Journal of Korean Institute of Intelligent Systems, Vol. 22, No. 5, October 2012, pp. 646-655
http://dx.doi.org/10.5391/JKIIS.2012.22.5.646

Interval-Valued Fuzzy Cosets

Keon Chang Lee*, **Kul Hur**** **and Pyung Ki Lim*****†

*Department of Computer Science, Dongshin University

,*Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University**

Abstract

First, we prove a number of results about interval-valued fuzzy groups involving the notions of interval-valued fuzzy cosets and interval-valued fuzzy normal subgroups which are analogs of important results from group theory. Also, we introduce analogs of some group-theoretic concepts such as characteristic subgroup, normalizer and abelian groups. Secondly, we prove that if A is an interval-valued fuzzy subgroup of a group G such that the index of A is the smallest prime dividing the order of G, then A is an interval-valued fuzzy normal subgroup. Finally, we show that there is a one-to-one correspondence the interval-valued fuzzy cosets of an interval-valued fuzzy subgroup A of a group G and the cosets of a certain subgroup H of G.

Key Words: interval-valued fuzzy normal subgroup, interval-valued fuzzy coset, interval-valued fuzzy characteristic fuzzy subgroup, normalizer, abelian group.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh[9], and in 1965, he[10] introduced the notion of interval-valued fuzzy set as a generalization of fuzzy sets. After that time, Mondal and Samanta[8], and choi et al.[3] applied it to topology. Also, several researchers [1,2, 4-7] applied one to algebra.

The present paper is a sequel to [4]. We obtain a number of further analogs of the properties of groups, thereby enriching the theory of interval-valued fuzzy groups and, in particular, corroborating the concept of interval-valued fuzzy normal subgroups and interval-valued fuzzy cosets introduced in [4,5]. Moreover, we obtain an analog of the following standard result from group theory that if θ is an automorphism of a group G which leaves invariant some normal subgroup N, then θ induces an automorphism of the quotient group G/N.

Some variations of this result are also considered, for which we obtain analogs for interval-valued fuzzy groups. Also we show that there is a natural one-to-one correspondence between the interval-valued fuzzy cosets of an interval-valued fuzzy subgroup A of a group G and the cosets of a subgroup G_A of G defined by $G_A = \{g \in G : A(g) = A(e)\}$, where e denotes, as usual, the identity element of the group G. Our analysis illustrates that the subgroup G_A defined above plays a significant role in investigating the structure of the corresponding interval-valued fuzzy subgroup.

접수일자 : 2012년 5월 23일
심사(수정)일자 : 2012년 10월 15일
게재확정일자 : 2012년 10월 16일
†교신저자

2. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\boldsymbol{a}$=$[a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))$ $(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$ (See [8]).

Definition 2.1 [8, 10]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, *IVS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point* of x to A. For any $[a, b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVSs in X as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2 [8]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^c = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha]$.

$\mathbf{Result 2.A [8, Theorem 1]}$. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $\tilde{\mathbf{0}} \subset A \subset \tilde{\mathbf{1}}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (A \cup A_\alpha)$.

(g) $(\tilde{\mathbf{0}})^c = \tilde{\mathbf{1}}$, $(\tilde{\mathbf{1}})^c = \tilde{\mathbf{0}}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha \in \Gamma} A_\alpha)^c = \bigcap_{\alpha \in \Gamma} A^c_\alpha$, $(\bigcap_{\alpha \in \Gamma} A_\alpha)^c = \bigcup_{\alpha \in \Gamma} A^c_\alpha$.

Definition 2.3 [8]. Let $f : X \to Y$ be a mapping, let $A = [A^L, A^U] \in D(I)^X$ and let $B = [B^L, B^U] \in D(I)^Y$. Then

(a) the *image* of A under f, denoted by $f(A)$, is an IVS in Y defined as follows: For each $y \in Y$,

$$f(A^L)(y) = \begin{cases} \bigvee_{y=f(x)} A^L(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$f(A^U)(y) = \begin{cases} \bigvee_{y=f(x)} A^U(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

(b) the *preimage* of B under f, denoted by $f^{-1}(B)$, is an IVS in Y defined as follows: For each $y \in Y$,

$$f^{-1}(B^L)(y) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B^U)(y) = (B^U \circ f)(x) = B^U(f(x)).$$

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

Result 2.B [8, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then

(a) $f^{-1}(B^c) = (f^{-1}(B))^c$, $\forall B \in D(I)^Y$.

(b) $[f(A)]^c \subset f(A^c)$, $\forall A \in D(I)^Y$.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.

(e) $f(f^{-1}(B)) \subset B, \forall B \in D(I)^Y$.

(f) $A \subset f(f^{-1}(A)), \forall A \in D(I)^Y$.

