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SCALAR CURVATURE FUNCTIONS OF ALMOST-KÄHLER METRICS ON A CLOSED SOLV-MANIFOLD

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Abstract. We discuss on the classification problem of symplectic manifolds into three families according to the scalar curvature functions of almost Kähler metrics they admit. We also present a 4-dimensional solv-manifold as an example which belongs to one of the three families.

1. Introduction

Kazdan and Warner classified closed smooth manifolds of dimension> 2 into three families according to what the scalar curvature functions can be on a manifold [2, Theorem 4.35]. In [6, 7] we studied an analogous classification question of closed symplectic manifolds of dimension> 2 according to the scalar curvature functions of almost Kähler metrics.

The purpose of this article is to elaborate this question, to discuss main problems and to supply more examples.

Let us recall symplectic homotopies [8]. Two symplectic forms ω_0 and ω_1 are called *deformation equivalent* if they can be joined by a

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smooth homotopy of sympelctic forms and *symplectomorphic* if there exists a diffeomorphism ψ such that $\omega_0 = \psi^*\omega_1$. And they are *weakly deformation equivalent* if there is a diffeomorphism ψ such that $\psi^*\omega_1$ is deformation equivalent to ω_0.

Now the above question can be re-stated as follows; does any weakly deformation equivalence class $[[\omega]]$ of a symplectic structure on a closed smooth manifold of dimension> 2 fall into one of the following families?

(a) Any smooth function is the scalar curvature of an almost Kähler metric in $[[\omega]]$.

(b) A smooth function is the scalar curvature of an almost Kähler metric in $[[\omega]]$ iff it is either identically zero or somewhere negative.

(c) A smooth function is the scalar curvature of an almost Kähler metric in $[[\omega]]$ iff it is negative somewhere.

There are a number of smooth manifolds which admit more than one weakly deformation equivalence classes. It is interesting to know if there exists a smooth manifold with more than one weakly deformation equivalence classes which belong to distinct families among (a), (b) and (c).

For the above classification, one should use scalar curvature deformation theory in L^p spaces, $p > 0$. An essential part is to show the surjectivity of the derivative DS of the scalar curvature functional S on the space of almost Kähler metrics compatible with a symplectic structure. We ask:

Question

1. Is DS surjective at generic constant scalar-curved almost Kähler metrics?
2. Is DS surjective at any non-constant scalar-curved almost Kähler metrics?

This question seems harder than the general Riemannian case in [2, 4.37] and so, only a few examples are shown to belong to some family: some symplectic tori [6, Section 5] and nil-manifolds [7]. We expect that most Kähler manifolds can belong to some classes but it is not strictly shown yet.

So, here we worked on a closed symplectic solv-manifold, described by Fernández and Gray in [4]. Together with afore-mentioned examples, it leads one to suspect the validity of Question 1 more. We proved the surjectivity of DS at some constant scalar-curved metric on it to get;

THEOREM 1.1. *For the weakly deformation equivalence class* $[[\omega]]$ *of the closed symplectic solv-manifold* (M,ω) *of Fernández-Gray, a smooth function is the scalar curvature of some almost-Kähler metrics in* $[[\omega]]$ *if and only if it is somewhere negative.*

2. Almost-Kähler metrics

For this subsection a good reference is [3]. An *almost-Kähler* metric on a smooth manifold M^{2n} of real dimension $2n$ is a Riemannian metric g compatible with a symplectic structure ω, i.e. $\omega(X,Y) = g(JX,Y)$ for an almost complex structure J, where X, Y are tangent vectors at a point of the manifold. Note that given ω, g determines J and vice versa. We call a Riemannian metric g ω-almost Kähler if g is compatible with ω and denote by $\Omega_\omega := \Omega_\omega(M)$ the set of all C^∞ ω-almost Kähler metrics on M. An almost-Kähler metric (g,ω,J) is Kähler if and only if J is integrable.

An almost complex structure J gives rise to a type decomposition of symmetric (0,2)-tensors. For any symmetric (0,2)-tensor field h, we have the splitting $h = h^+ + h^-$, where $h^+(X,Y) = \frac{1}{2}\{h(X,Y) + h(JX,JY)\}$ and $h^-(X,Y) = \frac{1}{2}\{h(X,Y) - h(JX,JY)\}$. A symmetric (0,2)-tensor field h is called J-invariant [or J-anti-invariant] if $h^- = 0$ [or $h^+ = 0$, respectively].

