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CUBIC IDEALS IN SEMIGROUPS

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Abstract. Operational properties of cubic sets are first investigated. The notion of cubic subsemigroups and cubic left (resp. right) ideals are introduced, and several properties are investigated. Relations between cubic subsemigroups and cubic left (resp. right) ideals are discussed. Characterizations of cubic left (resp. right) ideals are considered, and how the images or inverse images of cubic subsemigroups and cubic left (resp. right) ideals become cubic subsemigroups and cubic left (resp. right) ideals become cubic subsemigroups and cubic left (resp. right) ideals, respectively, are studied.

1. Introduction

Fuzzy sets are initiated by Zadeh [13]. In [14], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval [0,1] are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [9] in Medical diagnosis in thyroidian pathology, Kohout [8] also in Medicine, in a system CLINAID, Gorzalczany [10] in Approximate reasoning, Turksen [10, 11] in Interval-valued logic, in preferences modelling [12], etc. These works and others show the importance of these sets. Fuzzy sets deal with possibilistic uncertainty, connected with imprecision of states, perceptions

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and preferences. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [5] introduced a new notion, called a cubic set, and investigated several properties. Cubic set theory is applied to BCK/BCI-algebras (see [3, 4, 6, 7]) and Γ -semihypergroups (see [1]).

In this paper, we apply cubic sets to semigroups. We first investigates operational properties of cubic sets. We introduce the notion of cubic subsemigroups and cubic left (resp. right) ideals, and investigate several properties. We discuss relations between cubic subsemigroups and cubic left (resp. right) ideals. We consider characterizations of cubic left (resp. right) ideals, and study how the images or inverse images of cubic subsemigroups and cubic left (resp. right) ideals become cubic subsemigroups and cubic left (resp. right) ideals become cubic subsemigroups and cubic left (resp. right) ideals, respectively.

2. Preliminaries

A nonempty set S together with an associative binary operation "." is called a *semigroup*. A semigroup S is said to be *left* (resp. *right*) *zero* if xy = x (resp. xy = y) for all $x, y \in S$. A semigroup S is said to be *regular* if for each element $a \in S$, there exists an element x in S such that a = axa. A nonempty subset Q of a semigroup S is called a

- subsemigroup of S if $xy \in Q$ for all $x, y \in Q$,
- left (resp. right) ideal of S if $xa \in Q$ (resp. $ax \in Q$) for all $x \in S$ and $a \in Q$.

Let I be a closed unit interval, i.e., I = [0, 1]. By an *interval number* we mean a closed subinterval $\overline{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by D[0, 1] the set of all interval numbers. Let us define what are known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in D[0, 1]. We also define the symbols " \succeq ", " \preceq ", "=" in case of two elements in D[0, 1]. Consider two interval numbers $\overline{a}_1 := [a_1^-, a_1^+]$ and $\overline{a}_2 := [a_2^-, a_2^+]$. Then

$$\operatorname{rmin} \left\{ \overline{a}_1, \overline{a}_2 \right\} = \left[\min \left\{ a_1^-, a_2^- \right\}, \min \left\{ a_1^+, a_2^+ \right\} \right],$$
$$\operatorname{rmax} \left\{ \overline{a}_1, \overline{a}_2 \right\} = \left[\max \left\{ a_1^-, a_2^- \right\}, \max \left\{ a_1^+, a_2^+ \right\} \right],$$
$$\overline{a}_1 \succeq \overline{a}_2 \text{ if and only if } a_1^- \ge a_2^- \text{ and } a_1^+ \ge a_2^+,$$

and similarly we may have $\overline{a}_1 \leq \overline{a}_2$ and $\overline{a}_1 = \overline{a}_2$. To say $\overline{a}_1 \succ \overline{a}_2$ (resp. $\overline{a}_1 \prec \overline{a}_2$) we mean $\overline{a}_1 \succeq \overline{a}_2$ and $\overline{a}_1 \neq \overline{a}_2$ (resp. $\overline{a}_1 \leq \overline{a}_2$ and $\overline{a}_1 \neq \overline{a}_2$). Let

 $\overline{a}_i \in D[0,1]$ where $i \in \Lambda$. We define

$$\inf_{i \in \Lambda} \overline{a}_i = \begin{bmatrix} \inf_{i \in \Lambda} a_i^-, \inf_{i \in \Lambda} a_i^+ \end{bmatrix} \text{ and } \operatorname{rsup}_{i \in \Lambda} \overline{a}_i = \begin{bmatrix} \sup_{i \in \Lambda} a_i^-, \sup_{i \in \Lambda} a_i^+ \end{bmatrix}.$$

An *interval-valued fuzzy set* (briefly, *IVF set*) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A := \left\{ \left(x, \left[\mu_A^-(x), \mu_A^+(x) \right] \right) \mid x \in X \right\},\$$

which is briefly denoted by $\tilde{\mu}_A = \left[\mu_A^-, \mu_A^+\right]$ where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$. For any IVF set $\tilde{\mu}_A$ on X and $x \in X$, $\tilde{\mu}_A(x) = \left[\mu_A^-(x), \mu_A^+(x)\right]$ is called the degree of membership of an element x to $\tilde{\mu}_A$, in which $\mu_A^-(x)$ and $\mu_A^+(x)$ are refereed to as the lower and upper degrees, respectively, of membership of x to $\tilde{\mu}_A$.

3. Operational properties of cubic sets

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Definition 3.1 ([3]). Let X be a nonempty set. A *cubic set* \mathscr{A} in X is a structure

$$\mathscr{A} = \{ \langle x, \tilde{\mu}_A(x), f_A(x) \rangle : x \in X \}$$

which is briefly denoted by $\mathscr{A} = \langle \tilde{\mu}_A, f \rangle$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and f is a fuzzy set in X. In this case, we will use

$$\mathscr{A}(x) = \langle \tilde{\mu}_A(x), f_A(x) \rangle = \left\langle [\mu_A^-(x), \mu_A^+(x)], f_A(x) \right\rangle$$

for all $x \in X$.