(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)), \forall C \in D(I)^Z$.

(h) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.

(h) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) = \bigcap_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.

3. Interval-valued fuzzy subgroups

Definition 3.1 [1, 6]. Let G be a group with the identity e and let $A \in D(I)^G$. Then A is called an *interval-valued fuzzy subgroup*(in short, *IVG*) of G if

(i) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in G$.

(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$ for each $x \in G$.

We will denote the set of all IVGs of G as IVG(G).

Result 3.A [1, Proposition 3.1]. Let G be a group with the identity e and let $A \in$ IVG(G). Then $A(x^{-1}) = A(x)$ and $A^L(x) \leq A^L(e), A^U(x) \leq A^U(e)$ for each $x \in G$.

Result 3.B [6, Proposition 4.6]. If $A \in$ IVG(G), then $G_A = \{x \in G : A(x) = A(e)\}$ is a subgroup of G.

Result 3.C [6, Proposition 4.3]. Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IVG(G). Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVG(G).

Definition 3.2 [6]. Let G be a group with the identity e and let $A \in$ IVG(G). Then A is called an *interval-valued fuzzy normal subgroup*(in short, *IVNG*) of G if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the set of all IVNGs of G as IVNG(G).

Definition 3.3. Let A be an IVG of a group G and let $\theta : G \to G$ be a mapping. We define a mapping $A^\theta = [(A^\theta)^L, (A^\theta)^U] : G \to D(I)$ as follows : For each $g \in G$,

$$A^\theta(g) = A(\theta(g)).$$

For a group G, a subgroup K is called a *characteristic subgroup* if $\theta(K) = K$ for every automorphism θ of G. We now define an analog.

Definition 3.4. Let A be an IVG of a group G. Then A is called an *interval-valued fuzzy characteristic subgroup* of G if $A^\theta = A$ for every automorphism θ of G.

Proposition 3.5. Let G be a group, let $A \in D(I)^G$ and let $\theta : G \to G$ be a mapping.

(a) If $A \in$ IVG(G) and θ is a homomorphism, then $A^\theta \in$ IVG(G).

(b) If A is an interval-valued fuzzy characteristic subgroup of G, then $A \in$ IVNG(G).

Proof. (a) Let $x, y \in G$. Then

$$\begin{aligned} A^\theta(xy) &= A(\theta(xy)) \\ &= A(\theta(x)\theta(y)). \text{ [Since } \theta \text{ is a homomorphism]} \end{aligned}$$

Since $A \in$ IVG(G),

$$\begin{aligned} A^L(\theta(x)\theta(y)) &\geq A^L(\theta(x)) \wedge A^L(\theta(y)) \\ &= (A^\theta)^L(x) \wedge (A^\theta)^L(y). \end{aligned}$$

Similarly, we have that

$A^U(\theta(x)\theta(y)) \geq (A^\theta)^U(x) \wedge (A^\theta)^U(y)$.

Thus

$(A^\theta)^L(xy) \geq (A^\theta)^L(x) \wedge (A^\theta)^L(y)$

and

$(A^\theta)^U(xy) \geq (A^\theta)^U(x) \wedge (A^\theta)^U(y)$.

On the other hand,

$$\begin{aligned} A^\theta(x^{-1}) &= A(\theta(x^{-1})) \\ &= A(\theta(x)^{-1}) \text{ [Since } \theta \text{ is a homomorphism]} \\ &= A(\theta(x)) \text{ [By Result 3.}A\text{]} \\ &= A^\theta(x). \end{aligned}$$

Hence $A^\theta \in$ IVG(G).

(b) Let $\theta : G \to G$ be the automorphism of G defined by $\theta(g) = x^{-1}gx$ for each $g \in G$. Then clearly it is standard result that θ is an automorphism of G, called the $inner\ automorphism$ induced by x. Let $x, y \in G$. Since A is interval-valued fuzzy characteristic, $A^\theta = A$. Thus

$$\begin{aligned} A(xy) &= A^\theta(xy) = A(\theta(xy)) \\ &= A(x^{-1}(xy)x) \text{ [By the definition of } \theta] \\ &= A(yx). \end{aligned}$$

Hence $A \in$ IVNG(G). This completes the proof. □

Remark 3.6. Proposition 3.5(b) is an analog of the result that a characteristic subgroup of a group is normal.

Now we obtain analogs of the concepts of conjugacy, normalizer regarding a group, and their properties.

Definition 3.7. Let G be a group and let $A_1, A_2 \in$ IVG(G). Then we say that A_1 is *conjugate* to A_2 if there exists an $x \in G$ such that $A_1(g) = A_2(x^{-1}gx)$ for each $g \in G$.