The space Ω_ω is a smooth Fréchet manifold, and the tangent space $T_g\Omega_\omega$ at $g \in \Omega$ is exactly the set of J-anti-invariant symmetric (0,2)-tensor fields, where J is the almost complex structure corresponding to (g,ω).

3. Scalar curvature functions of almost Kähler metrics

In this section we recall the argument in [7].

3.1. Derivative of the scalar curvature functional. We consider the scalar curvature map defined on the space Ω_ω;

$$S_\omega(g) := \text{the scalar curvature of } g.$$

The differential at g, in the direction of a J-anti-invariant symmetric (0,2)-tensor h, of S_ω is given by

$$DS_\omega|_g(h) = \delta_g(\delta_g h) - g(r, h), \tag{1}$$

where r is the Ricci curvature tensor of g, $\delta_g h$ is the divergence of h which can be written in local coordinates as $(\delta_g h)_\lambda = -\nabla^\nu h_{\nu\lambda}$ and finally $\delta_g(\cdot)$ for 1-forms is the formal adjoint of the exterior differential on functions.

So $DS_\omega|_g$ is an under-determined elliptic operator for any $g \in \Omega_\omega$. The formal adjoint operator $(DS_\omega|_g)^* : C^\infty(M) \to T_g\Omega$ of $DS_\omega|_g$ with respect to the L^2 inner product induced from g is then as follows:

$$(DS_\omega|_g)^*(\psi) = \nabla^- d\psi - r^-\psi. \tag{2}$$

where $\nabla^- d\psi$ and r^- are the J-anti-invariant part of $\nabla d\psi$ and r, respectively, and J is the corresponding almost complex structure to (g, ω).

3.2. Scalar Curvature Map in L^p Setting. The scalar curvature map $S_\omega : \Omega_\omega \longrightarrow C^\infty(M)$ can be extended to a smooth map from the space of L^p_2 ω-almost-Kähler metrics, $L^p_2(\Omega_\omega)$, to the space of L^p functions, $L^p(M)$, if $p > \dim_{\mathbb{R}}(M)$, which will be assumed in this section.

Now at $g \in \Omega_\omega$ with J, consider the linearized map of S_ω, $DS_\omega|_g : L^p_2(T_g\Omega_\omega) \longrightarrow L^p(M)$. The space $L^p_2(T_g\Omega_\omega)$ consists of L^p_2 J-anti invariant symmetric 2-tensor fields h. As $DS_\omega|_g$ is an under-determined elliptic operator at any $g \in \Omega_\omega$, by the elliptic regularity theory [2, page 464], we have a decomposition:

$$L^p(M) = DS_\omega|_g(L^p_2(T_g\Omega_\omega)) \oplus \ker (DS_\omega|_g)^*.$$

and the kernel $\ker (DS_\omega|_g)^*$ of the formal adjoint map $(DS_\omega|_g)^*$ is finite dimensional and consists of C^∞ functions on M. Therefore in order to prove that $DS_\omega|_g$ is surjective, we need to show that $\ker (DS_\omega|_g)^*$ is zero.

Lemma 1, Lemma 2 and Proposition 1 below gives the framework of the argument; refer to [7] for the proof.

Lemma 1. *If $DS_\omega|_g$ is surjective at an almost-Kähler metric g, then the scalar curvature map S_ω is locally surjective at g, i.e. there exists $\epsilon > 0$ such that, if f is in $L^p(M)$ and $||f - S_\omega(g)||_{L^p} < \epsilon$, there is an L^p_2 almost-Kähler metric $\tilde{g}$ such that $f = S_\omega(\tilde{g})$. Furthermore if f is C^∞, so is $\tilde{g}$.*

Let $\mathcal{D}$ be the diffeomorphism group of M.

Lemma 2. *[5] If* $\dim_{\mathbb{R}}(M) \geq 2$ *and if* $f \in C^0(M)$*, then an* L^p *function* f_1 *belongs to the* L^p *closure of the set* $\{f \circ \phi, \phi \in \mathcal{D}\}$ *if and only if* $\inf f \leq f_1(x) \leq \sup f$ *almost everywhere.*

From above two lemmas, one gets;

Proposition 1. *Suppose that there exists an almost-Kähler metric* (g, ω) *with constant scalar curvature* s_g *and that* $DS_\omega|_g$ *is surjective at* g*, then any smooth function* f *with* $\inf f \leq s_g \leq \sup f$ *is the scalar curvature of an almost-Kähler metric* $\tilde{g}$ *for some symplectic form* $\phi^*\omega$*, where* ϕ *is a diffeomorphism.*

4. On a closed symplectic solv-manifold

In order to use Proposition 1, we need to have almost-Kähler metrics (g, ω) with constant scalar curvature such that $DS_\omega|_g$ is surjective. Here we consider the closed 4-dimensional symplectic solv-manifold from [4].