For any non-empty subset G of a set X, the *characteristic cubic set* of G in X is defined to be a structure

$$\chi_G = \{ \langle x, \tilde{\mu}_{\chi_G}(x), f_{\chi_G}(x) \rangle : x \in X \}$$

which is briefly denoted by $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ where

$$\tilde{\mu}_{\chi_G}(x) = \begin{cases} [1,1] & \text{if } x \in G, \\ [0,0] & \text{otherwise,} \end{cases} f_{\chi_G}(x) = \begin{cases} 0 & \text{if } x \in G, \\ 1 & \text{otherwise} \end{cases}$$

The whole cubic set S in a semigroup S is defined to be a structure

$$\mathcal{S} = \left\{ \left\langle x, \tilde{1}_S(x), 0_S(x) \right\rangle : x \in S \right\}$$

with $\tilde{1}_S(x) = [1, 1]$ and $0_S(x) = 0$ for all $x \in X$. It will be briefly denoted by $S = \langle \tilde{1}_S, 0_S \rangle$.

For two cubic sets $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ in a semigroup S, we define

$$\mathscr{A} \sqsubseteq \mathscr{B} \iff \tilde{\mu}_A \preceq \tilde{\mu}_B, \ f_A \ge f_B$$

and the *cubic product* of $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ is defined to be a cubic set

$$\mathscr{A} \ \textcircled{C} \ \mathscr{B} = \{ \langle x, (\tilde{\mu}_A \circ \tilde{\mu}_B)(x), (f_A \circ f_B)(x) \rangle : x \in S \}$$

which is briefly denoted by $\mathscr{A} \odot \mathscr{B} = \langle \tilde{\mu}_A \circ \tilde{\mu}_B, f_A \circ f_B \rangle$ where $\tilde{\mu}_A \circ \tilde{\mu}_B$ and $f_A \circ f_B$ are defined as follows, respectively:

$$\left(\tilde{\mu}_{A}\tilde{\circ}\tilde{\mu}_{B}\right)(x) = \begin{cases} \operatorname{rsup}_{x=yz} \left[\operatorname{rmin}\left\{\tilde{\mu}_{A}(y), \tilde{\mu}_{B}(z)\right\}\right] & \text{if } x = yz \text{ for some } y, z \in S, \\ [0,0] & \text{otherwise,} \end{cases}$$

and

$$(f_A \circ f_B)(x) = \begin{cases} \bigwedge_{\substack{x=yz \\ 1}} \max \left\{ f_A(y), f_B(z) \right\} & \text{if } x = yz \text{ for some } y, z \in S, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in S$. We also define the cap and union of two cubic sets as follows. Let \mathscr{A} and \mathscr{B} be two cubic sets in X. The *intersection* of \mathscr{A} and \mathscr{B} , denoted by $\mathscr{A} \sqcap \mathscr{B}$, is the cubic set

$$\mathscr{A} \sqcap \mathscr{B} = \langle \tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B, f_A \lor f_B \rangle$$

where $(\tilde{\mu}_A \cap \tilde{\mu}_B)(x) = \min \{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ and $(f_A \vee f_B)(x) = \max \{f_A(x), f_B(x)\}$.

The union of \mathscr{A} and \mathscr{B} , denoted by $\mathscr{A} \sqcup \mathscr{B}$, is the cubic set

 $\mathscr{A} \sqcup \mathscr{B} = \langle \tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B, f_A \wedge f_B \rangle$

where $(\tilde{\mu}_A \cup \tilde{\mu}_B)(x) = \operatorname{rmax} \{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ and $(f_A \wedge f_B)(x) = \min \{f_A(x), f_B(x)\}$.

Proposition 3.2. For any cubic sets $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$, $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ and $\mathscr{C} = \langle \tilde{\mu}_C, f_C \rangle$ in a semigroup S, we have

(1) $\mathscr{A} \sqcup (\mathscr{B} \sqcap \mathscr{C}) = (\mathscr{A} \sqcup \mathscr{B}) \sqcap (\mathscr{A} \sqcup \mathscr{C}),$

- $(2) \ \mathscr{A} \sqcap (\mathscr{B} \sqcup \mathscr{C}) = (\mathscr{A} \sqcap \mathscr{B}) \sqcup (\mathscr{A} \sqcap \mathscr{C}),$
- $(3) \ \mathscr{A} \ \mathbb{C}(\mathscr{B} \sqcup \mathscr{C}) = (\mathscr{A} \ \mathbb{C} \ \mathscr{B}) \sqcup (\mathscr{A} \ \mathbb{C} \ \mathscr{C}),$
- $(4) \ \mathscr{A} \ \textcircled{C} \ (\mathscr{B} \sqcap \mathscr{C}) \sqsubseteq (\mathscr{A} \ \textcircled{C} \ \mathscr{B}) \sqcap (\mathscr{A} \ \textcircled{C} \ \mathscr{C}).$

Proof. (1) and (2) are straightforward.

(3) Let x be any element of S. If x is not expressed as x = yz, then

$$(\tilde{\mu}_A \tilde{\circ} (\tilde{\mu}_B \tilde{\cup} \tilde{\mu}_C))(x) = [0, 0] = ((\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B) \tilde{\cup} (\tilde{\mu}_A \tilde{\circ} (\tilde{\mu}_C))(x))$$

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and

$$(f_A \circ (f_B \wedge f_C))(x) = 1 = ((f_A \circ f_B) \wedge (f_A \circ f_C))(x).$$