It is easy to show that the relation of conjugacy is an equivalence relation on IVG(G). Hence IVG(G) is a union of pairwise disjoint classes of interval-valued fuzzy subgroups each consisting of interval-valued fuzzy subgroups which are equivalent to one another. Now we shall obtain an expression giving the number of distinct conjugates of an interval-valued fuzzy subgroups.

Notation. Let G be a group, let $A \in$ IVG(G) and let $g \in G$. We define a mapping $A^g = [(A^g)^L, (A^g)^U] : G \to D(I)$ as follows : for each $u \in G$, $A^g(u) = A(g^{-1}ug)$, i.e., $(A^g)^L(u) = A^L(g^{-1}ug)$ and $(A^g)^U(u) = A^U(g^{-1}ug)$.

From Proposition 3.5(a), it is clear that $A^g \in$ IVG(G).

Definition 3.8. Let A be an IVG of a group G. Then the set $N(A) = \{g \in G : A^g = A\}$ is called the *normalizer* of A.

Proposition 3.9. Let A be an IVG of a group G. Then

(a) $N(A)$ is a subgroup of G.

(b) $A \in$ IVNG(G) id and only if $N(A) = G$.

(c) If G is a finite group, then the number of distinct conjugates of A is equal to the index of $N(A)$ in G.

Proof. (a) Let $g, h \in N(A)$ and let $u \in G$. Then $A^{gh}(u) = A((gh)^{-1}u(gh)) = A(h^{-1}(g^{-1}ug)h) = A^h(g^{-1}ug) = (A^h)^g(u)$. Thus $A^{gh} = (A^g)^h = A^h = A$. So $gh \in N(A)$. Let $x \in N(A)$ and let $y = x^{-1}$. Let $u \in G$. Then

$$\begin{aligned} A^y(u) &= A(y^{-1}uy) = A(xux^{-1}) = A((x^{-1}u^{-1}x)^{-1}) \\ &= A(x^{-1}u^{-1}x) \text{ [By Result 3.}A] \\ &= A^x(u^{-1}) \text{ [By the definition of } A^x] \\ &= A(u^{-1}) \text{ [Since } A^x = A] \\ &= A(u). \text{ [By Result 3.}A] \end{aligned}$$

Thus $A^y = A$. So $y = x^{-1} \in N(A)$. Hence $N(A)$ is a subgroup of G.

(b)($\Rightarrow$): Suppose $A \in$ IVNG(G) and let $g \in G$. Let $u \in G$. Then

$$\begin{aligned} A^g(u) &= A(g^{-1}ug) = A((g^{-1}u)g) \\ &= A(g(g^{-1}u)) \text{ [Since } A \in IVNG(G)] \\ &= A(u). \end{aligned}$$

Thus $A^g = A$. So $g \in N(A)$, i.e., $G \subset N(A)$. Hence $N(A) = G$.

($\Leftarrow$): Suppose $N(A) = G$ and let $x, y \in G$. Then

$$\begin{aligned} A(xy) &= A(xyxx^{-1}) = A(x(yx)x^{-1}) \\ &= A^{x^{-1}}(yx) \text{ [By the definition of } A^{x^{-1}}] \\ &= A(yx). \text{ [By the hypothesis]} \end{aligned}$$

Hence $A \in$ IVNG(G).

(c) Consider the decomposition of G as a union of cosets of $N(A)$,

$$G = x_1N(A) \cup x_2N(A) \cup \cdots \cup x_kN(A), \qquad (3.1)$$

where k is the number of distinct cosets, i.e., the index of $N(A)$ in G. Let $x \in N(A)$ and choose i such that $1 \leq i \leq$

k. Let $g \in G$. Then

$$\begin{aligned} A^{x_i x}(g) &= A((x_i x)^{-1} g(x_i x)) \\ &= A(x^{-1}(x_i^{-1} g x_i)x) \\ &= A^x(x_i^{-1} g x_i) \\ &= A(x_i^{-1} g x_i) \text{ [Since } x \in N(A)] \\ &= A^{x_i}(g). \end{aligned}$$

Thus $A^{x_i x} = A^{x_i}$ for each $x \in N(A)$ and $1 \le i \le k$. So any two elements of G which lie in the same coset $x_i N(A)$ give rise to the same conjugate A^{x_i} of A. Now we show that two distinct cosets give two distinct conjugates of A. Assume that $A^{x_i} = A^{x_j}$, where $i \ne j$ and $1 \le i \le k$, $1 \le j \le k$. Let $g \in G$. Then

$$A^{x_i}(g) = A^{x_j}(g), \text{ i.e., } A(x_i^{-1} g x_i) = A(x_j^{-1} g x_j). \quad (3.2)$$

