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# A REFINED ENUMERATION OF $p$-ARY LABELED TREES

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Abstract. Let $\mathcal{T}_n^{(p)}$ be the set of $p$-ary labeled trees on $\{1, 2, \ldots, n\}$. A maximal decreasing subtree of an $p$-ary labeled tree is defined by the maximal $p$-ary subtree from the root with all edges being decreasing. In this paper, we study a new refinement $\mathcal{T}_{n,k}^{(p)}$ of $\mathcal{T}_n^{(p)}$, which is the set of $p$-ary labeled trees whose maximal decreasing subtree has $k$ vertices.

## 1. Introduction

Let $p$ be a fixed integer greater than 1. A *$p$-ary tree* $T$ is a tree such that:

(i) Either $T$ is empty or has a distinguished vertex $r$ which is called the root of $T$, and
(ii) $T - r$ consists of a weak ordered partition $(T_1, \ldots, T_p)$ of $p$-ary trees.

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A 2-ary(resp. 3-ary) tree is called binary(resp. ternary) tree. Figure 1 exhibits all the ternary tree with 3 vertices. A *full $p$-ary tree* is a $p$-ary tree, where each vertex has either 0 or $p$ children. It is well known (see [6, 6.2.2 Proposition]) that the number of full $p$-ary trees with $n$ internal vertices is given by the $n$th order-$p$ Fuss-Catalan number [2, p. 361] $C_n^{(p)} = \frac{1}{pn+1}\binom{pn+1}{n}$. Clearly a full $p$-ary tree $T$ with $m$ internal vertices corresponds to a $p$-ary tree with $m$ vertices by deleting all the leaves in $T$, so the number of $p$-ary trees with $n$ vertices is also $C_n^{(p)}$.

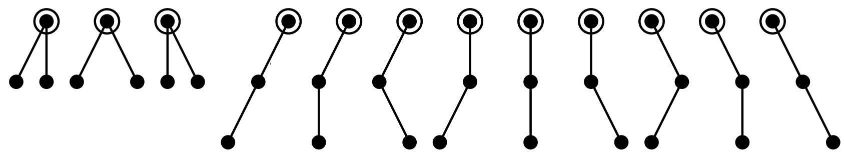

FIGURE 1. All 12 ternary trees with 3 vertices

An *$p$-ary labeled tree* is a $p$-ary tree whose vertices are labeled by distinct positive integers. In most cases, a $p$-ary labeled tree with $n$ vertices is identified with an $p$-ary tree on the vertex set $[n] := \{1, 2, \dots, n\}$. Let $\mathcal{T}_n^{(p)}$ be the set of $p$-ary labeled trees on $[n]$. Clearly the cardinality of $\mathcal{T}_n^{(p)}$ is given by

$$|\mathcal{T}_n^{(p)}| = n!\, C_n^{(p)} = (pn)_{(n-1)}, \tag{1}$$

where $m_{(k)} := m(m-1)\cdots(m-k+1)$ is a falling factorial.

For a given $p$-ary labeled tree $T$, a *maximal decreasing subtree* of $T$ is defined by the maximal $p$-ary subtree from the root with all edges being decreasing, denoted by $\mathrm{MD}(T)$. Figure 2 illustrates the maximal decreasing subtree of a given ternary tree $T$. Let $\mathcal{T}_{n,k}^{(p)}$ be the set of $p$-ary labeled trees on $[n]$ with its maximal decreasing subtree having $k$ vertices.

In this paper we present a formula for $|\mathcal{T}_{n,k}^{(p)}|$, which makes a refined enumeration of $\mathcal{T}_n^{(p)}$, or a generalization of equation (1). Note that a similar refinement for rooted labeled trees and ordered labeled trees were done before (see [4, 5]), but the $p$-ary case is much more complicated and has quite different features.

