Journal of the Society of Korea Industrial and Systems Engineering
Vol. 35, No. 3, pp.204－209, September 2012.

단일품목의 목표 In-Stock Ratio 조건을 충족시키기 위한 재고문제 최적해 알고리듬

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An Optimal Solution Algorithm of the Single Product Inventory Problem with Target In-Stock Ratio Constraint

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본 논문은 전국적인 소매업체의 각 지점별 고객 수요가 불확실한 상황에서 고객 서비스 목표 수준을 충족하는 최적 재고 수준을 결정하는 문제에 대해 연구하였다. 이를 위해 전국에 분포한 지점에서 물품을 판매하는 베스트바이, 월마트, 혹은 시어스와 같은 전국적인 소매업체 관점에서 사용할 수 있는 핵심 관리 지표(KPI)로서 ISR(In-Stock Ratio)를 정의하였으며, 전국적인 소매업체가 평균 ISR로 정의되는 고객 서비스 목표 수준을 충족하면서 각 지점 보유 재고의 총합을 최소화할 수 있는 최적화 모델을 수립하였다. 본 논문은 해당 모델에 항상 최적해가 존재함을 증명하고 해당 최적해를 Karush-Kuhn-Tucker 조건을 사용하여 고객 수요의 확률분포의 형태에 상관없이 일반화된 형태로 표현하였다. 또한 본 논문은 고객 수요가 정규분포와 같은 특정 확률분포를 따르는 경우에 대해 연구하였으며, 이 경우에 대한 최적 재고 수준을 나타내는 식을 도출하였다. 마지막으로 본 논문에서는 상기 기술된 상황에 대한 수리적인 예제를 통하여 최적 재고 수준과 확률분포 파라미터들간의 관계를 분석하였다.

Keywords : Logistics Quality, Single Period Inventory Problem, Newsvendor Model, Uncertain Supply

1. Introduction and Background

In the recent era of economic downturn the market is unstable and unpredictable due to consumers showing unprecedented spending behavior. Mostly, they seem to tighten up their household expenditure. On the other hand, major retailer, such as Best Buy, Walmart, SEARS, have been encountering more sophisticated problem of matching customer demand with more cost-efficient manner. There exist numerous activities for retailers to achieve efficient operations or to gain more competitive edge against their competing retailers. Such activities include inventory reduction, price drop down, marketing budget cut-off, the enhanced collaboration with manufacturers via various collaboration schemes such as Collaborative Production Forecasting and Replenishment (CPFR), Vendor Managed Inventory (VMI).

논문접수일 : 2012년 08월 15일　　게재확정일 : 2012년 09월 12일
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The fundamental philosophy for collaboration among supply chain partners (i.e., retailers, suppliers) is to assume that the entire supply chain can create more profits when it is operated by just one centralized party. Even though there exist numerous research papers on these collaboration schemes the most simple and efficient method to understand supply chain dynamics under this collaboration is to review research papers in the area of supply chain contract coordination. A supply chain contract is said to achieve a coordination when the partners local optimal decisions lead to the systemwide optimal performance. Refer [4, 7], and [10] for a basic concepts of supply chain contracts. In [10], contracts are grouped into six categories : quantity flexibility, commitment, quantity discount, price only, buyback, and revenue sharing. For detailed explanation with relevant research papers see [12].

It will not be too odd to say that one of the most critical obstacles in achieving the maximum sales or the maximum market-share is to observe frequent product shortages in retail stores. In this paper we try to give answers on how to minimize product shortages or how to maximize the product availability in meeting the dynamic customer demand.

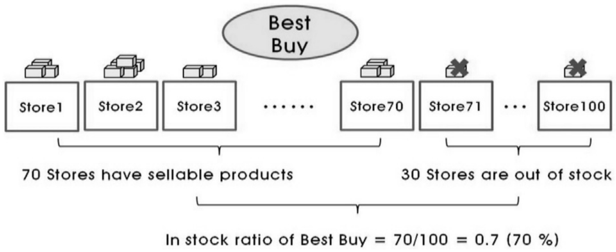
In the current market, customers seldom tolerate product shortages or backlogs. Their expectations for the product are persistently growing and growing. Furthermore customers are putting more and more emphasis on accompanying services even after their product purchases. These days there exist numerous websites available to customers that are providing customers with the comprehensive information on consumer products as well as potential companies that are selling those products. Companies that want to build good reputations from these evaluators should not ignore the frequent product shortages or the low product availability on their products in the market.