Let $h \in G$ such that $g = x_j h x_j^{-1}$. Then, by (3.2),

$$\begin{aligned} & A(x_i^{-1} x_j h x_j^{-1} x_i) = A(x_j^{-1} x_j h x_j^{-1} x_j) \\ \Rightarrow & A((x_i^{-1} x_j) h (x_j^{-1} x_i)) = A(h), \\ & \text{i.e., } A((x_j^{-1} x_i)^{-1} h (x_j^{-1} x_i)) = A(h) \\ \Rightarrow & A^{x_j^{-1} x_i}(h) = A(h), \text{ i.e., } A^{x_j^{-1} x_i} = A. \end{aligned}$$

Thus $x_j^{-1} x_i \in N(A)$. So $x_i N(A) = x_j N(A)$. Since (3.1) represent a partition of G into pairwise disjoint cosets and $i \ne j$, this is not possible. Hence the number of distinct conjugates of A is equal to the index of $N(A)$ in G. This completes the proof. □

Remark 3.10. Proposition 3.9(b) illustrates the motivation behind the term "normalizer" and it shows the analogy with the fact that a subgroup H of a group G is normal in G if and only if the normalizer of H in G is equal to G itself. And Proposition 3.9(c) is an analog of a basic result in group theory.

Definition 3.11 [4]. Let A be an IVG of a group G and let $x \in G$. We define two mappings $Ax = [Ax^L, Ax^U] : G \to D(I)$ and $xA = [xA^L, xA^U] : G \to D(I)$ as follows, respectively : For each $g \in G$, $Ax(g) = A(gx^{-1})$ and $xA(g) = A(x^{-1}g)$. Then Ax[resp.xA] is called the *interval-valued fuzzy right*[resp.*left*] *coset* of G determined by x and A.

Lemma 3.12. Let A be an IVG of a group G and let $K = \{x \in G : Ax = Ae\}$,
where e denotes the identity element of G. Then K is a subgroup of G. Furthermore, $G_A = K$.

Proof. Let $k \in K$ and let $g \in G$. Then $Ak(g) = Ae(g)$. Thus $A(gk^{-1}) = A(g)$. In particular, $A(ek^{-1}) = A(e)$, i.e., $A(k^{-1}) = A(e)$. Thus $k^{-1} \in G_A$. By Result 3.B, G_A is a subgroup of G. Thus $k \in G_A$. So $K \subset G_A$. Now let $h \in G_A$. Then

$$A(h) = A(e). \quad (3.3)$$

Let $g \in G$. Then $Ah(g) = A(gh^{-1})$ and $Ae(g) = A(g)$. Thus

$$\begin{aligned} A^L(gh^{-1}) &\ge A^L(g) \wedge A^L(h^{-1}) \\ &= A^L(g) \wedge A^L(h) \text{ [By Result 3.}A] \\ &= A^L(g) \wedge A^L(e) \text{ [By (3.3)]} \\ &= A^L(g). \text{ [By Result 3.}A] \end{aligned}$$

Similarly, we have that $A^U(gh^{-1}) \ge A^U(g)$. Also,

$$\begin{aligned} A^L(g) = A^L(gh^{-1}h) &\ge A^L(gh^{-1}) \wedge A^L(h) \\ &= A^L(gh^{-1}) \wedge A^L(e) \text{ [By (3.3)]} \\ &= A^L(gh^{-1}). \text{ [By Result 3.}A] \end{aligned}$$

Similarly, we have that $A^U(g) \ge A^U(gh^{-1})$. So $A(gh^{-1}) = A(g)$, i.e., $Ah = Ae$, i.e., $h \in K$. Hence $G_A \subset K$. Therefore $G_A = K$. This completes the proof. □

Corollary 3.12 [6, Proposition 5.4]. Let G be a group. If $A \in \text{IVNG}(G)$, then $G_A \lhd G$.

Proof. Let $g \in G$ and let $x \in G_A$. Then

$$\begin{aligned} A(g^{-1}xg) &= A(gg^{-1}x) \text{ [Since } A \in \text{IVNG}(G)] \\ &= A(x) \\ &= A(e). \text{ [Since } x \in G_A] \end{aligned}$$

Thus $g^{-1}xg \in G_A$. Hence $G_A \lhd G$. □

For a group G, the commutator $[x, y]$ of two elements x, y in G is defined as $[x, y] = x^{-1}y^{-1}xy$. If $xy = yx$, then obviously $[x, y] = e$. Thus G is abelian if $[x, y] = e$ for all $x, y \in G$. This motivates the following definition.

Remark 3.13. A special case of Lemma 3.12 is implicit in Theorem 2.12 in [4], where it was tacitly assumed that A is interval-valued fuzzy normal. But, as we see now, it is not necessarily to assume that A is interval-valued fuzzy normal, and this fact straightens the proof of the interval-valued fuzzy Lagrange's theorem [4, Theorem 4.12].

