An Analytic Solution to Projector Pose Estimation Problem

Joo-Haeng Lee

We present an analytic solution to the projector pose estimation problem for the pinhole projection model in which the source image is a centered rectangle with an unknown aspect ratio. From a single quadrilateral given as a target image, our solution gives the position and orientation of a projector as well as the aspect ratio of a source image. The proposed method decomposes the problem into two pose estimation problems of coupled line projectors aligned at each diagonal of the given quadrilateral and then computes the common solution that satisfies the relevant geometric constraints. The solution is formulated as simple analytic equations. We also provide a determinant of projectability of an arbitrary quadrilateral.

Keywords: Coupled line projectors, projector calibration, spatial augmented reality, projectability, inverse projection.

I. Introduction

A projector pose estimation problem aims to compute the position and orientation of a video projector using known information, such as a projection model, an image projected in the environment, and its correspondence with a 2D source image. It is a fundamental problem in spatial augmented reality [1] and projector-camera systems [2] in which the primary display device is a video projector.

This problem is similar to the camera pose estimation in photogrammetry and computer vision [3], which has been intensively studied to produce mature solutions: that is, the perspective-*n*-point (PnP) problem [4], quadrilateral targets [5], self-calibration [6], and practical methods [7].

The pose estimation of cameras and projectors is modeled using the same pinhole projection model. However, there exists a fundamental difference in geometric constraints. For example, in the case of a camera, geometric constraints, such as parallel lines and perpendicular angles, are specified in the scene geometry. In a projector, however, the constraint is set on the image side: that is, a source image should be a rectangle. Hence, an existing solution for a camera cannot be directly applied to a projector, which is the main motivation of this work.

Many previous methods for projector pose estimation and calibration rely on a calibrated camera to capture the geometry of projected patterns and then compute the homography to find a relative pose of a projector from a known camera pose [8]-[10]. However, little is known regarding the minimum geometric requirement to estimate the absolute pose of an uncalibrated projector.

In this letter, we propose an analytic solution to the projector pose estimation problem for the pinhole projection model in which the source image is a centered rectangle with an unknown aspect ratio. When a quadrilateral is given as a projected image in the environment, our solution gives the position and orientation of a projector as well as the aspect ratio of a source rectangle. It implies that a single quadrilateral is sufficient to estimate both intrinsic and extrinsic parameters of a pinhole projector.

In section II, we use the proposed method to decompose the problem into two pose estimation problems of line projectors aligned for each diagonal of the given quadrilateral. Then, in section III, we compute the common solution that satisfies the geometric constraints. The solution is formulated as simple equations. We also propose a determinant condition for projectability of an arbitrary quadrilateral.

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II. Pose Estimation of Line Projector

In this section, we describe the analytic solution to the pose

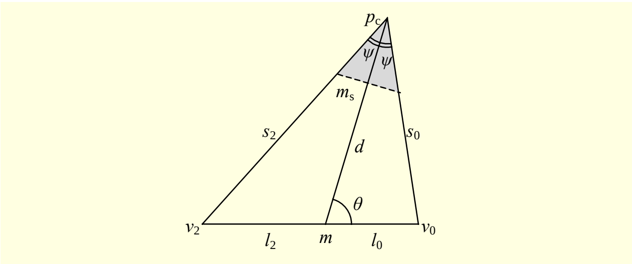

Fig. 1. Geometric configuration of line projector.

estimation problem for a line projector, which is the basis on which we derive the solution of the main problem in section III.

1. Configuration

When a source line segment L_s is projected on a target plane P, we get a target line segment L_t as a projected image of L_s (see Fig. 1). If we have prior knowledge of both intrinsic and extrinsic parameters of a line projector (that is, field of view ψ, center of projection p_c, and direction of the principal axis) as well as the geometry of a target plane P (that is, normal vector and origin), we can compute the target image L_t precisely.

2. Problem Definition

As an inverse process, the problem is to find both intrinsic and extrinsic parameters of a projector from a target image.

Let a given target image L_t be specified with two endpoints, v_0 and v_2, and a projected midpoint, $m \in L_t$, through which the principal axis passes.

