

t -SPLITTING SETS S OF AN INTEGRAL DOMAIN D SUCH THAT D_S IS A FACTORIAL DOMAIN

GYU WHAN CHANG

ABSTRACT. Let D be an integral domain, S be a saturated multiplicative subset of D such that D_S is a factorial domain, $\{X_\alpha\}$ be a nonempty set of indeterminates, and $D[\{X_\alpha\}]$ be the polynomial ring over D . We show that S is a splitting (resp., almost splitting, t -splitting) set in D if and only if every nonzero prime t -ideal of D disjoint from S is principal (resp., contains a primary element, is t -invertible). We use this result to show that $D \setminus \{0\}$ is a splitting (resp., almost splitting, t -splitting) set in $D[\{X_\alpha\}]$ if and only if D is a GCD-domain (resp., UMT-domain with $Cl(D[\{X_\alpha\}])$ torsion, UMT-domain).

1. Introduction

Let D be an integral domain with quotient field K , and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D . For each $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup \{I_v \mid J \in \mathbf{F}(D), J \subseteq I, \text{ and } J \text{ is finitely generated}\}$. An ideal $I \in \mathbf{F}(D)$ is called a t -ideal if $I_t = I$, and a t -ideal is a *maximal t -ideal* if it is maximal among proper integral t -ideals. It is well known that each nonzero principal ideal is a t -ideal; each proper integral t -ideal is contained in a maximal t -ideal; a prime ideal minimal over a t -ideal is a t -ideal; and each maximal

Received October 10, 2013. Revised November 28, 2013. Accepted December 3, 2013.

2010 Mathematics Subject Classification: 13A15 13G05.

Key words and phrases: t -splitting set, (almost) splitting set, UMT-domain, AGCD-domain, GCD-domain.

© The Kangwon-Kyungki Mathematical Society, 2013.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

t -ideal is a prime ideal. We say that an $I \in \mathbf{F}(D)$ is t -invertible if $(II^{-1})_t = D$; equivalently, if $II^{-1} \not\subseteq P$ for every maximal t -ideal P of D . Let $T(D)$ be the group of t -invertible fractional t -ideals of D under the t -multiplication $A * B = (AB)_t$, and let $\text{Prin}(D)$ be its subgroup of principal fractional ideals. The (t) -class group of D is an abelian group $Cl(D) = T(D)/\text{Prin}(D)$. The readers can refer to [12] for any undefined notation or terminology.

Let S be a saturated multiplicative subset of an integral domain D . As in [3], we say that S is a t -splitting set if for each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals A, B of D , where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. We say that S is an *almost splitting set* of D if for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n = sa$ for some $s \in S$ and $a \in N(S)$, where $N(S) = \{0 \neq x \in D \mid (x, s')_t = D \text{ for all } s' \in S\}$. A *splitting set* is an almost splitting set in which $n = n(d) = 1$ for every $0 \neq d \in D$. Let \bar{S} be the saturation of a multiplicative set S of D . Note that a splitting set is saturated, while both t -splitting sets and almost splitting sets need not be saturated. Also, note that S is t -splitting (resp., almost splitting) if and only if \bar{S} is; so we always assume that S is saturated. It is known that an almost splitting set is t -splitting [7, Proposition 2.3]; hence

$$\text{splitting set} \Rightarrow \text{almost splitting} \Rightarrow t\text{-splitting set}.$$

Moreover, if $Cl(D)$ is torsion, then a t -splitting set is almost splitting [7, Corollary 2.4] and if $Cl(D) = 0$, then splitting set \Leftrightarrow almost splitting $\Leftrightarrow t$ -splitting set.

Let X be an indeterminate over D and $D[X]$ be the polynomial ring over D . An *upper to zero* in $D[X]$ is a nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$, and D is called a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t -ideal of $D[X]$. We say that D is a *Prüfer v -multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t -invertible. As in [15], we say that D is an *almost GCD-domain* (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is principal. Clearly, GCD-domains are AGCD-domains. It is known that AGCD-domains are UMT-domains with torsion class group [5, Lemma 3.1]; D is a PvMD if and only if D is an integrally closed UMT-domain [13, Proposition 3.2]; and D is a GCD-domain if and only if D is a PvMD and $Cl(D) = 0$ [6, Corollary 1.5].