Therefore $\mathscr{A}(\mathbb{C}(\mathscr{B}\sqcup\mathscr{C}) = (\mathscr{A}(\mathbb{C}\mathscr{B})\sqcup(\mathscr{A}(\mathbb{C}\mathscr{C}))$. Assume that x is expressed as x = yz. Then

$$\begin{split} \left(\tilde{\mu}_{A}\tilde{\circ}(\tilde{\mu}_{B}\tilde{\cup}\tilde{\mu}_{C})\right)(x) &= \underset{x=yz}{\operatorname{rsup}}\left[\operatorname{rmin}\{\tilde{\mu}_{A}(y), (\tilde{\mu}_{B}\tilde{\cup}\tilde{\mu}_{C})(z)\}\right] \\ &= \underset{x=yz}{\operatorname{rsup}}\left[\operatorname{rmin}\{\tilde{\mu}_{A}(y), \operatorname{rmax}\{\tilde{\mu}_{B}(z), \tilde{\mu}_{C}(z)\}\}\right] \\ &= \underset{x=yz}{\operatorname{rsup}}\left[\operatorname{rmax}\{\operatorname{rmin}\{\tilde{\mu}_{A}(y), \tilde{\mu}_{B}(z)\}, \operatorname{rmin}\{\tilde{\mu}_{A}(y), \tilde{\mu}_{C}(z)\}\}\right] \\ &= \operatorname{rmax}\left\{\underset{x=yz}{\operatorname{rsup}}\left[\operatorname{rmin}\{\tilde{\mu}_{A}(y), \tilde{\mu}_{B}(z)\}, \operatorname{rmin}\{\tilde{\mu}_{A}(y), \tilde{\mu}_{C}(z)\}\right]\right\} \\ &= \left((\tilde{\mu}_{A}\tilde{\circ}\tilde{\mu}_{B})\tilde{\cup}(\tilde{\mu}_{A}\tilde{\circ}\tilde{\mu}_{C}))(x), \end{split}$$

and

$$(f_A \circ (f_B \wedge f_C))(x) = \bigwedge_{x=yz} \max \{ f_A(y), (f_B \wedge f_C)(z) \}$$

= $\bigwedge_{x=yz} \max \{ f_A(y), \min\{ f_B(z), f_C(z) \} \}$
= $\bigwedge_{x=yz} \min \{ \max\{ f_A(y), f_B(z) \}, \max\{ f_A(y), f_C(z) \} \}$
= $\min \left\{ \bigwedge_{x=yz} \max\{ f_A(y), f_B(z) \}, \bigwedge_{x=yz} \max\{ f_A(y), f_C(z) \} \right\}$
= $((f_A \circ f_B) \wedge (f \circ f_C))(x).$

Hence (3) holds.

(4) Let $x \in S$. If x is not expressed as x = yz, then it is clear. Assume that there exist $y, z \in S$ such that x = yz. Then

$$\begin{split} (\tilde{\mu}_A \tilde{\circ} (\tilde{\mu}_B \tilde{\cap} \tilde{\mu}_C))(x) &= \underset{x=yz}{\operatorname{rsup}} [\operatorname{rmin} \{ \tilde{\mu}_A(y), (\tilde{\mu}_B \tilde{\cap} \tilde{\mu}_C)(z) \}] \\ &= \underset{x=yz}{\operatorname{rsup}} [\operatorname{rmin} \{ \tilde{\mu}_A(y), \operatorname{rmin} \{ \tilde{\mu}_B(z), \tilde{\mu}_C(z) \} \}] \\ &= \underset{x=yz}{\operatorname{rsup}} [\operatorname{rmax} \{ \operatorname{rmin} \{ \tilde{\mu}_A(y), \tilde{\mu}_B(z) \}, \operatorname{rmin} \{ \tilde{\mu}_A(y), \tilde{\mu}_C(z) \} \}] \\ &\preceq \operatorname{rmin} \left\{ \underset{x=yz}{\operatorname{rsup}} [\operatorname{rmin} \{ \tilde{\mu}_A(y), \tilde{\mu}_B(z) \}], \underset{x=yz}{\operatorname{rsup}} [\operatorname{rmin} \{ \tilde{\mu}_A(y), \tilde{\mu}_C(z) \}] \right\} \\ &= ((\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B) \tilde{\cap} (\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_C))(x), \end{split}$$

and

$$(f_A \circ (f_B \lor f_C))(x) = \bigwedge_{x=yz} \max \{ f_A(y), (f_B \lor f_C)(z) \}$$

= $\bigwedge_{x=yz} \max \{ f_A(y), \max\{ f_B(z), f_C(z) \} \}$
= $\bigwedge_{x=yz} \max \{ \max\{ f_A(y), f_B(z) \}, \max\{ f_A(y), f_C(z) \} \}$
\ge max $\left\{ \bigwedge_{x=yz} \max\{ f_A(y), f_B(z) \}, \bigwedge_{x=yz} \max\{ f_A(y), f_C(z) \} \right\}$
= $((f_A \circ f_B) \lor (f \circ f_C))(x).$

Hence (4) holds.

Proposition 3.3. For any cubic sets $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle, \ \mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ and $\mathscr{C} = \langle \tilde{\mu}_C, f_C \rangle$ in a semigroup S, if $\mathscr{A} \sqsubseteq \mathscr{B}$, then $\mathscr{A} \odot \mathscr{C} \mathfrak{C} \sqsubseteq \mathscr{B} \odot \mathscr{C}$ and $\mathscr{C} \odot \mathscr{A} \sqsubseteq \mathscr{C} \odot \mathscr{B}$.

Proof. Straightforward.

