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From Visualization to Computer Animation Approaches in Mathematics Learning: the Legacy throughout History of Human Endeavours for Better Understanding

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Presently, there has been growing interests in using mathematics' history in teaching mathematics [Katz, V. & Tzanakis, C. (Eds.) (2011). *Recent Developments on Introducing a Historical Dimension in Mathematics Education.* Washington, DC: Mathematical Association of America]. Thus, this article introduces some work of scholars from ancient East Indian culture like Bhaskara (AD 1114–1185) and Arabic culture such as Ibn Qurrah (AD 9th c) that are related to Pythagoras Theorem. In addition, some Babylonian creative works related to Pythagorean triples found in a tablet known as 'Plimpton 322', and an application of the Pythagorean Theorem found in another tablet named 'Yale Tablet' are presented. Applications of computer animation of dissection Motion Operations concept in 2D and 3D using dynamic software like Geometer's-Sketchpad and Cabri-II-and-3D. Nowadays, creative minds are attracted by the recent stampede in the advances of technological applications in visual literacy; consequently, innovative environments that would help young students, gifted or not, acquiring meaningful conceptual understanding would immerge.

Keywords: history of math & math education; visualization; conceptual understanding, dynamic software; Pythagorean theorem
MESC Classification: A30, C70, D40, E50, G20, G30, G50; Q60; U60, U70
MSC2010 Classification: 97C70, 97D40, 97D50, 97E50, 97G30, 97U60; 97U70

INTRODUCTION

The Mathematical Association of America (MAA) has introduced its 'NOTES 78, 2011' entitled *Recent Developments on Introducing a Historical Dimension in Mathematics Education* (cf. Katz &Tzanakis, 2011). In the cited MAA 2011 publication, the article by Dematte & Furinghetti (2011) on the use of pictures of ancient documents in teaching is of interest here. Earlier in 2000, the International Commission on Mathematical Instruction (ICMI) had initiated and published its ICMI Study entitled *History in Mathematics Education* (cf. Fauvel & van Maanen, 2000).

Visualization through visual representations has been one of the human effective tools in creating conceptual arguments throughout mathematics history. It is interesting to recall what scholars of ancient civilizations such as the Babylonian, East Indian, Arabic, Chinese, and South East Asian civilizations had offered the world civilization in mathematics and other sciences. With the recent drive in the advances in technology, the focus of educators in technology and in mathematics education has been shifted to include computer animation environments. Larger students' populations, young gifted and others would have effective means for meaningful understanding of mathematical concepts. One such means could be through the use of Dynamic Software in the classroom such as the Geometer's Sketchpad (GSP, Los Angeles, USA-Based) and Capri-II & Capri-3D (Grenoble, France-Based). The main aim of this article is to present some authentic products in mathematics history that would exemplify the power and creativity in mathematics' history for mathematics teaching; this includes the use of hands-on manipulation, visual illustration, en route to the creation of computer animation of mathematical concepts.

East Indian Culture: Bhaskara (AD 1114–1185)

As a visual illustration, when the Hindu mathematician Bhaskara drew his famous diagram, he offered no further explanation than the word 'Behold' (Figure 1). He was not indicating that the lines in his diagram were terribly attractive to the eye; rather his specific arrangement delivered an elegant proof to the Pythagorean Theorem (Rahim, 2003, p. 144).

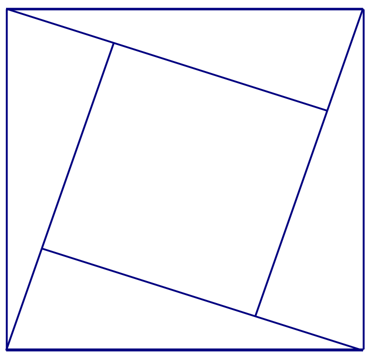

Figure 1. The Hindu mathematician Bhaskara's diagram: BEHOLD!

Bhaskara's 'BEHILD' would be an excellent topic of mathematics history for use in teaching mathematics.