We assume that the unknown source image L_s is aligned such that the principal axis passes through its midpoint m_s. Hence, two angles, $\angle m p_c v_0$ and $\angle m p_c v_2$, are identical as ψ. Note that ψ is an unknown *intrinsic parameter* that corresponds to the lens effect or field of view.

Let d be the length of the principal axis from p_c to m and the angle $\angle p_c m v_0$ be the orientation θ of the principal axis measured from the target plane P. The unknowns, d and θ, are extrinsic parameters that determine the pose of a projector.

3. Extrinsic Parameter

With two length coefficients from a target image, $l_0 = \| \overline{mv_0} \|$ and $l_2 = \| \overline{mv_2} \|$, the areas of two triangles, $\Delta p_c m v_0$ and $\Delta p_c m v_2$, have the ratio $l_0{:}l_2$. Since two angles, $\angle m p_c v_0$ and $\angle m p_c v_2$, are identical as ψ, the lengths of two sides, $s_0 = \| \overline{p_c v_0} \|$ and $s_2 = \| \overline{p_c v_2} \|$, have the same ratio $l_0{:}l_2$. Since s_0 and s_2 can be further described with l_0, l_2, d, and θ as

$$\frac{s_0^2}{s_2^2} = \frac{l_0^2}{l_2^2} = \frac{(d \sin\theta)^2 + (l_0 - d\cos\theta)^2}{(d \sin\theta)^2 + (l_2 + d\cos\theta)^2},$$

we can derive the following relation between d and θ:

$$\cos\theta = \left(\frac{l_2 - l_0}{2 l_0 l_2} \right) d = \alpha d, \tag{1}$$

where $\alpha = (l_2 - l_0) / 2 l_0 l_2$ is a coefficient solely defined by the target image.

4. Intrinsic Parameter

From the configuration of Fig. 1, we can derive the following relations:

$$s_0 s_2 \sin 2\psi = d(l_0 + l_2)\sin\theta,$$
$$s_0 = l_0 \sin\theta / \sin\psi, \qquad s_2 = l_2 \sin\theta / \sin\psi.$$

These relations and (1) can be combined to derive the relation between the extrinsic and intrinsic parameters, θ and ψ, as follows:

$$\tan\psi = 2\frac{l_0 l_2}{d(l_0 + l_2)} \sin\theta = \frac{l_2 - l_0}{l_0 + l_2} \tan\theta = \frac{1}{\beta} \tan\theta, \tag{2}$$

where $\beta = (l_0 + l_2)/(l_2 - l_0)$ is a coefficient solely defined by a given target image.

5. Remarks

Since (1) and (2) with three parameters (d, θ, and ψ) are under-constrained, we need to confine one parameter to determine the values of others. For example, when the length d is chosen such that $0 < d \le 1/|\alpha|$, we can find the orientation $\theta = \cos^{-1}(d\,\alpha)$ using (1). Then, the field of view ψ can be computed using (2).

III. Pose Estimation of a Rectangle Projector

Using the solution of section II, we describe the analytic solution to our main problem.

1. Configuration

We assume that the target image is given as an arbitrary convex quadrilateral, Q_t, with four vertices, v_i, and that its source image is a rectangle, Q_s, of an unknown aspect ratio (see Fig. 2). An additional condition is that the source image Q_s is well aligned such that the principal axis, which starts from an unknown center of projection P_c, passes through its midpoint m_s and is perpendicular to Q_s.

2. Problem Definition

The main problem in this letter is to find the pose of a projector while satisfying the standard pinhole projection

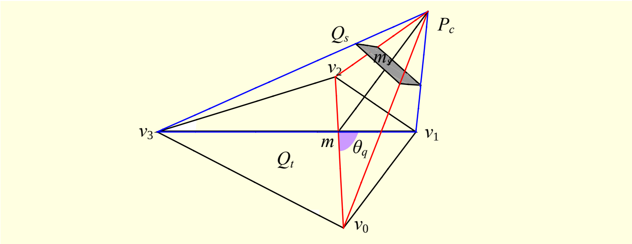

Fig. 2. Geometric configuration of rectangle projector.