Proposition 3.4. For non-empty subsets G and H of a semigroup S, we have

- $\begin{array}{l} (1) \ \chi_{G} \textcircled{\odot} \chi_{H} = \chi_{GH}, \ \text{i.e.}, \ \langle \tilde{\mu}_{\chi_{G}} \circ \tilde{\mu}_{\chi_{H}}, f_{\chi_{G}} \circ f_{\chi_{H}} \rangle = \langle \tilde{\mu}_{\chi_{GH}}, f_{\chi_{GH}} \rangle, \\ (2) \ \chi_{G} \sqcap \chi_{H} = \chi_{G \cap H}, \ \text{i.e.}, \ \langle \tilde{\mu}_{\chi_{G}} \cap \tilde{\mu}_{\chi_{H}}, f_{\chi_{G}} \lor f_{\chi_{H}} \rangle = \langle \tilde{\mu}_{\chi_{G \cap H}}, f_{\chi_{G \cap H}} \rangle, \\ (3) \ \chi_{G} \sqcup \chi_{H} = \chi_{G \cup H}, \ \text{i.e.}, \ \langle \tilde{\mu}_{\chi_{G}} \cup \tilde{\mu}_{\chi_{H}}, f_{\chi_{G}} \land f_{\chi_{H}} \rangle = \langle \tilde{\mu}_{\chi_{G \cup H}}, f_{\chi_{G \cup H}} \rangle. \end{array}$

Proof. (1) Let $a \in S$. If $a \in GH$, then $\tilde{\mu}_{\chi_{GH}}(a) = [1, 1], f_{\chi_{GH}}(a) = 0$ and a = bc for some $b \in G$ and $c \in H$. Thus

$$\begin{aligned} (\tilde{\mu}_{\chi_G} \tilde{\circ} \tilde{\mu}_{\chi_H}) (a) &= \sup_{a=xy} \left[\min \left\{ \tilde{\mu}_{\chi_G}(x), \tilde{\mu}_{\chi_H}(y) \right\} \right] \\ &\succeq \min \left\{ \tilde{\mu}_{\chi_G}(b), \tilde{\mu}_{\chi_H}(c) \right\} \\ &= [1, 1] \end{aligned}$$

and

$$(f_{\chi_G} \circ f_{\chi_H})(a) = \bigwedge_{a=xy} \left[\max \left\{ f_{\chi_G}(x), f_{\chi_H}(y) \right\} \right]$$
$$\leq \max \left\{ f_{\chi_G}(b), f_{\chi_H}(c) \right\}.$$
$$= 0$$

It follows that $(\tilde{\mu}_{\chi_G} \circ \tilde{\mu}_{\chi_H})(a) = [1, 1]$ and $(f_{\chi_G} \circ f_{\chi_H})(a) = 0$. Therefore $\left\langle \tilde{\mu}_{\chi_G} \tilde{\circ} \tilde{\mu}_{\chi_H}, f_{\chi_G} \circ f_{\chi_H} \right\rangle = \left\langle \tilde{\mu}_{\chi_{GH}}, f_{\chi_{GH}} \right\rangle,$

that is, $\chi_G \odot \chi_H = \chi_{GH}$. Assume that $a \notin GH$. Then $\tilde{\mu}_{\chi_{GH}}(a) = [0,0]$ and $f_{\chi_{GH}}(a) = 1$. Let $y, z \in S$ be such that a = yz. Then we know that $y \notin G$ or $z \notin H$. Assume that $y \notin G$. Then

$$(\tilde{\mu}_{\chi_G} \tilde{\circ} \tilde{\mu}_{\chi_H}) (a) = \sup_{a=yz} [\min \{ \tilde{\mu}_{\chi_G}(y), \tilde{\mu}_{\chi_H}(z) \}]$$
$$= \sup_{a=yz} [\min \{ [0,0], \tilde{\mu}_{\chi_H}(z) \}]$$
$$= [0,0] = \tilde{\mu}_{\chi_{GH}}(a)$$

and

$$(f_{\chi_G} \circ f_{\chi_H})(a) = \bigwedge_{a=yz} [\max \{ f_{\chi_G}(y), f_{\chi_H}(z) \}]$$
$$= \bigwedge_{a=yz} [\max \{ 1, f_{\chi_H}(z) \}]$$
$$= 1 = f_{\chi_{GH}}(a).$$

Similarly, if $z \notin H$, then $(\tilde{\mu}_{\chi_G} \tilde{\circ} \tilde{\mu}_{\chi_H})(a) = [0,0] = \tilde{\mu}_{\chi_{GH}}(a)$ and $(f_{\chi_G} \circ f_{\chi_H})(a) = 1 = f_{\chi_{GH}}(a)$. Therefore $\chi_G \textcircled{C} \chi_H = \chi_{GH}$. (2) and (3) are straightforward.

4. Cubic subsemigroups and ideals

Definition 4.1. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is called a *cubic subsemigroup* of S if it satisfies:

(4.1)
$$(\forall x, y \in S) \left(\begin{array}{c} \tilde{\mu}_A(xy) \succeq \min\left\{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \right\}, \\ f_A(xy) \le \max\left\{ f_A(x), f_A(y) \right\} \end{array} \right)$$

Example 4.2. Consider a semigroup $S = \{a, b, c, d, e, f\}$ with the following Cayley table (see Table 1).

TABLE 1. -- multiplication

Define a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in S as follows:

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S	$\tilde{\mu}_A(x)$	$f_A(x)$
a	[0.5, 0.7]	0.6
b	[0.3, 0.5]	0.7
c	[0.2, 0.4]	0.7
d	[0.6, 0.8]	0.3
e	[0.4, 0.6]	0.4
f	[0,0]	0.8

Then $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S.

Theorem 4.3. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is a cubic subsemigroup of S if and only if $\mathscr{A} \odot \mathscr{A} \sqsubseteq \mathscr{A}$.

Proof. Straightforward.

Definition 4.4. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is called a *left cubic ideal* of S if it satisfies:

(4.2)
$$(\forall a, b \in S) (\tilde{\mu}_A(ab) \succeq \tilde{\mu}_A(b), f_A(ab) \le f_A(b)).$$

Similarly, we say that a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is a *right cubic ideal* of S if $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ satisfies the following condition:

(4.3)
$$(\forall a, b \in S) (\tilde{\mu}_A(ab) \succeq \tilde{\mu}_A(a), f_A(ab) \le f_A(a)).$$

By a (two-sided) cubic ideal we mean a left and right cubic ideal.

Example 4.5. Consider a semigroup $S = \{a, b, c, d, e\}$ with the following Cayley table (see Table 2).