Definition 3.14. Let A be an IVG of a group G. Then A is said to be *interval-valued fuzzy abelian* if $A([x, y]) = A(e)$ for any $x, y \in G$.

Result 3.D [4, Theorem 2.12]. Let $A \in$ IVG(G). Then $A \in$ IVNG(G) if and only if $A^L([x, y]) \geq A^L(x)$ and $A^U([x, y]) \geq A^U(x)$ for any $x, y \in G$.

Analogous to some well-known properties of abelian group, we prove.

Theorem 3.15. (a) An interval-valued fuzzy abelian subgroup of a group is interval-valued fuzzy normal.

(b) Given an interval-valued fuzzy abelian subgroup of G, there is a normal subgroup N of G such that G/N is abelian.

Proof. (a) Let A be an interval-valued fuzzy abelian subgroup of G. Let $x, y \in G$. Then, by Result 3.A, $A^L([x, y]) = A^L(e) \geq A^L(x)$ and $A^U([x, y]) = A^U(e) \geq A^U(x)$. Hence, by Result 3.D, $A \in$ IVNG(G).

(b) Let A be an interval-valued fuzzy abelian subgroup of G. Then, by (a), $A \in$ IVNG(G). Thus, by Corollary 3.12, $G_A \lhd G$. Also, it is easy to see that $G' \subset G_A$, where G' denotes the commutator subgroup of G (i.e., the subgroup generated by all elements $[x, y]$, $x, y \in G$). Hence G/G_A is abelian. □

The following is the immediate result of Definition 3.2 and Result 3.C.

Proposition 3.16. If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a family of IVNGs of a group G, then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVNG(G). Furthermore, if $A, B \in$ IVNG(G), then $A \cap B \in$ IVNG(G).

It is a standard result in group theory that if G is a group, $H \leq G, K \leq G$ and $H \lhd G$, then $H \cap K \lhd K$ is normal in K. Now we derive an analog for interval-valued fuzzy subgroups.

Proposition 3.17. Let G be a group and let $A \in$ IVG(G), $B \in$ IVNG(G). Then $A \cap B$ is an interval-valued fuzzy normal subgroup of the group G_A.

Proof. It is clear that G_A is a subgroup of G by Result 3.B. By Proposition 3.16, $A \cap B \in$ IVG(G). Thus $A \cap B \in$ IVG(G_A). Let $x, y \in G_A$. Since G_A is a subgroup of G, $xy \in G_A$ and $yx \in G_A$. Thus $A(xy) = A(yx) = A(e)$. Since $B \in$ IVNG(G), $B(xy) = B(yx)$. So

$$\begin{aligned}(A \cap B)(xy) &= [A^L(xy) \wedge B^L(xy), A^U(xy) \wedge B^U(xy)]\\ &= [A^L(yx) \wedge B^L(yx), A^U(yx) \wedge B^U(yx)]\\ &= (A \cap B)(yx).\end{aligned}$$

Hence $A \cap B \in$ IVNG(G_A). □

4. Interval-valued fuzzy cosets

Result 4.A [4, Theorem 2.9]. Let A be an IVG of a group G. Then the followings are equivalent :

(a) $A^L(xyx^{-1}) \geq A^L(y)$ and $A^U(xyx^{-1}) \geq A^U(y)$ for any $x, y \in G$.

(b) $A(xyx^{-1}) = A(y)$ for any $x, y \in G$.

(c) $A \in$ IVNG(G).

(d) $xA = Ax$ for each $x \in G$.

(e) $xAx^{-1} = A$ for each $x \in G$.

Remark 4.1. We shall restrict ourselves in the subsequent discussion, without any loss of generality, with interval-valued fuzzy right cosets only(corresponding results for interval-valued fuzzy left cosets could be obtained without any difficulty). Consequently from now on we call an interval-valued fuzzy right coset an *interval-valued fuzzy coset* and denote it as Ax for each $x \in G$.

Definition 4.2 [4]. Let A be an IVG of a finite group G. Then the cardinality $|G/A|$ of G/A is called an *index* of A, where G/A denotes the set of all interval-valued fuzzy cosets of A.

Result 4.B [4, Proposition 3.7]. Let A be an IVNG of a group G. We define an operation $*$ on G/A as follows : For any $x, y \in G$, $Ax * Ay = Axy$. Then $(G/A, *)$ is a group. In this case, G/A is called the interval-valued fuzzy quotient group by A.

Result 4.C [4, Theorem 4.12]. Let A be an IVG of a finite group G. Then the index of A divides the order of G.

It is a well-known result in group theory that subgroup of index 2 is a normal subgroup. We now obtain an analog of a generalization of this result.

Proposition 4.3. Let A be an IVG of a finite group G such that the index of A is p, where p is the smallest prime dividing the order of G. Then $A \in$ IVNG(G).