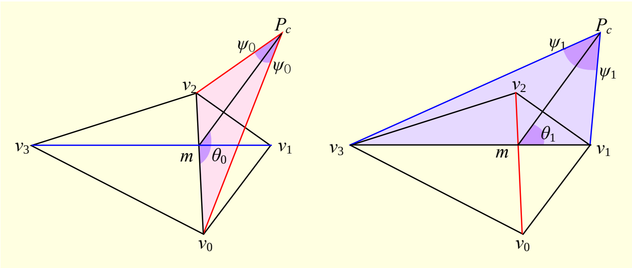

Fig. 3. Two line projectors defined for each diagonal.

constraint such that the source image is a rectangle although its aspect ratio is not constrained.

In a perspective projection, each diagonal of the source rectangle Q_s is projected to a diagonal of the target quadrilateral Q_t while preserving the projective correspondence between the midpoint m_s of Q_s and the intersection point m of two diagonals of the target quadrilateral Q_t.

This implies that we can decompose the problem into two subproblems of a line projector for each diagonal (as in Fig. 3) and that a common solution for these subproblems becomes the final solution to the pose estimation of a rectangle projector. The geometric constraint for a rectangle source image is satisfied when two field-of-view angles of each line projector are identical. We elaborate on the details in the following.

3. Formulation

For a target quadrilateral, Q_t, its projected midpoint m can be computed as an intersection of two diagonals: $g_0 = \overline{v_0 v_2}$ and $g_1 = \overline{v_1 v_3}$. A pair of coefficients, α_i and β_i, is defined for each diagonal, $g_i (i \in \{0,1\})$, and m, and the solution for each line projector is described using (1) and (2):

$$\cos\theta_i = \alpha_i d_i, \tag{3}$$

$$\tan\theta_i = \beta_i \tan\psi_i. \tag{4}$$

Since two line projectors of Fig. 3 share the center of projection, the lengths between m and P_c should be identical: $d_0 = d_1 \equiv d$. Moreover, the fields of view should also be identical to define a rectangle source image: $\psi_0 = \psi_1 \equiv \psi$. These geometric constraints can be formulated into two equations with two unknowns, θ_0 and θ_1, as

$$\alpha_1 \cos\theta_0 = \alpha_0 \cos\theta_1, \tag{5}$$

$$\beta_1 \tan\theta_0 = \beta_0 \tan\theta_1. \tag{6}$$

4. Solution

Using the trigonometric identity, $\cos^2\theta = 1/(1+\tan^2\theta)$, we can derive the following equation from (5) and (6):

$$A^2 \cos^2\theta_0 = 1/(1+B^2\tan^2\theta_0) \tag{7}$$

with the known values of *diagonal length coefficients* $A = \alpha_1/\alpha_0$ and $B = \beta_1/\beta_0$. While satisfying the field-of-view constraint $\tan\psi_1 = (1/\beta)\tan\theta_i > 0$, (7) can be further reduced to the following form:

$$\cos\theta_0 = \frac{\sqrt{A^2B^2-1}}{A\sqrt{B^2-1}}. \tag{8}$$

Now, we can apply (8) to (5) to compute $\cos\theta_1$.

Since we know the values of the orientation θ_i of two line projectors, we can find the values of distance d and field of view ψ using (3) and (4). Since the common solution of two subproblems is known, we can find the actual projector parameters as follows.

5. Projector Parameters

Now, the 3D coordinates of the center of projection $P_c = (x, y, z) \neq m$ can be computed as follows. Assume that the projected midpoint is the origin of the coordinate system: m=(0, 0, 0). Let the projection of P_c onto the xy-plane be $\overline{P_c} = (x, y, 0)$ and the angle between the x-axis and the diagonal vector (v_i–m) be ϕ_i. Now, the unit vector $\overline{v_i} = v_i / \| v_i \|$ can be represented as $\overline{v_i} = (\cos\phi_i, \sin\phi_i, 0)$. It is clear that the following equation holds for $i \in \{0,1\}$:

$$\langle P_c, \overline{v_i} \rangle = \langle \overline{P_c}, \overline{v_i} \rangle. \tag{9}$$

Using (9) and related definitions, we can derive three equations, which can be easily solved for P_c=(x, y, z):

$$x\cos\phi_0 + y\sin\phi_0 = d\cos\theta_0, \tag{10}$$

$$x\cos\phi_1 + y\sin\phi_1 = d\cos\theta_1, \tag{11}$$

$$z^2 = d^2 - (x^2 + y^2). \tag{12}$$

Since we have the complete projection frustum (in Fig. 2) defined by P_c and Q_t, it is straightforward to compute the other projector parameters. For example, we can find the canonical geometry of a source rectangle and its aspect ratio.