Arabic Culture: Thābit ibn Qurrah al-Ḫarrānī (9th Century)[1]

Through the Arab Islamic Empire and in particular during the 9th century, an Arab mathematician, Thābit ibn Qurrah al-Harrānī, had come with an impressive visual representation for proving Pythagoras Theorem (Figure 2). For further information, just click the link below:

http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Thabit.html

The Arab scholar, Thābit ibn Qurrah al-Harrānī was known as a mathematician, astronomer, physician, and philosopher.

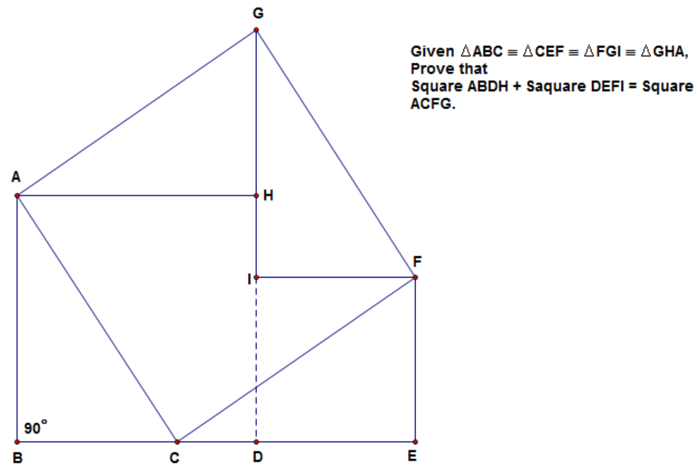

Figure 2. Thābit ibn Qurrah visual representation of his proof of Pythagoras Theorem

Thābit ibn Qurrah's proof shown in Figure 2 would be another interesting topic of math's history for use in teaching mathematics.

The Yale Tablet: YBC 7289

The famous 'root 2' tablet (Figure 3) would be an excellent topic too for using the history of mathematics in teaching mathematics; it was originally found in southern Mesopotamia, and presently identified in the Yale Babylonian Collection as YBC 7289 at Yale University, New Haven, USA. Historians have suggested that the Babylonians (1800–1600 B.C.) had prepared a calculation of the diagonal of a 30 by 30 units square (see Figure 3). This has been identified as proof that the Babylonians had known of Pythagoras

[1] Internet source for Arabic Culture: Thābit ibn Qurrah al-Ḥarrānī (9th Century): Retrieved May 2013.
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Thabit.html

Theorem more than one thousand years before Pythagoras — Pythagoras had lived in Alexandria, Egypt about 500 B.C. (Neugebauer & Sachs, 1945). The link

http://www.stat.auckland.ac.nz/~paul/ItDT/HTML/node37.html

will present the figure shown below resembling the Yale 7289 Tablet.

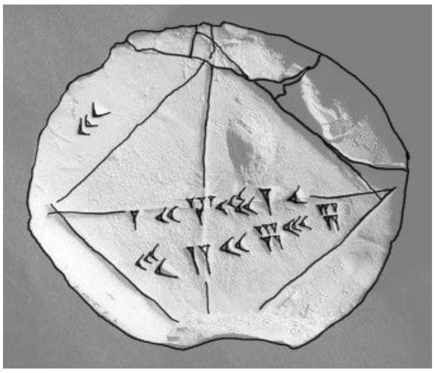

Figure 3. Yale 7289 Tablet

Explanation: Figure 4 below presents an explanation of the YBC 7289. The left part of Figure 4 below shows the Babylonian's calculation of √2 represented as

|<<||||<<<<<|<

in Babylonian cuneiforms, just above the diagonal AC of the square ABCD with, **|** , corresponds to

,

the cuneiform symbol characterized the unit (number one) and, **<** , corresponds to

the cuneiform symbol represented *number ten.* Historians stated that the Babylonian numeration system has no 'decimal point' to separate the whole and the fractional parts in a given fractional number, rather the context of the problem determines whether the number at hand is a whole or fractional number within the base–60 Babylonian numeration system (Neugebauer & Sachs, 1945); see Charts 1 and 2 below with √1 < √2 < √4 in mind.