TABLE 2. -- multiplication

a	b	c	d	e
a	a	a	a	a
a	a	a	a	a
a	a	c	c	e
a	a	c	d	e
a	a	c	c	e
	$\begin{bmatrix} a \\ a \\ a \\ a \\ a \\ a \end{bmatrix}$	a b a a a a a a a a a a a a a a a a	a b c a a a a a a a a c a a c a a c a a c a a c	a b c d a a a a a a a a a a a a c c a a c d a a c c a a c c

Define a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in S as follows:

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S	$\tilde{\mu}_A(x)$	$f_A(x)$
a	[0.6, 0.8]	0.1
b	[0, 0.2]	0.7
c	[0.2, 0.4]	0.2
d	[0.1, 0.3]	0.6
e	[0.4, 0.6]	0.4

It is easy to verify that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic ideal of S.

Obviously, every left (resp. right) cubic ideal is a cubic subsemigroup. But the converse may not be true as seen in the following example.

TABLE 3 multiplication	n
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	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Example 4.6. Consider a semigroup $S = \{0, a, b, c\}$ with the Cayley table (see Table 3). Define a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in S as follows:

$S \mid$	$\tilde{\mu}_A(x)$	$f_A(x)$
0	[0.5, 0.8]	0.2
a	[0.3, 0.6]	0.6
b	[0.5, 0.8]	0.4
c	[0.2, 0.4]	0.6

It is easy to verify that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S, but it is not a left cubic ideal of S since $\tilde{\mu}_A(cb) = \tilde{\mu}_A(a) = [0.3, 0.6] \not\geq [0.5, 0.8] = \tilde{\mu}_A(b)$ and/or $f_A(cb) = f_A(a) = 0.6 > 0.4 = f_A(b)$.

Theorem 4.7. For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup *S*, the following are equivalent:

(1) $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left cubic ideal of S. (2) $\mathscr{S} \odot \mathscr{A} \sqsubseteq \mathscr{A}$.

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left cubic ideal of S. Let $a \in S$. If $(\mathcal{S} \odot \mathscr{A})(a) = \langle [0,0], 1 \rangle$, then it is clear that $\mathcal{S} \odot \mathscr{A} \sqsubseteq \mathscr{A}$. Otherwise, there exists $x, y \in S$ such that a = xy. Since $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left cubic

ideal of S, we have

$$(\tilde{1}_S \tilde{\circ} \tilde{\mu}_A) (a) = \operatorname{rsup}_{a=xy} \left[\operatorname{rmin} \left\{ \tilde{1}_S(x), \tilde{\mu}_A(y) \right\} \right]$$
$$\leq \operatorname{rsup}_{a=xy} \left[\operatorname{rmin} \left\{ [1, 1], \tilde{\mu}_A(xy) \right\} \right]$$
$$= \operatorname{rsup} \tilde{\mu}_A(a) = \tilde{\mu}_A(a)$$

and

$$(0_S \circ f)(a) = \bigwedge_{a=xy} \max \{0_S(x), f_A(y)\}$$
$$\geq \bigwedge_{a=xy} f_A(xy) = \bigwedge f_A(a) = f_A(a).$$

Therefore $\mathcal{S}_{\bigcirc}\mathscr{A} \sqsubseteq \mathscr{A}$.

Conversely, suppose that $\mathcal{S} \odot \mathscr{A} \sqsubseteq \mathscr{A}$. For any elements x and y of S, let a = xy. Then

$$\tilde{\mu}_A(xy) = \tilde{\mu}_A(a) \succeq (\mathbf{1}_S \tilde{\circ} \tilde{\mu}_A) (a)$$
$$= \sup_{a=bc} \left[\operatorname{rmin} \left\{ \tilde{\mathbf{1}}_S(b), \tilde{\mu}_A(c) \right\} \right]$$
$$\succeq \operatorname{rmin} \left\{ \tilde{\mathbf{1}}_S(x), \tilde{\mu}_A(y) \right\} = \tilde{\mu}_A(y)$$

and

$$f_A(xy) = f_A(a) \le (0_S \circ f) (a)$$
$$= \bigwedge_{a=bc} \max \{ 0_S(b), f_A(c) \}$$
$$\le \max \{ 0_S(x), f_A(y) \} = f_A(y).$$

Hence $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left cubic ideal of S.

Similarly, we can induces the following theorem.

Theorem 4.8. For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup *S*, the following are equivalent:

(1) $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a right cubic ideal of S. (2) $\mathscr{A} \oplus \mathscr{S} \sqsubseteq \mathscr{A}$.

Theorem 4.9. If $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic set in a semigroup *S*, then $\mathscr{S} \odot \mathscr{A}$ (resp. $\mathscr{A} \odot \mathscr{S}$) is a left (resp. right) cubic ideal of *S*.

Proof. Since $\mathcal{S}(\mathbb{C}(\mathcal{A})) = (\mathcal{S}(\mathbb{C}(\mathcal{S})) \otimes \mathcal{A} \sqsubseteq \mathcal{S}(\mathbb{C}(\mathcal{A}))$, it follows from Theorem 4.7 that $\mathcal{S}(\mathbb{C}(\mathcal{A}))$ is a left cubic ideal of S. Similarly, $\mathcal{A}(\mathbb{C}(\mathcal{S}))$ is a right cubic ideal of S.

Now we will consider conditions for a left (resp. right) cubic ideal to be constant.

Proposition 4.10. Let U be a left zero subsemigroup of a semigroup S. If $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left cubic ideal of S, then $\mathscr{A}(x) = \mathscr{A}(y)$ for all $x, y \in U$.

Proof. Let $x, y \in U$. Then xy = x and yx = y. Thus

$$\tilde{\mu}_A(x) = \tilde{\mu}_A(xy) \succeq \tilde{\mu}_A(y) = \tilde{\mu}_A(yx) \succeq \tilde{\mu}_A(x)$$

and

$$f_A(x) = f_A(xy) \le f_A(y) = f_A(yx) \le f_A(x).$$

Therefore $\mathscr{A}(x) = \mathscr{A}(y)$ for all $x, y \in U$.