We recall the solvable Lie group G of dimension 3 consisting of matrices of the form

$$\begin{bmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $x, y, z \in \mathbb{R}$ and k is a real number such that $e^k + e^{-k}$ is an integer different from 2.

Then x, y, z are global coordinates for G and $\{dx - kxdz, dy + kydz, dz\}$ is a basis of right invariant 1-forms on G. There exists a co-compact (i.e. uniform) discrete subgroup Γ of G so that $N = G/\Gamma$ is a manifold and the forms $dx - kxdz, dy + kydz, dz$ descend to 1-forms on N. We consider the product 4-manifold $M = N \times S^1$. The group $\Gamma \times \mathbb{Z}$ acts naturally on its universal cover is $G \times \mathbb{R}$, naturally identified with $\mathbb{R}^4 = \{(x, y, z, t) |\ x, y, z, t \in \mathbb{R}\}$.

The symplectic form is $\omega = (dx - kxdz) \wedge (dy + kydz) + dz \wedge dt = dx \wedge dy + dz \wedge dt + kydx \wedge dz + kxdy \wedge dz$. We consider the almost Kähler metric $g = (dx - kxdz)^2 + (dy + kydz)^2 + dz^2 + dt^2$.

Consider the g-orthonormal frame fields $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = kx\frac{\partial}{\partial x} - ky\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, $e_4 = \frac{\partial}{\partial t}$, which are the dual to $\{dx - kxdz, dy + kydz, dz, dt\}$.

The corresonding almost complex structure J is then given by $J(e_1) = e_2,\ J(e_2) = -e_1,\ J(e_3) = e_4,\ J(e_4) = -e_3$.

These ω, g, J are all right invariant on $G \times \mathbb{R}$ and descends to an almost Kähler structure on M.

The discrete subgroup $\Gamma \times \mathbb{Z}$ is generated [1, Theorem 4 (4)] by $\{\gamma_i, i = 1,2,3,4\}$ where

$\gamma_1(x,y,z,t) = (x + u_1 e^{kz}, y + u_2 e^{-kz}, z, t)$,

$\gamma_2(x,y,z,t) = (x + v_1 e^{kz}, y + v_2 e^{-kz}, z, t)$ for some $\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \neq 0$,

$\gamma_3(x,y,z,t) = (x,y,z+n,t)$ for some $n \in Z$,

$\gamma_4(x,y,z,t) = (x,y,z,t+1)$.

So, any function ψ on M satisfies

$$\psi(p) = \psi(\gamma_i \cdot p), \quad i = 1,2,3,4 \quad \text{at any point } p \in M. \tag{3}$$

By routine computation one can find the components r_{ij} of Ricci curvature as follows; $r_{11} = 0, r_{22} = 0, r_{33} = -2k^2, r_{44} = 0$, and $r_{ij} = 0$ for $i \neq j$. Then the components $r^-_{ij} = r^-(e_i, e_j) = \frac{1}{2}\{r(e_i,e_j) - r(Je_i, Je_j)\}$ of the J-anti-invariant part of the Ricci tensor are as follows: $r^-_{11} = 0$, $r^-_{22} = 0$, $r^-_{33} = -k^2$, $r^-_{44} = k^2$, and $r^-_{ij} = 0$ for $(i \neq j)$.