<table>
<tr><td>…60^3</td><td>60^2</td><td>60</td><td>1</td><td>1/60</td><td>$1/60^2$</td><td>$1/60^3$ …</td></tr>
<tr><td></td><td></td><td></td><td>|</td><td><<||||</td><td><<<<<|</td><td><</td></tr>
</table>

Chart 1. Babylonian Base-60 Numeration System layout

…10^3	10^2	10	1•	1/10	$1/10^2$	$1/10^3$…
			1•	4	01	004 …

Chart 2. Hindu-Arabic Base-10 Numeration System's Layout

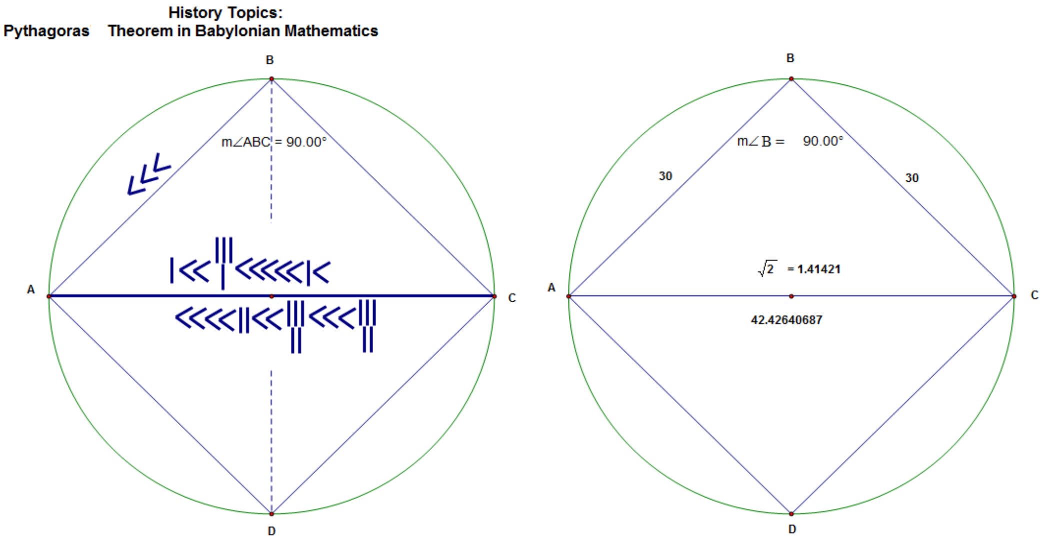

Figure 4. Yale Tablet's Replica shows the diagonal length AC of 30 by 30 ABCD square in Babylonian Cuneiforms

Accordingly, in the problem shown at the left part of Figure 4 above:

First, the expression,

|<<||||<<<<<|<

just above the diagonal AC of the square ABCD equals the following value in the Hindu-Arabic base-10 numeration system:

$$1 + 24/60 + 51/60^2 + 10/60^3 = 1.414212963$$

(by calculating the expression) to get the Babylonian value for √2 as

1.414212963.

That is,

$$|{<}{<}||||{<}{<}{<}{<}{<}|{<} = 1 + 24/60 + 51/60^2 + 10/60^3 = 1.414212963$$

as the Babylonian value for √2.
In contrast, the Graphic Calculator, TI-83, for √2 gives √2 = 1.414213562.
It is clear how very close these two values for √2 are!

Second, the expression

<<<<||<<|||||<<<|||||

just under the diagonal AC of the square ABCD equals the following value in the Hindu-Arabic base-10 numeration system:

$$42 + 25/60 + 35/60^2 = 42.42638889$$

to get the Babylonian value for 30√2.

Clearly the roles of visual representations have been thorough throughout.