In [9, Theorem 2.8], the authors proved that if D_S is a principal ideal domain (PID), then S is a t -splitting set of D if and only if every nonzero prime ideal of D disjoint from S is t -invertible. They used this result to show that $D \setminus \{0\}$ is a t -splitting set of $D[X]$ if and only if D is a UMT-domain [9, Corollary 2.9]. Also, in [8, Theorem 2], the author showed that if D_S is a PID, then S is an almost splitting set of D if and only if every nonzero prime ideal of D disjoint from S contains a primary element. (A nonzero element $a \in D$ is said to be *primary* if aD is a primary ideal.) The purpose of this paper is to show that the results of [9, Theorem 2.8] and [8, Theorem 2] are also true when D_S is a factorial domain (note that a PID is a factorial domain). Precisely, we show that if D_S is a factorial domain, then S is a splitting (resp., almost splitting, t -splitting) set in D if and only if every nonzero prime t -ideal of D disjoint from S is principal (resp., contains a primary element, is t -invertible). Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D , and note that $D[\{X_\alpha\}]_{D \setminus \{0\}}$ is a factorial domain. Hence, we then use the results we obtained in this paper to show that $D \setminus \{0\}$ is a splitting (resp., almost splitting, t -splitting) set in $D[\{X_\alpha\}]$ if and only if D is a GCD-domain (resp., UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion, UMT-domain).

2. Main Results

Let D be an integral domain, $D^* = D \setminus \{0\}$, $\{X_\alpha\}$ be a nonempty set of indeterminates over D , and $D[\{X_\alpha\}]$ be the polynomial ring over D .

We begin this section with nice characterizations of splitting sets, almost splitting sets, and t -splitting sets which appear in [2, Theorem 2.2], [4, Proposition 2.7], and [3, Corollary 2.3], respectively.

LEMMA 1. *Let S be a saturated multiplicative subset of D .*

1. *S is a splitting (resp., t -splitting) set of D if and only if $dD_S \cap D$ is principal (resp., t -invertible) for every $0 \neq d \in D$.*
2. *S is an almost splitting set of D if and only if for every $0 \neq d \in D$, there is a positive integer $n = n(d)$ such that $d^n D_S \cap D$ is principal.*

Note that if D_S is a PID, then every nonzero prime ideal P of D disjoint from S has height-one, and thus P is a t -ideal. Hence, our first result is a generalization of [9, Theorem 2.8] that if D_S is a PID, then S is a t -splitting set in D if and only if every nonzero prime ideal of D disjoint

from S is t -invertible. The proof is similar to those of [9, Theorem 2.8] and [8, Theorem 2].

THEOREM 2. *Let D be an integral domain and S be a saturated multiplicative subset of D such that D_S is a factorial domain. Then S is a t -splitting set in D if and only if every prime t -ideal of D disjoint from S is t -invertible.*

Proof. (\Rightarrow) Assume that S is a t -splitting set of D , and let P be a prime t -ideal of D with $P \cap S = \emptyset$. Then $(PD_S)_t = PD_S$ [3, Theorem 4.9], and hence $PD_S = pD_S$ for some $p \in P$ since D_S is a factorial domain. Thus, by Lemma 1, $P = PD_S \cap D = pD_S \cap D$ is t -invertible.

(\Leftarrow) Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S$ for some $p_i \in D$ and positive integers e_i such that every p_i is a prime element in D_S and $p_i D_S \neq p_j D_S$ if $i \neq j$. Let P_i be the prime ideal of D such that $P_i D_S = p_i D_S$. Clearly, P_i is minimal over dD , and hence P_i is a t -ideal. Moreover, $P_i \cap S = \emptyset$; so P_i is t -invertible by assumption (and hence a maximal t -ideal [13, Proposition 1.3]). Note that $(P_i^{e_i})_t$ is P_i -primary [1, Lemma 1] because P_i is a maximal t -ideal. Also, $(P_i^{e_i})_t D_S = p_i^{e_i} D_S$, and thus $P_i^{e_i} D_S \cap D = (P_i^{e_i})_t$ and $(P_i^{e_i})_t$ is t -invertible. Hence