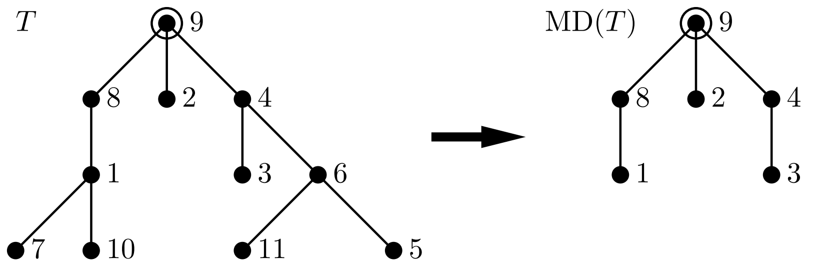


FIGURE 2. The maximal decreasing subtree of the ternary labeled tree $T$

## 2. Main results

From now on we will consider labeled trees only. So we will omit the word "labeled". Recall that $\mathcal{T}_{n,k}^{(p)}$ is the set of $p$-ary trees on $[n]$ with its maximal decreasing ordered subtree having $k$ vertices. Let $\mathcal{Y}_{n,k}^{(p)}$ be the set of $p$-ary trees $T$ on $[n]$, where $T$ is given by attaching additional $(n-k)$ increasing leaves to a decreasing tree with $k$ vertices. Let $\mathcal{F}_{n,k}^{(p)}$ be the set of (non-ordered) forests on $[n]$ consisting of $k$ $p$-ary trees, where the $k$ roots are not ordered. In Figure 3, the first two forests are the same, but the third one is a different forest in $\mathcal{F}_{4,2}^{(2)}$.

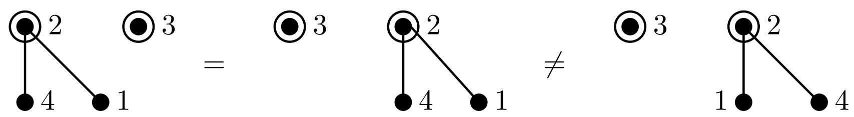


FIGURE 3. Forests in $\mathcal{F}_{4,2}^{(3)}$

Define the numbers

$$t(n,k) = \left|\mathcal{T}_{n,k}^{(p)}\right|,$$
$$y(n,k) = \left|\mathcal{Y}_{n,k}^{(p)}\right|,$$
$$f(n,k) = \left|\mathcal{F}_{n,k}^{(p)}\right|.$$

We will show that a $p$-ary tree can be "decomposed" into a $p$-ary tree in $\bigcup_{n,k} \mathcal{Y}^{(p)}_{n,k}$ and a forest in $\bigcup_{n,k} \mathcal{F}^{(p)}_{n,k}$. Thus it is important to count the numbers $y(n, k)$ and $f(n, k)$.

LEMMA 2.1. *For $0 \le k < n$, the number $y(n, k)$ satisfies the recursion:*

$$y(n+1, k+1) = \sum_{m=0}^{p} \binom{n}{m} p_{(m)} \,(kp - n + m + 1)\, y(n - m, k) \tag{2}$$

*with the following boundary conditions:*

$$y(n, n) = \prod_{j=0}^{n-1} (1 + (p-1)j) \qquad \text{for } n \ge 1 \tag{3}$$

$$y(n, k) = 0 \qquad \text{for } k < \max\left(\frac{n-1}{p}, 1\right). \tag{4}$$

*Proof.* Consider a tree $Y$ in $\mathcal{Y}^{(p)}_{n+1,k+1}$. The tree $Y$ with $n+1$ vertices consists of its maximal decreasing tree with $k+1$ vertices and the number of increasing leaves is $n-k$. Note that the vertex 1 is always contained in $\mathrm{MD}(Y)$.

If the vertex 1 is a leaf of $Y$, consider the tree $Y'$ by deleting the leaf 1 from $Y$. The number of vertices in $Y'$ and $\mathrm{MD}(Y')$ are $n$ and $k$, respectively. So the number of possible trees $Y'$ is $y(n, k)$. Since we cannot attach the vertex 1 to $n-k$ increasing leaves of $Y'$, there are $kp - (n-1)$ ways of recovering $Y$. Thus the number of $Y$ with the leaf 1 is

$$(kp - n + 1) \cdot y(n, k). \tag{5}$$

If the vertex 1 is not a leaf of $Y$, then the vertex 1 has increasing leaves $\ell_1, \ldots, \ell_m$, where $1 \le m \le p$. Consider the tree $Y''$ obtained by deleting $\ell_1, \ldots, \ell_m$ from $Y$. Clearly 1 is a leaf of $Y''$ and the number of vertices in $Y''$ and $\mathrm{MD}(Y'')$ are $n-m+1$ and $k+1$, respectively. Thus by (5), the number of possible trees $Y''$ is $(kp - (n-m) + 1) \cdot y(n-m, k)$. To recover $Y$ is to relabel all the vertices except 1 of $Y''$ with the label set $\{2, 3, \ldots, n+1\} \setminus \{\ell_1, \ldots, \ell_m\}$ and to attach the leaves $\ell_1, \ldots, \ell_m$ to the vertex 1 of $Y''$. Clearly $\ell_1, \ldots, \ell_m$ is a subset of $\{2, 3, \ldots, n+1\}$. It is obvious that a way of attaching $\ell_1, \ldots, \ell_m$ to vertex 1 can be regarded as an injection from $\ell_1, \ldots, \ell_m$ to $[p]$. Thus the number of $Y$ without the