6. Existence of Solution

Existence of the analytic solution to (8) can be determined with the following condition that is solely defined with the diagonal length coefficients A and B of (7):

$$(A^2 > 1 \text{ and } B^2 < 1 \text{ and } A^2B^2 < 1)$$
$$\text{or} \quad (A^2 < 1 \text{ and } B^2 > 1 \text{ and } A^2B^2 > 1). \tag{13}$$

This condition can be used as a *determinant of projectability* for a given target quadrilateral before explicit computation of pose estimation and projector parameters.

The final condition for existence of the solution is as follows:

$$|\theta_0 - \theta_1| \le \theta_q \le \theta_0 + \theta_1, \tag{14}$$

in which θ_q is the angle between two diagonals (see Fig. 2): $\theta_q = \angle v_0 m v_1$. Equation (14) is the *angular constraint* that combines two subproblems in a geometric manner. Note that we can find the common solution of two subproblems as in subsection III.4 regardless of satisfying (14). However, the center of projection $P_c=(x, y, z)$ in subsection III.5 cannot be found without satisfying this last condition (14).

IV. Examples of Projector Pose Estimation

In Fig. 4, the aspect ratio of a source rectangle is denoted by r. Each quadrilateral Q_t (in the right column of Fig. 4) is represented as a canonical form: two diagonals intersect at the origin m=(0, 0, 0), and the first diagonal is aligned on the x-axis and has the unit length $\|g_0\|$=1. The unknown four parameters, that is, the *diagonal parameters*, are as follows: the length of the second diagonal $\|g_1\|$, the angle $\theta_q = \angle v_0 m v_1$, and two ratios defined for each diagonal $t_i=\|mv_i\|/\|g_i\|$.

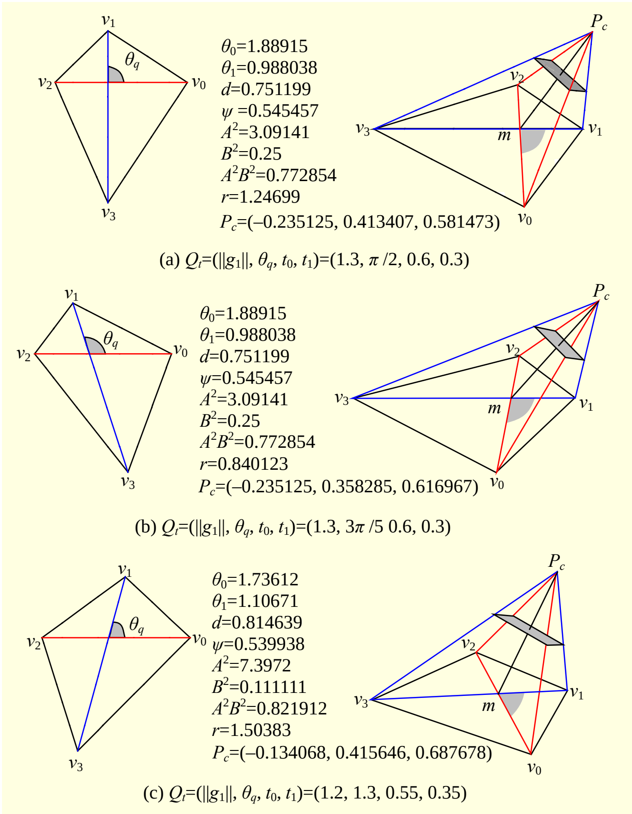

Fig. 4. Examples of pose estimation for different quadrilaterals.

V. Conclusion

In this letter, we presented an analytic solution to the projector pose estimation problem by decomposing the problem into two subproblems of coupled line projectors. We also presented the determinant of projectability for an arbitrary quadrilateral.

The proposed solution can be the theoretic basis of more practical applications [1], [2]. For example, an interesting future work will be planning multiple projectors to cover a complex indoor area. We must extend our framework to handle the case in which the projection screen is nonplanar [11].

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