In the Hindu-Arabic Base-10 numeration system the equivalent value for each of the values,

|<<||||<<<<<|< and <<<<||<<|||||<<<|||||

is shown at the right part of Figure 4 above.

The material presented above would be another topic of using math's history in teaching mathematics.

The Plimpton 322 Tablet[2]

Neugebauer & Sachs (1945) have described the cuneiform tablet Plimpton 322, dated from 1800 – 1600 BC, as a clear demonstration that the Babylonians were familiar with at least fifteen different sets of Pythagorean triples. The size of some of these triples such as (12709; 13500; 18541) indicates that these were not obtainable by trial and error (pp. 38–41); also see Eves (1964, pp. 35–37). Further, Edwards (1978) stated that,

> Pythagorean triples are named for their relationship to the Pythagorean Theorem, which states that in a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. It is not easy to find triples by trial and error, particularly those that involve large numbers. The Babylonians probably had a method for finding them. Most likely their interest stemmed from the geometric applications of the set of numbers. Therefore they knew of the Pythagorean Theorem one thousand years before Pythagoras. (p. 104)

Babylonian Numeration System - A brief description

The Babylonian numeration system is a place-value base-60 system consists of two symbols representing the unit (number one) and number 10. The two symbols were produced "by use of a stylus with a triangular end, which produced the characteristic wedge-shaped, or cuneiform, impressions" (Baumgart, Deal, Vogeli, & Hallerberg, 1969, p. 37).

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[2] Internet source for Yale Tablet 'Yale YBC 7289:
http://www.stat.auckland.ac.nz/~paul/ItDT/HTML/node37.html

The Babylonian system is briefly described by Vogeli (1969, pp. 36–37):

> Sometime before 2000 B.C. the Babylonians developed a base-sixty or sexagesimal system of numeration which employed the positional principle. Actually, this system was a mixture of base ten and base sixty in which numbers less than 60 were represented by using a simple base-ten grouping system and numbers 60 and greater were designated by the principle of position with **the base of sixty**. … **A stylus that looks like a wedge-shaped triangle** was used to produce
>
> 
>
> as the symbol characterized the unit (number one) which in turn was repeatedly used to produce the numbers 1, 2, 3, up to 9 in addition to a second digit
>
> 
>
> represented *number ten*. These two symbols were repeatedly used to represent numbers from 11 to 59. Numbers 60 and beyond, were represented in terms of these two symbols for numbers from 1 to 59, using the principle of position to specify multiples of powers of 60. … In the Old Babylonian texts (1800-1600 B.C.) no symbol for zero was used, but a blank space was left for any missing power of 60. Some of the texts from the Seleucid period (first three centuries B.C.) contain a separation symbol,
>
> ,
>
> used to indicate such an empty space between digits.

Animation[3]

Animation is a powerful visual technique for explaining math concepts. The Microsoft Power Point animating feature has been in a widespread use by many scholars and teachers in presenting teaching materials and article in classrooms and in professional meetings.

Recently, it has been argued that Dissection-Motion Operations (DMO), both in 2D and 3D, have been an appropriate means that fits naturally within any Dynamic Geometry Software. The literature offers an expanding knowledge-base on the use of technology in mathematics (de Villiers, 1997; Healy & Hoyles, 2001; Rahim, 2011; Rahim & Sawada, 1986; Whiteley, 2000). Further, computer's animation through other software means such as the Geometer's Sketchpad (GSP), Cabri-II, Cabri-3D and AutoCAD are effective tools especially in showing samples of "Proofs without Words" (in MAA's terminology) of mathematical propositions through object-to-object transforms. A sample of 'Proof without Words" for the proposition: "*Every regular pentagonal region is transformable through Dissection Motion Operations into a rectangular region of equal area"* through a series of figures is provided below. The last 3 figures (Figures 12, 13, and 14) exemplify object-to-object transformation in 3D and interrelationships among shapes. For example,

[3] Internet source for to view animations of some math propositions: http://mrahim.lakeheadu.ca/

the proposition: *Any regular pentagonal region* is transformable through Dissection Motion Operation into a *rectangular region of equal area*, is shown in the following sequence of Figures 5–11.