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | |
| 2 | 0 | 2 | 2 | | | | | | | |
| 3 | 0 | 2 | 10 | 6 | | | | | | |
| 4 | 0 | 0 | 24 | 56 | 24 | | | | | |
| 5 | 0 | 0 | 24 | 256 | 360 | 120 | | | | |
| 6 | 0 | 0 | 0 | 640 | 2672 | 2640 | 720 | | | |
| 7 | 0 | 0 | 0 | 720 | 11824 | 28896 | 21840 | 5040 | | |
| 8 | 0 | 0 | 0 | 0 | 30464 | 196352 | 330624 | 201600 | 40320 | |
| 9 | 0 | 0 | 0 | 0 | 35840 | 857728 | 3177600 | 4032000 | 2056320 | 362880 |

TABLE 1. $y(n,k)$ with $p=2$

leaf 1 is

$$\binom{n}{m}\binom{p}{m} m!\,(kp-(n-m)+1)\cdot y(n-m,k). \tag{6}$$

Since $m$ may be the number from 1 to $p$ and substituting $m=0$ in (6) yields (5), we have the recursion (2).

Since $\mathcal{Y}^{(p)}_{n,n}$ is the set of decreasing $p$-ary trees on $[n]$, the equation (3) holds (see [1]). If the inequality $pk-(k-1)<n-k$ holds, $\mathcal{Y}^{(p)}_{n,k}$ should be empty. For $n \geq 1$ and $k=0$, $\mathcal{Y}^{(p)}_{n,k}$ is also empty. Thus the equation (4) also holds. $\square$

The table for $y(n,k)$ with $p=2$ is shown in Table 1.

Now we calculate $f(n,k)$ which is the number of forests on $[n]$ consisting of $k$ $p$-ary trees, where the $k$ components are not ordered. Here we use the convention that the empty product is 1.

LEMMA 2.2. *For $0 \leq k \leq n$, we have*

$$f(n,k)=\binom{n}{k} pk \prod_{i=1}^{n-k-1}(pn-i) \quad \text{if } n>k, \tag{7}$$

*else $f(n,n)=1$.*

*Proof.* Consider a forest $F$ in $\mathcal{F}^{(p)}_{n,k}$. The forest $F$ consists of (non-ordered) $p$-ary trees $T_1,\ldots,T_k$ with roots $r_1,r_2,\ldots,r_k$, where $r_1<r_2<\cdots<r_k$. The number of ways for choosing roots $r_1,r_2,\cdots,r_k$ from $[n]$ is equal to $\binom{n}{k}$. From the *reverse Prüfer algorithm (RP Algorithm)* in [3],

the number of ways for adding $n-k$ vertices successively to $k$ roots $r_1, r_2, \cdots, r_k$ is equal to

$$pk(pn-1)(pn-2)\cdots(pn-n+k+1)$$

for $0<k<n$, thus the equation (7) holds. For $0=k<n$, $\mathcal{F}^{(p)}_{n,0}$ is empty, so $f(n,0)=0$ included in (7). For $0\le k=n$, $\mathcal{F}^{(p)}_{n,n}$ is the set of forests with no edges, so $f(n,n)=1$. □

Since the number $y(n,k)$ is determined by the recurrence relation (2) in Lemma 2.1, we can count the number $t(n,k)$ with the following theorem.

THEOREM 2.3. *For $n\ge 1$, we have*

$$t(n,k)=\sum_{m=k}^{n}\binom{n}{m}\frac{m-k}{n-k}(pn-pk)_{(n-m)}\,y(m,k) \quad \text{if } 1\le k<n, \tag{8}$$

*else* $t(n,n)=\prod_{j=0}^{n-1}(pj-j+1)$, *where* $a_{(\ell)}:=a(a-1)\cdots(a-\ell+1)$ *is a falling factorial.*

*Proof.* Given a $p$-ary tree $T$ in $\mathcal{T}^{(p)}_{n,k}$, let $Y$ be the subtree of $T$ consisting of $\mathrm{MD}(T)$ and its increasing leaves. If $Y$ has $m$ vertices, then $Y$ is a subtree of $T$ with $(m-k)$ increasing leaves. Also, the induced subgraph $Z$ of $T$ generated by the $(n-k)$ vertices not belonging to $\mathrm{MD}(T)$ is a (non-ordered) forest consisting of $(m-k)$ $p$-ary trees whose roots are increasing leaves of $Y$. Figure 4 illustrates the subgraph $Y$ and $Z$ of a given ternary tree $T$.