$$\begin{aligned} dD_S \cap D &= p_1^{e_1} \cdots p_k^{e_k} D_S \cap D \\ &= (p_1^{e_1} D_S \cap \cdots \cap p_k^{e_k} D_S) \cap D \\ &= (P_1^{e_1} D_S \cap \cdots \cap P_k^{e_k} D_S) \cap D \\ &= (P_1^{e_1} D_S \cap D) \cap \cdots \cap (P_k^{e_k} D_S \cap D) \\ &= (P_1^{e_1})_t \cap \cdots \cap (P_k^{e_k})_t \\ &= ((P_1^{e_1})_t \cdots (P_k^{e_k})_t)_t. \end{aligned}$$

Thus, S is a t -splitting set by Lemma 1. \square

The next result is a generalization of [9, Corollary 2.9] that D^* is a t -splitting set in $D[X]$, where X is an indeterminate over D , if and only if D is a UMT-domain.

COROLLARY 3. *D^* is a t -splitting set in $D[\{X_\alpha\}]$ if and only if D is a UMT-domain.*

Proof. (\Rightarrow) Let $X \in \{X_\alpha\}$, and let P be a nonzero prime ideal of $D[X]$ with $P \cap D = (0)$. Then P is a prime t -ideal of $D[X]$, and hence $Q := P[Y]$, where $Y = \{X_\alpha\} \setminus \{X\}$, is a prime t -ideal of $D[\{X_\alpha\}]$ [11, Lemma 2.1(1)] such that $Q \cap D^* = \emptyset$. Hence, Q is t -invertible by Theorem 2 because $D[\{X_\alpha\}]_{D^*}$ is a factorial domain. Note that $D[\{X_\alpha\}] =$

$(QQ^{-1})_t = ((P[Y])(P[Y])^{-1})_t = ((P[Y])(P^{-1}[Y]))_t = ((PP^{-1})[Y])_t = (PP^{-1})_t[Y]$ [11, Lemma 2.1(1)]. Hence, P is t -invertible, and thus P is a maximal t -ideal of $D[X]$.

(\Leftarrow) Let Q be a prime t -ideal of $D[\{X_\alpha\}]$ such that $Q \cap D^* = \emptyset$. Since $Q \neq (0)$, there are $X_1, \dots, X_n \in \{X_\alpha\}$ such that $Q \cap D[X_1, \dots, X_{n-1}] = (0)$, but $Q \cap D[X_1, \dots, X_n] \neq (0)$. Let $R = D[X_1, \dots, X_{n-1}]$ and $P = Q \cap R[X_n]$. Then R is a UMT-domain [11, Theorem 2.4] and P is an upper to zero in $R[X_n]$. Hence, P is a t -invertible prime t -ideal. Let $Z = \{X_\alpha\} \setminus \{X_1, \dots, X_n\}$, and note that $P[Z] \subseteq Q$ and $P[Z]$ is a t -invertible prime t -ideal of $D[\{X_\alpha\}]$ (see the proof of (\Rightarrow) above). Hence, $P[Z]$ is a maximal t -ideal of $D[\{X_\alpha\}]$, and thus $Q = P[Z]$ and Q is t -invertible. Thus, by Theorem 2, D^* is a t -splitting set. \square

We next give an almost splitting set analog of Theorem 2. Even though the proof is a word for word translation of the proof of [8, Theorem 2], we give it for the completeness of this paper.

THEOREM 4. *Let D be an integral domain and S be a saturated multiplicative subset of D such that D_S is a factorial domain. Then S is an almost splitting set in D if and only if every prime t -ideal of D disjoint from S contains a primary element.*

Proof. (\Rightarrow) Assume that S is an almost splitting set of D , and let P be a prime t -ideal of D disjoint from S . Then $PD_S = pD_S$ for some $p \in P$ (see the proof of Theorem 2), and since S is almost splitting, by Lemma 1, there is a positive integer n such that $P \supseteq p^n D_S \cap D = qD$ for some $q \in D$. Clearly, q is a primary element. Thus, P contains a primary element q .