Similarly, we have the following proposition.

Proposition 4.11. Let U be a right zero subsemigroup of a semigroup S. If $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a right cubic ideal of S, then $\mathscr{A}(x) = \mathscr{A}(y)$ for all $x, y \in U$.

Theorem 4.12. Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a left cubic ideal of a semigroup S. If the set of all idempotent elements of S forms a left zero subsemigroup of S, then $\mathscr{A}(u) = \mathscr{A}(v)$ for all idempotents elements u and v of S.

Proof. Let Idm(S) be the set of all idempotent elements of S and assume that Idm(S) is a left zero subsemigroup of S. For any $u, v \in Idm(S)$, we have uv = u and vu = v. Hence

$$\tilde{\mu}_A(u) = \tilde{\mu}_A(uv) \succeq \tilde{\mu}_A(v) = \tilde{\mu}_A(vu) \succeq \tilde{\mu}_A(u)$$

and

$$f_A(u) = f_A(uv) \le f_A(v) = f_A(vu) \le f_A(u).$$

Therefore $\mathscr{A}(x) = \mathscr{A}(y)$ for all $u, v \in Idm(S)$.

Similarly, we have the following theorem.

Theorem 4.13. Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a right cubic ideal of a semigroup S. If the set of all idempotent elements of S forms a right zero subsemigroup of S, then $\mathscr{A}(u) = \mathscr{A}(v)$ for all idempotents elements u and v of S.

Theorem 4.14. Let S be a semigroup. Then the following properties hold.

- (1) The intersection of two cubic subsemigroups of S is a cubic subsemigroup of S.
- (2) The intersection of two left (resp. right) cubic ideals of S is a left (resp. right) cubic ideal of S.

Proof. (1) Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ be cubic subsemigroups of S. Let x and y be any elements of S. Then

$$(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B) (xy) = \min \{ \tilde{\mu}_A(xy), \tilde{\mu}_B(xy) \}$$

$$\succeq \min \{ \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \}, \min \{ \tilde{\mu}_B(x), \tilde{\mu}_B(y) \} \}$$

$$= \min \{ \min \{ \tilde{\mu}_A(x), \tilde{\mu}_B(x) \}, \min \{ \tilde{\mu}_A(y), \tilde{\mu}_B(y) \} \}$$

$$= \min \{ (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B) (x), (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B) (y) \}$$

and

$$(f_A \lor f_B) (xy) = \max\{f_A(xy), f_B(xy)\} \\ \leq \max\{\max\{f_A(x), f_A(y)\}, \max\{f_B(x), f_B(y)\}\} \\ = \max\{\max\{f_A(x), f_B(x)\}, \max\{f_A(y), f_B(y)\}\} \\ = \max\{(f_A \lor f_B) (x), (f_A \lor f_B) (y)\}.$$

Therefore $\mathscr{A} \sqcap \mathscr{B} = \langle \tilde{\mu}_A \cap \tilde{\mu}_B, f_A \lor f_B \rangle$ is a cubic subsemigroup of S. The second property can be proved in a similar manner. \Box

Proposition 4.15. If $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a right cubic ideal and $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ is a left cubic ideal of a semigroup S, then $\mathscr{A} \odot \mathscr{B} \sqsubseteq \mathscr{A} \sqcap \mathscr{B}$.

Proof. Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a right cubic ideal and $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ any left cubic ideal of S. Then by Theorems 4.7 and 4.8, we have $\mathscr{A} \odot \mathscr{B} \sqsubseteq \mathscr{A} \odot \mathscr{S} \sqsubseteq \mathscr{A}$ and $\mathscr{A} \odot \mathscr{B} \sqsubseteq \mathscr{S} \odot \mathscr{B} \sqsubseteq \mathscr{B}$. Thus $\mathscr{A} \odot \mathscr{B} \sqsubseteq \mathscr{A} \sqcap \mathscr{B}$.

Proposition 4.16. If S is a regular semigroup, then $\mathscr{A} \oplus \mathscr{B} = \mathscr{A} \sqcap \mathscr{B}$ for every right cubic ideal $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and every left cubic ideal $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ of S.

Proof. Let a be any element of S. Since S is regular, there exist an element $x \in S$ such that a = axa. Hence we have

$$(\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B)(a) = \sup_{a=yz} [\min\{\tilde{\mu}_A(y), \tilde{\mu}_B(z)\}]$$

$$\succeq \min\{\tilde{\mu}_A(ax), \tilde{\mu}_B(a)\}$$

$$\succeq \min\{\tilde{\mu}_A(a), \tilde{\mu}_B(a)\}$$

$$= (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(a)$$

and

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$$(f_A \circ f_B)(a) = \bigwedge_{a=yz} \max\{f_A(y), f_B(z)\}$$

$$\leq \max\{f_A(ax), f_B(a)\}$$

$$\leq \max\{f_A(a), f_B(a)\}$$

$$= (f_A \lor f_B)(a),$$

and so $\mathscr{A} \ \mathbb{C} \mathscr{B} \supseteq \mathscr{A} \sqcap \mathscr{B}$. It follows from Proposition 4.15 that $\mathscr{A} \ \mathbb{C} \mathscr{B} = \mathscr{A} \sqcap \mathscr{B}$.

We now discuss the converse of Proposition 4.16. We first consider the following lemmas.

Lemma 4.17 ([2]). For a semigroup S, the following conditions are equivalent.

- (1) S is regular.
- (2) $R \cap L = RL$ for every right ideal R of S and every left ideal L of S.

Lemma 4.18. For a non-empty subset G of a semigroup S, we have (1) G is a subsemigroup of S if and only if the characteristic subi-

- (1) G is a subsemigroup of S if and only if the characteristic cubic set $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ of G in S is a cubic subsemigroup of S.
- (2) G is a left (right) ideal of S if and only if the characteristic cubic set $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ of G in S is a left (resp. right) cubic ideal of S.

Proof. Straightforward.