Previous research closely related to our research can be categorized into three groups. In the first group, after turning it into inventory problems they find the optimal level of inventory of the retailer who is facing either stochastic demand or stochastic supply or facing the both uncertainties. In these research, the optimal solution maximizes (or minimizes) the retailer's total profit (or total cost) through optimal trade-off between the shortage and the surplus. The most famous model in this kind is the newsvendor model. In newsvendor model, every morning, the owner of a corner newspaper stand determines the size of an order for newspapers to sell for that day. If he/she orders too many newspapers, some papers will be remaining unsold increasing excess inventory and they might have to be sold as scrap papers at the end of the day at lower than the original price. On the other hand, if the newsboy does not order enough newspapers, some potential customers will be disappointed resulting in loss of both the associated sales opportunity and the certain portion of the potential profit. The goal of the newsvendor is to find the most profitable (or optimal) quantity of newspapers that maximizes his expected (average) profit given that the demand distribution and cost parameters are known. For the detailed analysis on the newsvendor model readers can refer [8].

The second group deals with the shelf space optimization problem where they generate optimal location for each product so that the total sales amount can be maximized. If customers were completely brand-loyal, they will look for the specific item and buy it when the product is available and they would delay their purchase when the product is not available at the store. Thus, space allocated to a product would have no effect on its sales [2]. However, marketing research shows that most customer decisions are made at the point of purchase (see, e.g., [9]). In addition, [5] discovers that, except in relatively short time periods buyers of any particular brand therefore buy other brands more often than the brand itself. This indicates that the product choice of customers may be influenced by in-store factors including shelf space allocated to a product. With a well-designed shelf space management system, retailers can attract customers, prevent stockouts and, more importantly, increase the financial performance of the store while reducing operating costs [11].

The third group focuses on developing KPI (Key Performance Index) for accurately measuring the product availability for a product or a group of products over a certain period. Supply Chain Performance refers to the extended supply chains activities in meeting end-customer requirements, including product availability, on-time delivery, and all the necessary inventory and capacity in the supply chain to deliver that performance in a more responsive manner. In reality, it might not be possible to exactly define this KPI for the product availability without adding appropriate assumptions to simplify the complex real market dynamics. See [6] for comprehensive metric definitions over a given supply chain. He introduced service, assets and speed as three key dimensions for the supply chain matrics and suggested every supply chain should have at least one performance measure on each of these three key dimensions.

In our paper, as in the third group we first define a new product availability KPI, *ISR* (*In-Stock Ratio*) which can be defined as the percentage of stores having any sellable products available in stock. Suppose that Best Buy operates 100 stores nation-wide. As shown in <Figure 1>, if there are 100 stores and overall 70 stores have sellable products while 30 stores don't have any, then the *ISR* of Best Buy will be 70%.



<Figure 1> *ISR* : Illustrative Example

Our KPI will be appropriate to be used by nationwide retailers such as Best Buy, Walmart, and Sears that are selling products via their number of stores located across the entire nation. And then we introduce an optimization model where the objective is to minimize the inventory level at each individual stores of the retailers while the overall product availability is kept above the target level.

The rest of this paper is organized as following : In Section 2, we describe the mathematical model to maximize *ISR* with product quantity's constraint and the optimality conditions and simple example are shown in Section 3. Section 4 summarizes the contribution of this paper and discusses future research perspective.

2. Mathematical Model

In this study, we consider a nationwide retailer such as Best Buy, Walmart who is running N stores nationwide. We also assume that each store receives products only through the retailer's central warehouses at the first day of each week and each store i has random weekly demand D_i with $F_i(x)$ and $f_i(x)$ as cumulative distribution function and probability density function, respectively.

To make our problem more tractable we assume that the retailer's inventory management cycle is weekly based. This means that *ISR*, our performance measure for the retailer, will be computed only once at the end of each week for the simplification of our model. *ISR*, in this research, is the percentage of stores having any sellable product on their shelf at the end of each week.

Let $I_i(x_i)$ be the indicate variable taking value '1' only when there exists any product on the shelf in store i at the end of each week where x_i represents the initial inventory level, then

$$E[I_i(x_i)] = \int_0^{x_i} f_i(x)dx = F_i(x_i)$$

This means that $E(I_i)$ equals to the probability for the initial inventory x_i to cover the store i's weekly demand, that is demand fill-rate of the store i.

To describe the entire supply chain, let us define $X = (x_1, \cdots, x_N)$ be the initial inventory level vector at the beginning of each week, then we can define *ISR* as

$$ISR_X = \frac{\sum_{i=1}^{N} I_i(x_i)}{N}$$

and, the expected value of *ISR* is calculated as follows:

$$E(ISR_X) = \frac{\sum_{i=1}^{N} F_i(x_i)}{N} \tag{1}$$

Then the problem can be defined in two ways. One approach is to maximize *ISR* while the total inventory level cannot exceed certain target level and the other is to minimize each retailer's initial inventory level while meeting the desired *ISR* level. It's interesting to note that both problems exhibit the primal-dual relationship. In our paper, we focus on the latter approach since the most important mission of the central distribution centers is to effectively distribute products to the stores. Here, the optimal inventory level of each store will be the necessary input to accomplish the mission. In order words, if the optimal inventory level is $X^* = (x_1^*, \cdots, x_N^*)$ and the current inventory level is $X_{cur} = (x_1^{cur}, \cdots, x_N^{cur})$, then the central distribution centers need to deliver $x_i^* - x_i^{cur}$ units to retailer i and hold $\sum_{i=1}^{N}(x_i^* - x_i^{cur})$ units to meet the desired target *ISR* level.