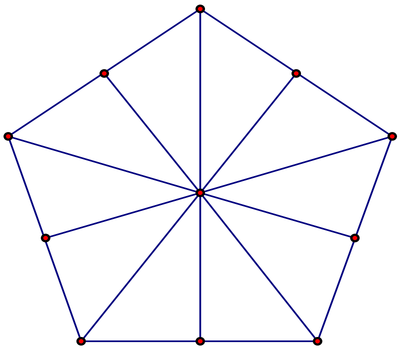

Figure 5. Dissection of a regular pentagon region into 10 congruent right triangle subregions

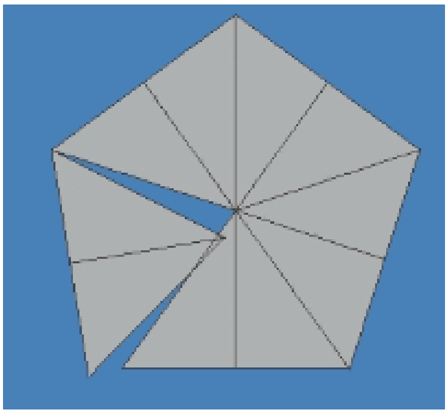

Figure 6. Motion starts

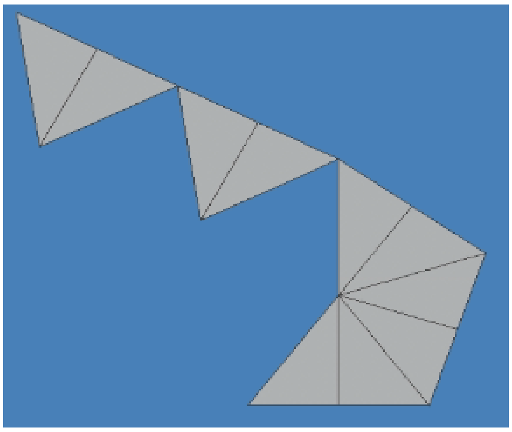

Figure 7. Motion of the pieces continues

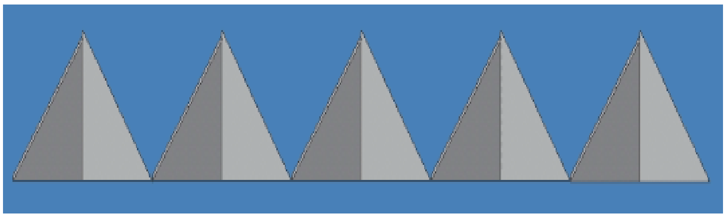

Figure 8. The ten triangles are arranged along a horizontal line by the above rotations

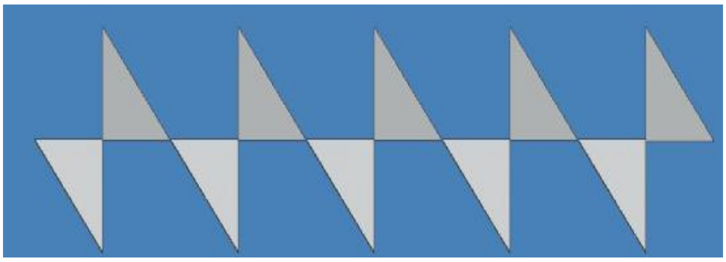

Figure 9. Five traingles were reflected along the horizontal line

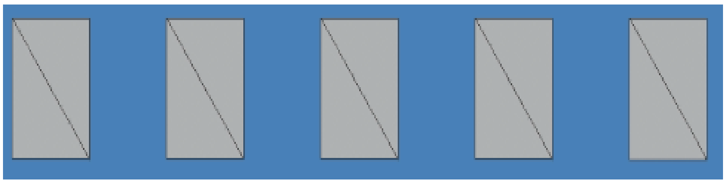

Figure 10. The five triangles were translated to make five small rectangles

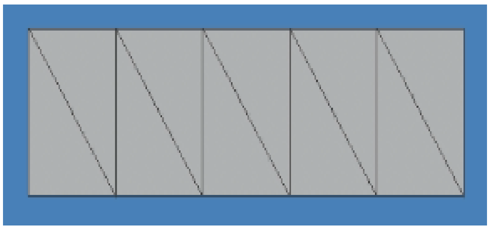

Figure 11. The ten trianglar regions through DMO form an area equivalent rectangular region

