \tag{3.5}$$

The function $\overline{H}$ makes sense and defines an analytic function of z in the following two cases [3]:

(i) $0 < |z| < \infty$ and

$$\mu_1 = \sum_{j=1}^{m} |\beta_j| + \sum_{j=1}^{n} |\alpha_j A_j| - \sum_{j=m+1}^{q} |\beta_j B_j| - \sum_{j=n+1}^{p} |\alpha_j| > 0; \tag{3.6}$$

(ii) $\mu_1 = 0$, $0 < |z| < \tau^{-1}$ and

$$\tau := \left\{ \prod_{j=1}^{m} (\beta_j)^{-\beta_j} \right\} \left\{ \prod_{j=1}^{n} (\alpha_j)^{A_j \alpha_j} \right\} \left\{ \prod_{j=n+1}^{p} (\alpha_j)^{\alpha_j} \right\} \left\{ \prod_{j=m+1}^{q} (\beta_j)^{-B_j \beta_j} \right\}. \tag{3.7}$$

4. Functional Relations Involving Generalized Mellin-Barnes Type of Contour Integral

Here we give two interesting double summation formulas involving the $\overline{H}$-function asserted by the following theorem.

Theorem 4. *If each of the series involved converges absolutely, the following formulas hold:*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n} \sum_{l=0}^{\infty} \frac{(\alpha+n)_l (\alpha+\beta)_l (-\eta)_l}{(\alpha)_l \, l!} \\
&\quad \cdot \overline{H}^{m,n+2}_{p+2,q+2} \left[z \left| \begin{matrix} (1-\mu,C;1),(1-\mu-\alpha-\beta-\eta,C;1),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q},(1-\mu-\eta,C;1),(1-\mu-\alpha-\beta-n-l,C;1) \end{matrix} \right. \right] \\
&= \sum_{h=1}^{m} \sum_{k=0}^{\infty} \frac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h} \left[\psi(\mu + C\varsigma(h,k)) + \psi(\mu+\alpha+\beta+\eta + C\varsigma(h,k)) \right. \\
&\qquad \left. -\psi(\mu+\beta+C\varsigma(h,k)) - \psi(\mu+\eta+C\varsigma(h,k)) \right] \frac{z^{\varsigma(h,k)}}{k!}
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=0}^{\infty} \frac{(\alpha+\beta)_l (-\eta)_l}{l!} \\
&\quad \cdot \overline{H}^{m,n+3}_{p+3,q+2} \left[z \left| \begin{matrix} (1-\mu,C;1),(1-\alpha-n-l,C;1),(1-\alpha,C;1),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q},(1-\alpha-l,C;1),(1-\mu-\alpha-\beta-n-l,C;1) \end{matrix} \right. \right] \\
&= \sum_{h=1}^{m} \sum_{k=0}^{\infty} \frac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h} \frac{\Gamma(\alpha + C\varsigma(h,k)) \Gamma(\mu+\eta+C\varsigma(h,k))}{\Gamma(\mu+\alpha+\beta+\eta+C\varsigma(h,k))} \\
&\qquad \cdot \left[\psi(\mu + C\varsigma(h,k)) + \psi(\mu+\alpha+\beta+\eta+C\varsigma(h,k)) \right. \\
&\qquad \left. -\psi(\mu+\beta) - \psi(\mu+\eta+C\varsigma(h,k)) \right] \frac{z^{\varsigma(h,k)}}{k!},
\end{aligned} \tag{4.2}$$

where $C > 0$, $\Re(\mu+\beta) > 0$, $\Re(\mu+\eta) > 0$, $\mu \notin \mathbb{Z}_0^-$, *and* $\overline{f}(\varsigma)$ *and* $\varsigma(h,k)$ *are given in* (3.4) and (3.5), *respectively.*

Proof. In view of the $\overline{H}$-function representation (3.3), we apply Theorem 2 by setting $\rho \equiv \rho(k) = C\,\varsigma(h,k)$ $(C>0)$ and $R_k = \dfrac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h}$ $(k \in \mathbb{N}_0)$, where $\overline{f}(\varsigma)$ and $\varsigma(h,k)$ are defined by (3.4) and (3.5), respectively. Then we have