Suppose that a smooth function ψ belongs to $\mathrm{Ker}(\mathrm{D_g S})^*$. Equivalently, it satisfies ${\nabla_g}^- d\psi - \psi r^-_g = 0$. Now one compute the J-anti-invariant part $\nabla^- d\psi$ of the Hessian of ψ. For convenience we denote $\frac{\partial \psi}{\partial x}$ by ψ_x and $\frac{\partial^2 \psi}{\partial x \partial y}$ by ψ_{yx}, etc.. The Riemannian connection ∇ can be computed by the formula $2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle$;

$$\nabla_{e_1} e_1 = -ke_3, \quad \nabla_{e_1} e_3 = ke_1, \quad \nabla_{e_2} e_2 = ke_3, \quad \nabla_{e_2} e_3 = -ke_2$$

and others are all zero. We compute $\nabla d\psi(X,Y) = X(Y\psi) - (\nabla_X Y)\psi$ and set $\alpha_{ij} = \nabla d\psi(e_i, e_j)$;

$$\begin{aligned}
&\alpha_{11} = \psi_{xx} + k^2 x\psi_x - k^2 y\psi_y + k\psi_z, \quad \alpha_{22} = \psi_{yy} - k^2 x\psi_x + k^2 y\psi_y - k\psi_z,\\
&\alpha_{33} = k^2(x^2\psi_{xx} + y^2\psi_{yy} + x\psi_x + y\psi_y - 2xy\psi_{xy})\\
&\qquad\quad + \psi_{zz} + 2kx\psi_{xz} - 2ky\psi_{yz},\\
&\alpha_{44} = \psi_{tt}, \qquad \alpha_{12} = \psi_{xy}, \qquad \alpha_{13} = kx\psi_{xx} - ky\psi_{xy} + \psi_{xz},\\
&\alpha_{14} = \psi_{xt}, \qquad \alpha_{23} = kx\psi_{xy} - ky\psi_{yy} + \psi_{yz},\\
&\alpha_{24} = \psi_{yt}, \qquad \alpha_{34} = kx\psi_{xt} - ky\psi_{yt} + \psi_{zt}.
\end{aligned}$$

We compute $\nabla^- d\psi(X,Y) = \frac{1}{2}\{\nabla d\psi(X,Y) - \nabla d\psi(JX,JY)\}$ and set $\beta_{ij} = 2\nabla^- d\psi(e_i,e_j)$;

$$\begin{aligned}
\beta_{11} &= \psi_{xx} - \psi_{yy} + 2k^2x\psi_x - 2k^2y\psi_y + 2k\psi_z,\\
\beta_{33} &= k^2(x^2\psi_{xx} + y^2\psi_{yy} + x\psi_x + y\psi_y - 2xy\psi_{xy})\\
&\quad +\psi_{zz} - \psi_{tt} + 2k(x\psi_{xz} - y\psi_{yz}),\\
\beta_{12} &= 2\psi_{xy}, \qquad \beta_{13} = kx\psi_{xx} - ky\psi_{xy} + \psi_{xz} - \psi_{yt},\\
\beta_{14} &= kx\psi_{xy} - ky\psi_{yy} + \psi_{xt} + \psi_{yz}, \qquad \beta_{34} = 2kx\psi_{xt} - 2ky\psi_{yt} + 2\psi_{zt}.
\end{aligned}$$

We deduce that the equation ${\nabla_g}^- d\psi - \psi r_g^- = 0$ is equivalent to the following system of six differential equations.

$$\begin{aligned}
&\langle 1\rangle\ \psi_{xx} - \psi_{yy} + 2k^2x\psi_x - 2k^2y\psi_y + 2k\psi_z = 0,\\
&\langle 2\rangle\ k^2(x^2\psi_{xx} + y^2\psi_{yy} + x\psi_x + y\psi_y - 2xy\psi_{xy}) + \psi_{zz} - \psi_{tt}\\
&\qquad\qquad +2kx\psi_{xz} - 2ky\psi_{yz} = -2k^2\psi,\\
&\langle 3\rangle\ kx\psi_{xx} - ky\psi_{xy} + \psi_{xz} - \psi_{yt} = 0,\\
&\langle 4\rangle\ kx\psi_{xy} - ky\psi_{yy} + \psi_{xt} + \psi_{yz} = 0,\\
&\langle 5\rangle\ kx\psi_{xt} - ky\psi_{yt} + \psi_{zt} = 0, \qquad \langle 6\rangle\ \psi_{xy} = 0.
\end{aligned}$$

Now the surjectivity of D_gS follows by showing that ψ should be necessarily zero. The computation is elementary and it goes as follows.