In 3-D, the object-to-object transforms through Dissection Motion Operation, are exemplified through *prism-to-prism* transforms as shown in Figures 12, 13, and 14 where a rhombus prism is transformed into a rectangular prism.

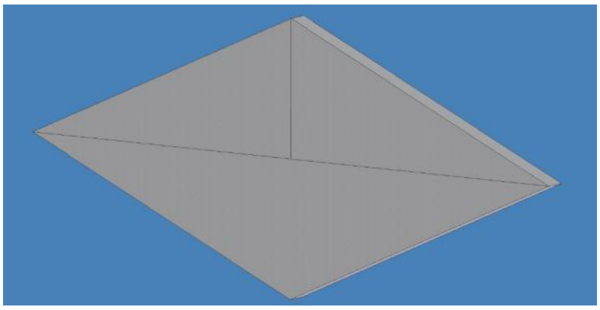

Figure 12. A rhombus prism dissected into 3 triangular prisms

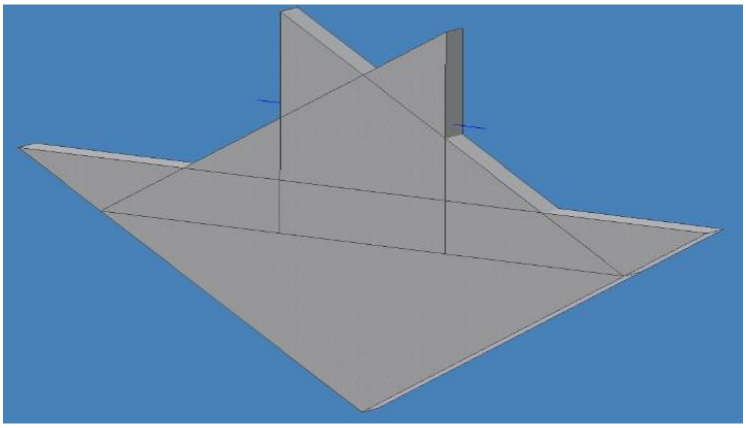

Figure 13. Translation of the two right triangular prisms to form an equivalent rectangular prism

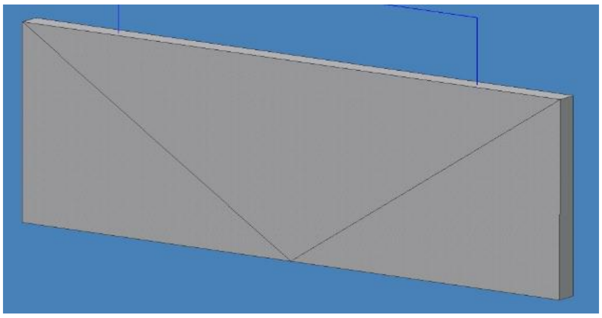

Figure 14. Rectangular prism as the final product of DMO

Remark. To view animations of some math propositions discussed, just click he link below:

http://mrahim.lakeheadu.ca/

CONCLUSION

In conclusion, the main purpose for the use of mathematics' history in teaching mathematics through the use of visual representations, both electronically animated, and physically modeled such as the use of pictures of ancient document in teaching mathematics, is to create a rich environment that would help the learners, gifted and not gifted, for conceptual better understanding in proving and sense making of mathematical propositions (Usiskin, 1982; van Hiele, 1985, pp. 243–252). On the Babylonian aspect, one can have an access to a host of detailed information of the Babylonian numeration system by searching the internet; e.g. one may try the following link (Rothstein, 2010):

http://www.nytimes.com/2010/11/27/arts/design/27tablets.html?_r=0

REFERENCES

Baumgart, J.; Deal, D.; Vogeli, B. & Hallerberg, A. (Eds.) (1969). *Historical Topics for the Mathematics Classroom.* Reston, VA: National Council of Teachers of Mathematics (NCTM). ME **1989l.**00931 for 2nd ed. (1989)