Now let us count the number of $p$-ary trees $T\in\mathcal{T}^{(p)}_{n,k}$ with $|V(Y)|=m$ where $V(Y)$ is the set of vertices in $Y$. First of all, the number of ways for selecting a set $V(Y)\subset[n]$ is equal to $\binom{n}{m}$. By attaching $(m-k)$ increasing leaves to a decreasing $p$-ary tree with $k$ vertices, we can make a $p$-ary trees on $V(Y)$. So there are exactly $y(m,k)$ ways for making such a $p$-ary subtree on $V(Y)$. Since all the roots of $Z$ are determined by $Y$, by the definition of $\mathcal{F}^{(p)}_{n,k}$ and Lemma 2.2, the number of ways for constructing the other parts on $V(T)\setminus V(\mathrm{MD}(T))$ is equal to

$$f(n-k,m-k)\Big/\binom{n-k}{m-k}=\frac{m-k}{n-k}(pn-pk)_{(n-m)}.$$

Since the range of $m$ is $k\le m\le n$, the equation (8) holds.

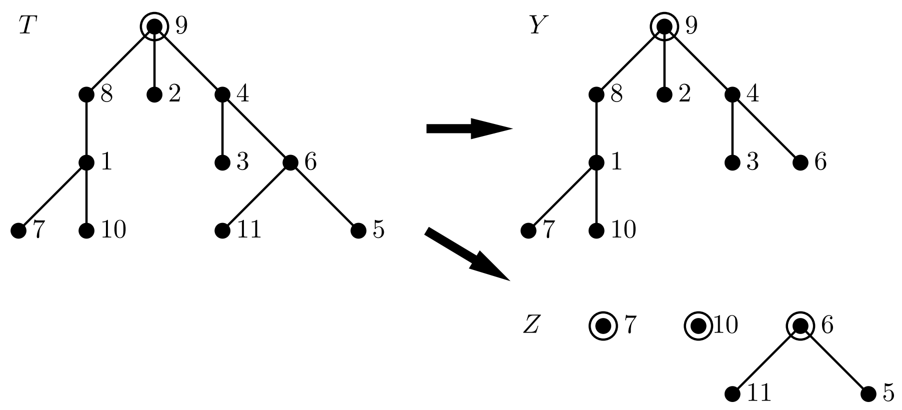


FIGURE 4. Decomposition of $T$ into $Y$ and $Z$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $n!C_n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 0 | 1 | | | | | | | 1 |
| 2 | 0 | 2 | 2 | | | | | | 4 |
| 3 | 0 | 14 | 10 | 6 | | | | | 30 |
| 4 | 0 | 152 | 104 | 56 | 24 | | | | 336 |
| 5 | 0 | 2240 | 1504 | 816 | 360 | 120 | | | 5040 |
| 6 | 0 | 41760 | 27744 | 15184 | 6992 | 2640 | 720 | | 95040 |
| 7 | 0 | 942480 | 621936 | 342768 | 162240 | 65856 | 21840 | 5040 | 2162160 |

TABLE 2. $t(n,k)$ with $p = 2$

Finally, $\mathcal{T}^{(p)}(n,n)$ is the set of decreasing $p$-ary trees on $[n]$, so

$$t(n,n) = y(n,n) = \prod_{j=0}^{n-1} (pj - j + 1)$$

holds for $n \geq 1$. □

The sequence $t(n,k)$ with $p = 2$ is listed in Table 2. Note that each row sum is equal to $n!C_n^{(p)}$ with $p = 2$.

REMARK. Due to Lemma 2.1 and Theorem 2.3, we can calculate $t(n,k)$ for all $n$, $k$. In particular we express $t(n,k)$ as a linear combination of $y(k,k), y(k+1,k), \ldots, y(n,k)$. However a closed form, a recurrence relation, or a (double) generating function of $t(n,k)$ have not been found yet.

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