Proof. By Result 3.B, G_A is a subgroup of G. Since A is an IVG of G such that the index of A is p, by Lemma 3.12 and Result 4.C, G_A has index p in G, i.e., G_A has p distinct (right) cosets, say, $\{G_A x_i : 1 \leq i \leq p\}$. Now consider the permutation representation of G on the cosets of G_A given by the map $\pi : x \to \pi_{x^{-1}}$, where $\pi_{x^{-1}} : G_A x_i \to G_A x_i x^{-1}$, $1 \leq i \leq p$. Since the index of G_A in G is p, π is an isomorphism of G into the symmetric group S_p. Furthermore, $Ker\pi = Core(G_A)$, where $Core(G_A)$ denotes the intersection of all the conjugates $g^{-1}G_A g$, $g \in G$. By the fundamental theorem of homomorphisms of groups and using Lagrange's theorem, the order of $G/Core(G_A)$ divides $p!$ which is the order of S_p. Furthermore,

$$G/Core(G_A) \cong (G/G_A)(G_A/Core(G_A))$$

and the order of G/G_A is p. Thus it follows that the order of $G_A/Core(G_A)$ divides $(p-1)!$. Since the order of G_A divides the order of G, $G_A = Core(G_A)$; otherwise we get a contradiction to the fact that p is the smallest prime dividing the order of G. Since $Core(G_A)$ is a normal subgroup of G, G_A is a normal subgroup of G. Now consider the quotient group G/H. Since the order of G/G_A is p, G/G_A is abelian. Let $x, y \in G$. Then $(G_A x)(G_A y) = (G_A y)(G_A x)$. Thus $G_A xy = G_A yx$. So there exists an $h \in G_A$ such that $xy = hyx$. Then

$$\begin{aligned} A^L(xy) &= A^L(hyx) \geq A^L(h) \wedge A^L(yx) \\ &= A^L(e) \wedge A^L(yx) = A^L(yx). \end{aligned}$$

Similarly, we have that $A^U(xy) \geq A^U(yx)$. Also, we have that $A^L(yx) \geq A^L(xy)$ and $A^U(yx) \geq A^U(xy)$. So $A(xy) = A(yx)$ for any $x, y \in G$. Hence $A \in$ IVNG(G). This completes the proof. □

The following is the immediate result of Proposition 4.3.

Corollary 4.3. Let A be an IVG of a group G such that the index of A is 2, then $A \in$ IVNG(G).

It is well-known in group theory that θ is a homomorphism of a group G into itself whose kernel is N, then θ induces a homomorphism from G/N into itself. Now we derive an analog of the following result.

Proposition 4.4. Let A be an IVNG of a group G and let θ be an homomorphism of G into itself such that $\theta(G_A) = G_A$. Then θ induces a homomorphism $\bar{\theta}$ of the interval-valued fuzzy cosets of A defined as follows : $\bar{\theta}(Ax) = A\theta(x)$ for each $x \in G$.

Proof. Suppose $x, y \in G$ such that $Ax = Ay$. Then $Ax(x) = Ay(x)$ and $Ax(y) = Ay(y)$. Thus $A(e) = A(xy^{-1}) = A(yx^{-1})$. So $xy^{-1}, yx^{-1} \in G_A$. Since

$\theta(G_A) = G_A$, $\theta(xy^{-1}), \theta(yx^{-1}) \in G_A$. Then

$$A(\theta(xy^{-1})) = A(\theta(yx^{-1})) = A(e). \qquad (4.1)$$

Let $g \in G$. Then

$$\begin{aligned}(A\theta(x))^L(g) &= A^L(g\theta(x)^{-1})\\ &= A^L(g\theta(x^{-1})) \text{ [Since } \theta \text{ is a homomorphism]}\\ &= A^L(g\theta(y^{-1})\theta(y)\theta(x^{-1}))\\ &\geq A^L(g\theta(y^{-1})) \wedge A^L(\theta(y)\theta(x^{-1}))\\ &\quad \text{[Since } A \in \text{ IVG(G)]}\\ &= A^L(g\theta(y^{-1})) \wedge A^L(\theta(yx^{-1}))\\ &\quad \text{[Since } \theta \text{ is a homomorphism]}\\ &= (A\theta(y))^L(g) \wedge A^L(e) \text{ [By (4.1)]}\\ &= (A\theta(y))^L(g). \text{ [By Result 3.}A]\end{aligned}$$