Dematte, A. & Furinghetti, F. (2011). History, Figures and Narratives in Mathematics Teaching. In: V. Katz and C. Tzanakis (Eds.), *Recent Developments on Introducing a Historical Dimension in Mathematics Education* (pp. 103–112). Washington, DC: Mathematical Association of America (MAA).

de Villiers, M., (1997). The role of proof in investigation, computer-based geometry: some personal reflections. In: J. King and D. Schattschneider (Eds.), *Geometry Tuned On! Dynamic Software in Learning, Teaching, and Research* (pp.15–24). Washington, DC: Mathematical Association of America. ME **1997e.**03111

Edwards, H. M. (1978). Fermat's Last Theorem. *Scientific American* **239(4)**, 104–122.

Fauvel, E, & van Maanen, J. (2000). *History in Mathematics Education: The ICMI Study.* Dordrecht: Kluwer Academic Publishers.

Healy, L. & Hoyles, C. (2001). Software tools for geometry problem solving: Potentials and pitfalls. *Intern. J. Comput. Math. Learn.* 6(3), 235–256. ME **2002b.**03111

Katz, V. & Tzanakis, C. (Eds.) (2011). *Recent Developments on Introducing a Historical Dimension in Mathematics Education.* Washington, DC: Mathematical Association of America

Neugebauer, O. & Sachs, A. (1945). *Mathematical Cuneiform Texts*. New Haven, Conn.: American Oriental Society and the American Schools of Oriental Research.

O'Daffer, P. G. & Clemens, S. R. (1992). *Geometry: An Investigative Approach* (2nd Ed). Don Mills, Ontario, Canada: Addison-Wesley.

Rahim, M. H. (1986). Laboratory investigations in geometry: A piece-wise congruence approach: Preliminary propositions, applications, and generalizations. *Int. J. Math. Educ. Sci. Technol.* **17(4)**, 425–447. ME **1986i.**02532

_____ (2003). A new proof of the Pythagorean using a compass and unmarked straight edge. *Int. J. Math. Educ. Sci. Technol.* **34(1)**, 144–150. ME **2003b.**01477

_____ (2011). Dynamic Geometry Software-based dissection-motion-operations: A visual medium for proof and proving. *Learning and Teaching Mathematics* **9**, 3–7.

Rahim, M. H. & Sawada, D. (1986). Revitalizing school geometry through Dissection Motion Operations. *Sch. Sci. Math.* **86(3)**, 235–246. ME **1986x.**00472

Rothstein, E. (2010). Masters of Math, From Old Babylon. *New York Times* **2010(November 26)**. Retrieved October 3, 2013:
http://www.nytimes.com/2010/11/27/arts/design/27tablets.html?_r=0

Usiskin, Z. (1982), *Van Hiele Levels and Achievement in Secondary School Geometry*. University of Chicago. Available
http://www.blog.republicofmath.com/wp-content/uploads/2011/04/Van_Hiele_Levels_Usiskin.pdf

van Hiele, P. (1985) [1959]. *The Child's Thought and Geometry*, Brooklyn, NY: City University of New York, pp. 243-252.

Whiteley, W. (2000). *Dynamic Geometry and the Practice of Geometry.* Paper distributed at the Ninth International Congress on Mathematical Education (ICME 9), held at Tokyo, Japan; 31 July – 7 August. Available at
http://www.math.yorku.ca/Who/Faculty/Whiteley/Dynamic.pdf