From the equation $\langle 6\rangle$, $\psi(x,y,z,t)$ can be written as $\psi = a(x,z,t) + b(y,z,t)$. Put it into $\langle 4\rangle$ to get

$$\langle 4'\rangle \qquad - kyb_{yy} + b_{yz} = -a_{xt}.$$

LHS depends on y,z,t and RHS on x,z,t. So both sides depends only on z,t. We get $a_{xtx} = 0$. From $\frac{\partial}{\partial x}\langle 5\rangle$, we have $k\psi_{xt} + kx\psi_{xtx} + \psi_{ztx} = ka_{xt} + (a_{xt})_z = 0$. We solve for a_{xt} to get $a_{xt} = c(x,t)e^{-kz}$. By (3) ψ satisfies $\psi(x,y,z,t) = \psi(x,y,z+n,t)$. Then a_{xt} also satisfies $a_{xt}(x,y,z,t) = a_{xt}(x,y,z+n,t)$. This implies $a_{xt} = 0$.

If we repeat the same argument with $\langle 3\rangle$ and $\frac{\partial}{\partial y}\langle 5\rangle$, we can get $b_{yt} = 0$. Then $\langle 4'\rangle$ gives $kyb_{yy} = b_{yz}$. And $\langle 3\rangle$ gives $kxa_{xx} = -a_{xz}$.

$\frac{\partial}{\partial x}\langle 1\rangle$ gives $\psi_{xxx} + 2k^2\psi_x + 2k^2x\psi_{xx} + 2k\psi_{xz} = \psi_{xxx} + 2k^2\psi_x = 0$. Solving this for ψ_x, we get $\psi_x = c_1(z)\cos(\sqrt{2}kx) + c_2(z)\sin(\sqrt{2}kx)$, where we used $\psi_x = a_x$ and $\psi_{xt} = a_{xt} = 0$.

By (3), $\psi(x,y,z,t) = \psi(x+ae^{kz}, y+be^{-kz}, z, t)$ for some $a \neq 0$. So, $\psi_x(x,y,z,t) = \psi_x(x+ae^{kz}, y+be^{-kz}, z, t)$. Setting $e(z) = \sqrt{2}kae^{kz}$, this becomes

$$\begin{aligned}
\psi_x(x,y,z,t) &= c_1(z)\cos(\sqrt{2}kx) + c_2(z)\sin(\sqrt{2}kx)\\
&= \psi_x(x+ae^{kz}, y+be^{-kz}, z, t)\\
&= c_1(z)\cos(\sqrt{2}kx + e(z)) + c_2(z)\sin(\sqrt{2}kx + e(z))\\
&= \cos(\sqrt{2}kx)\{c_1(z)\cos(e(z)) + c_2(z)\sin(e(z))\}\\
&\quad +\sin(\sqrt{2}kx)\{-c_1(z)\sin(e(z)) + c_2(z)\cos(e(z))\}.
\end{aligned}$$

Comparing coefficients we have $c_1(z) = c_1(z)\cos(e(z)) + c_2(z)\sin(e(z))$ and $c_2(z) = -c_1(z)\sin(e(z)) + c_2(z)\cos(e(z))$. Then we get $c_1(z) = c_2(z) = 0$. So we get $\psi_x = 0$. Similarly we get $\psi_y = 0$, using $\frac{\partial}{\partial y}\langle 1\rangle$.

From $\langle 1\rangle$, we get $\psi_z = 0$. From $\langle 2\rangle$, we get $-\psi_{tt} = -2k^2\psi$. Then $\psi = \psi(t) = c_1 e^{\sqrt{2}kt} + c_2 e^{-\sqrt{2}kt}$.

As $\psi(x,y,z,t) = \psi(x,y,z,t+1)$, $\psi = 0$. So $\mathrm{Ker}(D_gS)^* = 0$. and the linearized map $DS_\omega|_g$ is surjective.

The scalar curvature of $c^2g, c > 0$ can be any negative constant and c^2g is an almost Kähler metric compatible with the symplectic structure $c^2\omega$, which is deformation equivalent to ω. Clearly $DS_{c^2\omega}|_{c^2g}$ is also surjective. So from Proposition 1 we get the 'if' part of Theorem 1.1. Next, M cannot admit a Riemannian metric with nonnegative scalar curvature for a similar reason as the Kodaira-Thurston manifold, see [7]. This proves Theorem 1.1.

Remark 1 Remarkably, the proof shows that there exists a non-zero local (though not global on M) solution of $(D_gS)^*\psi = 0$: $\psi = c_1 e^{\sqrt{2}kt} + c_2 e^{-\sqrt{2}kt}$. This is in contrast to the metrics on Kodaira-Thurston manifold.

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