Similarly, we have that $(A\theta(x))^U(g) \geq (A\theta(y))^U(g)$. Also, we have that $(A\theta(y))^L(g) \geq (A\theta(x))^L(g)$ and $(A\theta(y))^U(g) \geq (A\theta(x))^U(g)$. Thus $A\theta(x) = A\theta(y)$. So $\bar{\theta}$ is well-defined. Now let $x, y \in G$. Then

$$\begin{aligned}\bar{\theta}(Ax * Ay) &= \bar{\theta}(Axy) \text{ [By Result 4.}B]\\ &= A\theta(xy) \text{ [By the definition of } \bar{\theta}]\\ &= A\theta(x)\theta(y) \text{ [Since } \theta \text{ is a homomorphism]}\\ &= A\theta(x) * A\theta(y) \text{ [By Result 4.}B]\\ &= \bar{\theta}(Ax) * \bar{\theta}(Ay). \text{ [By the definition of } \bar{\theta}]\end{aligned}$$

Hence $\bar{\theta}$ is a homomorphism. This completes the proof. □

Corollary 4.4-1. In the same hypothesis as in Proposition 4.4, if θ is an automorphism and G is finite, then $\bar{\theta}$ is an automorphism.

Proof. Since G has finite order, it is easy to see that θ has finite order. Suppose that θ has order k. Then $\theta^k = id_G$, where id_G denotes the identity mapping. Let $x, y \in G$ such that $\bar{\theta}(Ax) = \bar{\theta}(Ay)$. Then, by the definition of $\bar{\theta}$, $A\theta(x) = A\theta(y)$.
Since $\theta^k = id_G$, $\theta^k(x) = x$ and $\theta^k(y) = y$. Thus $Ax = A\theta^k(x) = A\theta^k(y) = Ay$.
So $\bar{\theta}$ is injective. Hence $\bar{\theta}$ is an automorphism. □

Corollary 4.4-2. In the same hypothesis as in Proposition 4.4, if $\bar{\theta}$ is an automorphism and $G_A = (e)$, then θ is an automorphism.

Proof. Let $x, y \in G$ such that $\theta(x) = \theta(y)$. Then $A\theta(x) = A\theta(y)$, i.e., $\bar{\theta}(Ax) = \bar{\theta}(Ay)$. Since $\bar{\theta}$ is injective, $Ax = Ay$. Then $Ax(y) = Ay(y)$. Thus $A(yx^{-1}) = A(e)$. So $yx^{-1} \in G_A$. Since $G_A = (e)$, $yx^{-1} = e$. Thus $x = y$. So θ is injective. Hence θ is an automorphism. □

The motivation of the following result stems from the standard theorem in group theory that if θ is an automorphism of G and N is a normal subgroup of G such that $N^\theta \subset N$, then θ induces an automorphism of the quotient group G/N into itself.

Remark 4.5. In Proposition 4.4, we have assumed A to be interval-valued fuzzy normal instead of assuming only that A is an interval-valued fuzzy subgroup. This has been done to ensure that the law of composition of interval-valued fuzzy cosets is well-defined, and this fact is used in the proof of Proposition 4.4 to show that $\bar{\theta}$ is a homomorphism(refer to Result 4.B). However, it is clear from the proof that to show $\bar{\theta}$ is well-defined it is not necessary to assume A to be interval-valued fuzzy normal.

Proposition 4.6. Let A be an IVNG of a group G and let θ be an automorphism of G such that $A^\theta = A$(recall the definition of A^θ given by Definition 3.3). Then θ induces an automorphism $\bar{\theta}$ of G/A defined as follows : for each $x \in G$, $\bar{\theta}(Ax) = A\theta(x)$.

Proof. Let $x, y \in G$ such that $Ax = Ay$. We show that $\bar{\theta}(Ax) = \bar{\theta}(Ay)$, i.e., $A\theta(x)(g) = A\theta(y)(g)$ for each $g \in G$. Let $g \in G$. Since θ is an automorphism of G, there exists a $g^* \in G$ such that $\theta(g^*) = g$. Since $Ax = Ay$, $Ax(g^*) = Ay(g^*)$, i.e., $A(g^*x^{-1}) = A(g^*y^{-1})$. Since $A^\theta = A$, $A^\theta(g^*x^{-1}) = A^\theta(g^*y^{-1})$. By Definition 3.3, $A(\theta(g^*x^{-1})) = A(\theta(g^*y^{-1}))$. Since θ is an automorphism of G, $A(\theta(g^*)\theta(x^{-1})) = A(\theta(g^*)\theta(y^{-1}))$. Thus $A(g\theta(x^{-1})) = A(g\theta(y^{-1}))$, i.e., $A\theta(x)(g) = A\theta(y)(g)$. So $\bar{\theta}(Ax) = \bar{\theta}(Ay)$. Hence $\bar{\theta}$ is well-defined. The proof of the fact that $\bar{\theta}$ is a homomorphism is analogous to the corresponding part of the proof of Proposition 4.4, and thus we omit the details. Now suppose $Ax \in Ker\bar{\theta}$ for each $x \in G$. Then $\bar{\theta}(Ax) = A\theta(x) = Ae$. Let $g \in G$. Then $A\theta(x)(\theta(g)) = Ae\theta(g)$, i.e., $A(\theta(g)\theta(x^{-1})) = A\theta(g)$. Thus $A\theta(gx^{-1}) = A\theta(g)$, i.e., $A^{\theta}(gx^{-1}) = A^{\theta}(g)$. Since $A^{\theta} = A$, $A(gx^{-1}) = A(g)$. Then $Ax(g) = Ae(g)$. Thus $Ax = Ae$, i.e., $Ker\bar{\theta} = \{Ae\}$. So $\bar{\theta}$ is injective. Hence $\bar{\theta}$ is an automorphism of G/A. This completes the proof. □