(\Leftarrow) Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S$, where every e_i is a positive integer and the p_i 's are non-associate prime elements in D_S (see the proof of Theorem 2). Let P_i be the prime ideal of D such that $P_i D_S = p_i D_S$. Then P_i is a prime t -ideal of D and $P_i \cap S = \emptyset$; so P_i contains a primary element q_i . Clearly, $q_i D_S = p_i^{n_i} D_S$ for some positive integer n_i . Let $n = n_1 \cdots n_k$ and $m_i = \frac{n}{n_i} e_i$. Then $p_i^{ne_i} D_S = q_i^{m_i} D_S$, and

hence

$$\begin{aligned}
 d^n D_S \cap D &= ((p_1^{n e_1}) D_S \cap \cdots \cap (p_k^{n e_k}) D_S) \cap D \\
 &= ((q_1^{m_1} D_S) \cap \cdots \cap (q_k^{m_k} D_S)) \cap D \\
 &= (q_1^{m_1} D_S \cap D) \cap \cdots \cap (q_k^{m_k} D_S \cap D) \\
 &= (q_1^{m_1} D \cap \cdots \cap (q_k^{m_k} D) \\
 &= (q_1^{m_1} \cdots q_k^{m_k}) D,
 \end{aligned}$$

where the fourth and last equalities follow from the fact that each $q_i^{m_i}$ is a primary element with $\sqrt{q_i^{m_i} D} \neq \sqrt{q_j^{m_j} D}$ for $i \neq j$. Therefore, S is an almost splitting set by Lemma 1. \square

Let $N(D^*) = \{f \in D[\{X_\alpha\}] \mid (f, d)_v = D[\{X_\alpha\}] \text{ for all } d \in D^*\}$. It is clear that $(f, d)_v = D[\{X_\alpha\}]$ for all $d \in D^*$ if and only if $c(f)_v = D$, where $c(f)$ is the ideal of D generated by the coefficients of f . Hence, $Cl(D[\{X_\alpha\}]_{N(D^*)}) = 0$ [14, Theorem 2.14]. The next result is a generalization of [5, Theorem 2.4].

COROLLARY 5. *D^* is an almost splitting set in $D[\{X_\alpha\}]$ if and only if D is a UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion.*

Proof. (\Rightarrow) If D^* is an almost splitting set in $D[\{X_\alpha\}]$, then $Cl(D[\{X_\alpha\}]_{D^*}) = Cl((D[\{X_\alpha\}])_{N(D^*)}) = 0$. Thus, $Cl(D[\{X_\alpha\}])$ is torsion [7, Theorem 2.10(2)]. Also, since almost splitting sets are t -splitting sets, D is a UMT-domain by Corollary 3.

(\Leftarrow) Assume that D is a UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion. Then D^* is a t -splitting set by Corollary 3, and since $Cl(D[\{X_\alpha\}])$ is torsion, D^* is an almost splitting set. \square

COROLLARY 6. *If D is integrally closed, then D^* is an almost splitting (resp., a t -splitting) set in $D[\{X_\alpha\}]$ if and only if D is an AGCD-domain (resp., a PvMD).*

Proof. Note that $Cl(D[\{X_\alpha\}]) = Cl(D)$ [10, Corollary 2.13]; an integrally closed UMT-domain is a PvMD; and an integrally closed AGCD-domain is a PvMD with torsion class group. Hence, the result follows directly from Corollaries 3 and 5. \square

THEOREM 7. *Let D be an integral domain and S be a saturated multiplicative subset of D such that D_S is a factorial domain. Then S is a splitting set in D if and only if every prime t -ideal of D disjoint from S is principal.*

Proof. (\Rightarrow) Let P be a prime t -ideal of D with $P \cap S = \emptyset$. Then $PD_S = pD_S$ for some prime element p of D_S (see the proof of Theorem 2), and thus $PD_S \cap D = pD_S \cap D$ is principal by Lemma 1.