Theorem 4.19. For every right cubic ideal $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and every left cubic ideal $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ of a semigroup S, if $\mathscr{A} \odot \mathscr{B} = \mathscr{A} \sqcap \mathscr{B}$, then S is regular.

Proof. Assume that $\mathscr{A} \odot \mathscr{B} = \mathscr{A} \sqcap \mathscr{B}$ for every right cubic ideal $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and every left cubic ideal $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ of a semigroup S. Let R and L be any right ideal and any left ideal of S, respectively. In order to see that $R \cap L \subseteq RL$ holds, let a be any element of $R \cap L$. Then the characteristic cubic sets $\chi_R = \langle \tilde{\mu}_{\chi_R}, f_{\chi_R} \rangle$ and $\chi_L = \langle \tilde{\mu}_{\chi_L}, f_{\chi_L} \rangle$ are a right cubic ideal and a left cubic ideal of S, respectively, by Lemma 4.18(2). It follows from the hypothesis and Proposition 3.4 that

$$\begin{split} \tilde{\mu}_{\chi_{RL}}(a) &= \left(\tilde{\mu}_{\chi_R} \tilde{\circ} \tilde{\mu}_{\chi_L}\right)(a) \\ &= \left(\tilde{\mu}_{\chi_R} \tilde{\cap} \tilde{\mu}_{\chi_L}\right)(a) \\ &= \tilde{\mu}_{\chi_{R\cap L}}(a) = [1,1] \end{split}$$

and

$$f_{\chi_{RL}}(a) = (f_{\chi_R} \circ f_{\chi_L})(a)$$
$$= (f_{\chi_R} \lor f_{\chi_L})(a)$$
$$= f_{\chi_{R\cap L}}(a) = 0$$

and so that $a \in RL$. Thus $R \cap L \subseteq RL$. Since the inclusion in the other direction always holds, we obtain that $R \cap L = RL$. It follows from Lemma 4.17 that S is regular.

Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a cubic set in X. For any $r \in [0, 1]$ and $[s, t] \in D[0, 1]$, we define $U(\mathscr{A}; [s, t], r)$ as follows:

$$U(\mathscr{A}; [s,t], r) = \{x \in X \mid \tilde{\mu}_A(x) \succeq [s,t], f_A(x) \le r\},\$$

and we say it is a *cubic level set* of $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ (see [6]).

Theorem 4.20. For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup *S*, the following are equivalent:

- (1) $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S.
- (2) Every nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a subsemigroup of S.

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S. Let $x, y \in U(\mathscr{A}; [s,t], r)$ for all $r \in [0,1]$ and $[s,t] \in D[0,1]$. Then $\tilde{\mu}_A(x) \succeq [s,t], f_A(x) \leq r, \tilde{\mu}_A(y) \succeq [s,t]$ and $f_A(y) \leq r$. It follows from (4.1) that $\tilde{\mu}_A(xy) \succeq \min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \succeq [s,t]$ and $f_A(xy) \leq$ $\max\{f_A(x), f_A(y)\} \leq r$. Hence $xy \in U(\mathscr{A}; [s,t], r)$ and thus $U(\mathscr{A}; [s,t], r)$ is a subsemigroup of S.

Conversely, let $r \in [0, 1]$ and $[s, t] \in D[0, 1]$ be such that $U(\mathscr{A}; [s, t], r) \neq \emptyset$, and $U(\mathscr{A}; [s, t], r)$ is a subsemigroup of S. Suppose that (4.1) is false. Then there exist $a, b \in S$ such that $\tilde{\mu}_A(ab) \not\succeq \min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ or $f_A(ab) \not\leq \max\{f_A(a), f_A(b)\}$. If $\tilde{\mu}_A(ab) \not\succeq \min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$, then $\tilde{\mu}_A(ab) \prec [s_0, t_0] \preceq \min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ for some $[s_0, t_0] \in D[0, 1]$. Hence $a, b \in U(\mathscr{A}; [s_0, t_0], \max\{f_A(a), f_A(b)\})$, but

$$ab \notin U(\mathscr{A}; [s_0, t_0], \max\{f_A(a), f_A(b)\}).$$

This gives a contradiction. If $f_A(ab) \leq \max\{f_A(a), f_A(b)\}$, then there exists $r_0 \in [0, 1]$ such that

$$f_A(ab) > r_0 \ge \max\{f_A(a), f_A(b)\}.$$

Thus $a, b \in U(\mathscr{A}; \min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$, and $ab \notin U(\mathscr{A}; \min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$. This is a contradiction. Assume that there exist $[s_0, t_0] \in D[0, 1]$ and $r_0 \in [0, 1]$ such that $\tilde{\mu}_A(ab) \prec [s_0, t_0] \preceq \min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and $f_A(ab) > r_0 \ge \max\{f_A(a), f_A(b)\}$. Then $a, b \in U(\mathscr{A}; [s_0, t_0], r_0)$ but

 $ab \in U(\mathscr{A}; [s_0, t_0], r_0)$, which gives a contradiction. Hence (4.1) is valid, and therefore $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S.

Theorem 4.21. For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup *S*, the following are equivalent:

- (1) $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left (resp. right) cubic ideal of S.
- (2) Every nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a left (resp. right) ideal of S.