Theorem 4.7. Let A be an IVG of a finite group G and let $x, y \in G$. Then $G_A x = G_A y$ if and only if $Ax = Ay$.

Proof. By Result 3.B and Lemma 3.12, G_A is a subgroup of G and $G_A = \{x \in G : Ax = Ae\}$.

($\Rightarrow$): Suppose $G_A x = G_A y$ for any $x, y \in G$. Then $xy^{-1} \in G_A$. Thus $Axy^{-1} = Ae$. Let $g \in G$. Then $Axy^{-1}(g) = Ae(g)$, i.e., $A(gyx^{-1}) = A(g)$. Replacing g by gy^{-1}, which is also an arbitrary element of G, we get that $A(gx^{-1}) = A(gy^{-1})$ for each $y \in G$. Thus $Ax(g) = Ay(g)$ for each $y \in G$. So $Ax = Ay$.

($\Leftarrow$): Suppose $Ax = Ay$ for any $x, y \in G$ and let $g \in G$. Then $Ax(g) = Ay(g)$, i.e., $A(gx^{-1}) = A(gy^{-1})$. In particular, $A(yx^{-1}) = A(yy^{-1}) = A(e)$. Thus $yx^{-1} \in G_A$. So $G_A x = G_A y$. This completes the proof. □

Remark 4.8. Proposition 4.6 shows that there is a one-to-one correspondence between the (right) cosets of G_A in G and the interval-valued fuzzy cosets of A, given by the mapping $x \leftrightarrow Ax$ for each $x \in G$. Hence we see that the subgroup G_A plays a key role in the analysis of interval-valued fuzzy cosets.

References

[1] R. Biswas, "Rosenfeld's fuzzy subgroups with interval-valued membership functions," *Fuzzy set and systems*, vol. 63, pp. 87-90, 1995.

[2] M. S. Cheong and K. Hur, "Interval-valued fuzzy generalized bi-ideals of a semigroup," *International J.Fuzzy Logic and Intelligent Systems*, vol. 11, pp. 259-266, 2011.

[3] J. Y. Choi, S. K. Kim and K. Hur, "Interval-valued smooth topological spaces," *Honam Math.J.*, vol. 32, pp. 711-738, 2010.

[4] S. Y. Jang, K. Hur and P. K. Lim, "Interval-valued fuzzy normal subgroups," *International J.Fuzzy Logic and Intelligent Systems*, vol. 12, pp. 205-214, 2012.

[5] H. W. Kang, "Interval-valued fuzzy subgroups and homomorphisms," *Honam Math.J.*, vol.33, 2011.

[6] H. W. Kang and K. Hur, "Interval-valued fuzzy subgroups and subrings," *Honam Math.J.*, vol. 32, pp. 543-617, 2010.

[7] K. C. Lee, H. W. Kang and K. Hur, "Interval-valued fuzzy generalited bi-ideals of a semigroup," *Honam Math.J.*, vol. 33, 2011.

[8] T. K. Mondal and S. K. Samanta, "Topology of interval-valued fuzzy sets," *Indian J.Pure Appl.Math.*, vol. 30, pp. 20-38, 1999.

[9] L. A. Zadeh, "Fuzzy sets," *Inform. and Control*, vol. 8, 1965. 338-353.

[10] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning I," *Inform. Sci.*, vol. 8, pp. 199-249, 1975.

저 자 소 개

Keon Chang Lee
Professor in Dongshin University
His research interests are Category Theory, Hyperspace and Topology.
Phone : 061-330-3313
E-mail : kclee@dsu.ac.kr

Kul Hur
Professor in Wonkwang University
His research interests are Category Theory, Hyperspace and Topology.
Phone : 063-850-6081
E-mail : kulhur@wonkwang.ac.kr

Pyung Ki Lim
Professor in Wonkwang University
His research interests are Category Theory, Hyperspace and Topology.
Phone : 063-850-6559
E-mail : pklim@wonkwang.ac.kr