(\Leftarrow) An argument similar to the proof (\Leftarrow) of Theorem 4 shows that $dD_S \cap D$ is principal for every $0 \neq d \in D$. Thus, by Lemma 1, S is a splitting set. \square

Let X be an indeterminate over D . In [9, p. 77] (cf. [2, Example 4.7]), it was noted that D^* is a splitting set in $D[X]$ if and only if D is a GCD-domain.

COROLLARY 8. D^* is a splitting set in $D[\{X_\alpha\}]$ if and only if D is a GCD-domain.

Proof. If D^* is a splitting set in $D[\{X_\alpha\}]$, then $Cl(D) = Cl(D[\{X_\alpha\}]) = 0$ [2, Corollary 3.8] because $Cl(D[\{X_\alpha\}]_{D^*}) = Cl((D[\{X_\alpha\}])_{N(D^*)}) = 0$. Hence, D is integrally closed [10, Corollary 2.13] and D is a UMT-domain by Corollary 3. Thus, D is a GCD domain because D is an integrally closed UMT-domain with $Cl(D) = 0$. Conversely, assume that D is a GCD-domain. Then D^* is a t -splitting set in $D[\{X_\alpha\}]$ by Corollary 3 and $Cl(D[\{X_\alpha\}]) = Cl(D) = 0$. Thus, D^* is a splitting set. \square

Let S be a saturated multiplicative subset of an integral domain D such that D_S is a factorial domain. The proofs of Theorems 2, 4, and 7 show that S is splitting (resp., almost splitting, t -splitting) if and only if for every nonzero prime element p of D_S , the ideal $pD_S \cap D$ is principal (resp., contains a primary element, t -invertible).

Acknowledgement. The author would like to thank the referees for their several helpful comments and suggestions. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0007069).

References

- [1] D.D. Anderson, D.F. Anderson, and M. Zafrullah, *Atomic domains in which almost all atoms are prime*, Comm. Algebra **20** (1992), 1447–1462.
- [2] D.D. Anderson, D.F. Anderson, and M. Zafrullah, *Splitting the t -class group*, J. Pure Appl. Algebra **74** (1991), 17–37.

- [3] D.D. Anderson, D.F. Anderson, and M. Zafrullah, *The ring $D + XD_S[X]$ and t -splitting sets*, Commutative Algebra Arab. J. Sci. Eng. Sect. C Theme Issues **26** (1) (2001), 3–16.
- [4] D.D. Anderson, T. Dumitrescu, and M. Zafrullah, *Almost splitting sets and AGCD domains*, Comm. Algebra **32** (2004), 147–158.
- [5] D.F. Anderson and G.W. Chang, *Almost splitting sets in integral domains, II*, J. Pure Appl. Algebra **208** (2007), 351–359.
- [6] A. Bouvier and M. Zafrullah, *On some class groups of an integral domain*, Bull. Soc. Math. Grèce (N.S.) **29** (1988), 45–59.
- [7] G.W. Chang, *Almost splitting sets in integral domains*, J. Pure Appl. Algebra **197** (2005), 279–292.
- [8] G.W. Chang, *Almost splitting sets S of an integral domain D such that D_S is a PID*, Korean J. Math. **19** (2011), 163–169.
- [9] G.W. Chang, T. Dumitrescu, and M. Zafrullah, *t -splitting sets in integral domains*, J. Pure Appl. Algebra **187** (2004), 71–86.
- [10] S. El Baghdadi, L. Izelgue, and S. Kabbaj, *On the class group of a graded domain*, J. Pure Appl. Algebra **171** (2002), 171–184.
- [11] M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra **26** (1998), 1017–1039.
- [12] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.
- [13] E. Houston and M. Zafrullah, *On t -invertibility, II*, Comm. Algebra **17** (1989), 1955–1969.
- [14] B.G. Kang, *Prüfer v -multiplication domains and the ring $R[X]_{N_v}$* , J. Algebra **123** (1989), 151–170.
- [15] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. **51** (1985), 29–62.

Department of Mathematics
Incheon National University
Incheon 406-772, Korea
E-mail: whan@incheon.ac.kr