(4.3)
$$\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n} \sum_{k=0}^{\infty} \frac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h} \frac{\Gamma(\mu + C\,\varsigma(h,k))\,\Gamma(\mu+\alpha+\beta+\eta+C\,\varsigma(h,k))}{\Gamma(\mu+\eta+C\,\varsigma(h,k))\Gamma(\mu+\alpha+\beta+n+C\,\varsigma(h,k))}$$
$$\cdot\,{}_3F_2\left[\ \alpha+n,\ \ \alpha+\beta,\ \ -\eta; \mu+\alpha+\beta+n+C\,\varsigma(h,k),\ \alpha;\ 1\right] \frac{z^k}{k!}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h} \left[\psi(\mu + C\,\varsigma(h,k)) + \psi(\mu+\alpha+\beta+\eta+C\,\varsigma(h,k))\right.$$
$$\left. -\psi(\mu+\beta+C\,\varsigma(h,k)) - \psi(\mu+\eta+C\,\varsigma(h,k))\right] \frac{z^k}{k!}.$$

Now, replacing z by z^{1/B_h} in (4.3) and multiplying each side of equality (4.3) by z^{b_h/B_h} , then summing both sides of the resulting equations from $h=1$ to $h=m$ $(\leq q)$, we get

(4.4)
$$\sum_{h=1}^{m} \left[\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n} \sum_{k=0}^{\infty} \frac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h} \frac{\Gamma(\mu + C\,\varsigma(h,k))\,\Gamma(\mu+\alpha+\beta+\eta+C\,\varsigma(h,k))}{\Gamma(\mu+\eta+C\,\varsigma(h,k))\Gamma(\mu+\alpha+\beta+n+C\,\varsigma(h,k))}\right.$$
$$\left.\cdot\,{}_3F_2\left[\ \alpha+n,\ \ \alpha+\beta,\ \ -\eta; \mu+\alpha+\beta+n+C\,\varsigma(h,k),\ \alpha;\ 1\right] \frac{z^{k+b_h/B_h}}{k!}\right]$$
$$= \sum_{h=1}^{m} \Big[\sum_{k=0}^{\infty} \frac{(-1)^k \overline{f}(\varsigma(h,k))}{B_h} \Big[\psi(\mu + C\,\varsigma(h,k)) + \psi(\mu+\alpha+\beta+\eta+C\,\varsigma(h,k))$$
$$- \psi(\mu+\beta+C\,\varsigma(h,k)) - \psi(\mu+\eta+C\,\varsigma(h,k))\Big]\Big] \frac{z^{k+b_h/B_h}}{k!}.$$

This, in view of (3.3), proves the required result (4.1).

A similar argument as in the proof of (4.1) will establish the formula (4.2). This completes the proof of Theorem 4. $\square$

5. Special Cases and Concluding Remarks

In this section we briefly consider another variation of the results derived in the preceding sections. On account of the most general nature of the $\overline{H}$-function in our main results given by (4.1) and (4.2), a large number of infinite series relations involving simpler functions can be easily obtained as their special cases. Yet, as an illustration, a few interesting special cases will be considered as follows:

(i) For $\alpha + \beta = \nu$, $\eta = -\beta$, and $\mu = \lambda - \nu$, the $\overline{H}$-function reduces to the familiar Fox H-function. Then the functional relations (4.1) and (4.2) yield equalities (3.10) and (3.11) in Chen and Srivastava [5, p. 385].

(ii) If we set $\eta = 1$ in (4.1) and (4.2) and give some suitable parametric replacement in the resulting identities, we can arrive at the equalities (4.1) and (4.2) in Saxena [10, pp. 128-129].

(iii) If we set $\alpha = \nu$, $\mu = \lambda - \alpha$, $\eta = -\beta$ and $\nu + \beta = \alpha$ in Theorems 2 and 3, we are led to Theorems 3 and 4 in [5], respectively.

Theorem 4 gives a further generalization of the functional relations (3.10) and (3.11) in [5, p. 385].

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