Proof. It can be easily verified by the similar way to the proof of Theorem 4.20. $\hfill \Box$

Denote by $\mathcal{C}(X)$ the family of cubic sets in a set X. Let X and Y be given classical sets. A mapping $h: X \to Y$ induces two mappings $\mathcal{C}_h :$ $\mathcal{C}(X) \to \mathcal{C}(Y), \quad \mathscr{A} \mapsto \mathcal{C}_h(\mathscr{A}), \text{ and } \mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X), \quad \mathscr{B} \mapsto \mathcal{C}_h^{-1}(\mathscr{B}),$ where $\mathcal{C}_h(\mathscr{A})$ is given by

$$\mathcal{C}_{h}(\tilde{\mu}_{A})(y) = \begin{cases} \operatorname{rsup} \tilde{\mu}_{A}(x) & \text{if } h^{-1}(y) \neq \emptyset\\ y = h(x)\\ [0,0] & \text{otherwise} \end{cases}$$
$$\mathcal{C}_{h}(f_{A})(y) = \begin{cases} \inf_{y = h(x)} f_{A}(x) & \text{if } h^{-1}(y) \neq \emptyset\\ 1 & \text{otherwise} \end{cases}$$

for all $y \in Y$; and $\mathcal{C}_h^{-1}(\mathscr{B})$ is defined by $\mathcal{C}_h^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(h(x))$ and $\mathcal{C}_h^{-1}(f_B)(x) = f_B(h(x))$ for all $x \in X$. Then the mapping \mathcal{C}_h (resp. \mathcal{C}_h^{-1}) is called a cubic transformation (resp. inverse cubic transformation) induced by h. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in X has the *cubic property* if for any subset T of X there exists $x_0 \in T$ such that $\tilde{\mu}_A(x_0) = \operatorname{rsup} \tilde{\mu}_A(x)$

and $f_A(x_0) = \inf_{x \in T} f_A(x)$.

Theorem 4.22. For a homomorphism $h: X \to Y$ of semigroups, let $\mathcal{C}_h: \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_h^{-1}: \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by h.

- (1) If $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle \in \mathcal{C}(X)$ is a cubic subsemigroup of X which has the cubic property, then $\mathcal{C}_h(\mathscr{A})$ is a cubic subsemigroup of Y.
- (2) If $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle \in \mathcal{C}(Y)$ is a cubic subsemigroup of Y, then $\mathcal{C}_h^{-1}(\mathscr{B})$ is a cubic subsemigroup of X.

Proof. (1) Given $h(x), h(y) \in h(X)$, let $x_0 \in h^{-1}(h(x))$ and $y_0 \in h^{-1}(h(y))$ be such that

$$\tilde{\mu}_A(x_0) = \underset{a \in h^{-1}(h(x))}{\operatorname{rsup}} \tilde{\mu}_A(a), \ f_A(x_0) = \inf_{a \in h^{-1}(h(x))} f_A(a),$$

and

$$\tilde{\mu}_A(y_0) = \sup_{b \in h^{-1}(h(y))} \tilde{\mu}_A(b), \ f_A(y_0) = \inf_{b \in h^{-1}(h(y))} f_A(b),$$

respectively. Then

$$\begin{split} \mathcal{C}_{h}(\tilde{\mu}_{A})(h(x)h(y)) &= \underset{z \in h^{-1}(h(x)h(y))}{\operatorname{rsup}} \tilde{\mu}_{A}(z) \\ &\succeq \tilde{\mu}_{A}(x_{0}y_{0}) \succeq \operatorname{rmin}\{\tilde{\mu}_{A}(x_{0}), \tilde{\mu}_{A}(y_{0})\} \\ &= \operatorname{rmin}\left\{\underset{a \in h^{-1}(h(x))}{\operatorname{rsup}} \tilde{\mu}_{A}(a), \underset{b \in h^{-1}(h(y))}{\operatorname{rsup}} \tilde{\mu}_{A}(b)\right\} \\ &= \operatorname{rmin}\{\mathcal{C}_{h}(\tilde{\mu}_{A})(h(x)), \mathcal{C}_{h}(\tilde{\mu}_{A})(h(y))\}, \\ \mathcal{C}_{h}(f)(h(x)h(y)) &= \underset{z \in h^{-1}(h(x)h(y))}{\operatorname{inf}} f_{A}(z) \\ &\leq f_{A}(x_{0}y_{0}) \leq \operatorname{max}\{f_{A}(x_{0}), f_{A}(y_{0})\} \\ &= \operatorname{max}\left\{\underset{a \in h^{-1}(h(x))}{\operatorname{inf}} f_{A}(a), \underset{b \in h^{-1}(h(y))}{\operatorname{inf}} f_{A}(b)\right\} \\ &= \operatorname{max}\{\mathcal{C}_{h}(h(x)), \mathcal{C}_{h}(h(y))\}. \end{split}$$

Therefore $\mathcal{C}_h(\mathscr{A})$ is a cubic subsemigroup of Y.

(2) For any $x, y \in X$, we have

$$\mathcal{C}_{h}^{-1}(\tilde{\mu}_{B})(xy) = \tilde{\mu}_{B}(h(xy)) = \tilde{\mu}_{B}(h(x)h(y))$$

$$\succeq \min\{\tilde{\mu}_{B}(h(x)), \tilde{\mu}_{B}(h(y))\}$$

$$= \min\{\mathcal{C}_{h}^{-1}(\tilde{\mu}_{B})(x), \mathcal{C}_{h}^{-1}(\tilde{\mu}_{B})(y)\},$$

$$\mathcal{C}_{h}^{-1}(f_{B})(xy) = f_{B}(h(xy)) = f_{B}(h(x)h(y))$$

$$\leq \max\{f_{B}(h(x)), f_{B}(h(y))\}$$

$$= \max\{\mathcal{C}_{h}^{-1}(f_{B})(x), \mathcal{C}_{h}^{-1}(f_{B})(y)\}.$$

Hence $\mathcal{C}_h^{-1}(\mathscr{B})$ is a cubic subsemigroup of X.

By the similar way to the proof of Theorem 4.22, we have the following theorem.

Theorem 4.23. For a homomorphism $h: X \to Y$ of semigroups, let $\mathcal{C}_h: \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_h^{-1}: \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by h.

- (1) If $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle \in \mathcal{C}(X)$ is a left (resp. right) cubic ideal of X which has the cubic property, then $\mathcal{C}_h(\mathscr{A})$ is a left (resp. right) cubic ideal of Y.
- (2) If $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle \in \mathcal{C}(Y)$ is a left (resp. right) cubic ideal of Y, then $\mathcal{C}_h^{-1}(\mathscr{B})$ is a left (resp. right) cubic ideal of X.

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