This problem can be represented as the following model, minimizing the sum of the initial inventory level for the entire retail stores while the expected value of *ISR* is kept above the pre-determined target *ISR* level.

$$\text{Model 1 :} \quad \min \sum_{i=1}^{N} x_i$$

$$\text{s.t. } E[ISR_X] \geq \alpha$$

$$x_i \geq E[D_i] \qquad \text{for } i = 1, \cdots, N$$

The second constraint in this model is inserted since most retailers keep at least the expected weekly demands for initial inventory level in reality. This model is similar to the general knapsack problem which is an infamous problem to solve. Moreover, since the expected value of ISR is a nonlinear function of X in general, it is not easy to solve the problem explicitly. Therefore we propose the method to solve the problem in a relaxed environment in the next section.

3. Optimal Distribution Policy

As mentioned in Section 2, since Model 1 is an integer nonlinear programming problem, it is difficult to solve the problem directly. In order to change this problem into a more tractable one, we assume that $F_i(x)$ and $f_i(x)$ are continuous functions of x and x_i which can take real nonnegative values. If the product demand at each retailer is scarce, the integer constraint is very important. However, if the product demand at each retailer is relatively large (e.g., over 100), the integer constraint can be neglected.

Since Model 1 is a nonlinear programming problem, the optimality conditions can be obtained by the Karush-Kuhn-Tucker (KKT) condition as follows :

$$\lambda(\alpha - E(ISR_X)) = 0 \tag{2}$$

$$\left(1 - \frac{\lambda}{N} f_i(x_i)\right)(E(D_i) - x_i) = 0 \ \text{ for } \ \forall i \tag{3}$$

$$\alpha - E(ISR_X) \le 0 \tag{4}$$

$$E(D) - x_i \le 0$$

where λ is the Lagrangian multiplier for the expected value of ISR which means the change of product amount when α is changed.

Let us consider a simple case where the demand distribution of each store is independently and identically distributed. In this case, since $F(x) = F_i(x) = F_j(x)$ for $\forall i, j$, simply $E(ISR) = F(x)$ and $E(D_i) = E(D)$. Moreover, the objective function is a linear function and the feasible region established by the two constraints is a convex set since $F(x)$ is an increasing function. These information guarantee that any solution which satisfies the optimality condition is a global optimal solution of the problem.

By the optimality condition in equation (2), we get $\lambda = 0$ or $E(ISR_X) = \alpha$.

If $\lambda = 0$, then $x_i = E(D)$ for $\forall i$ by equation (3). Therefore, if $F(E(D)) \ge \alpha$, the optimal value is $x_i^* = E(D)$ for $\forall i$, and otherwise λ cannot be zero. This means that if $F(E(D)) \ge \alpha$, the optimal value is $x_i^* = E(D)$ for $\forall i$ and otherwise the optimal value is $x_i^* > E(D)$.

If $\lambda \ne 0$, then $E(ISR_X) = \alpha$. Since $E(ISR_X) = F(x)$,

$$x^* = F^{-1}(\alpha).$$

In addition, the corresponding λ^* is also calculated by equation (3) if $x_i^* \ne E(D)$,

$$\lambda^* = \frac{N}{f_i(x^*)}.$$

Summarizing the two cases discussed above, we conclude that

$$x^* = \max\{E(D), F^{-1}(\alpha)\}$$

Therefore, the minimum total inventory level is Nx^* and the central distribution centers need to prepare $Nx^* - \sum_{i=1}^{N} x_i^{cur}$ units of products for delivering to retailer stores. Moreover, if we change α into $\alpha + \delta$, then the optimal value is increased by $\delta\lambda^*$ if λ^* is not equal to zero.

In this study, we only consider the case where the weekly demand follows a normal distribution with mean μ_i and standard deviation σ_i. Therefore,

$$f_i(x) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}.$$

Any normal distribution function is a decreasing function in $[\mu_i, \infty]$, and this property makes sure that the feasible region of our problem is convex. Since the objective function is a linear function, we can guarantee that the solution driven by optimality conditions is the global optimum.

First of all, let $\alpha = 0.5$, then $x_i = \mu_i$ for $\forall i$ satisfies all optimality conditions. Therefore we can say that $x_i^* = \mu_i$ for $\forall i$ and the objective value is $\sum_{i=1}^{N} \mu_i$.

When $\alpha > 0.5$, λ^* has to be strictly positive since $x_i^* > \mu_i$, $\exists i$, and if $\lambda^* < \lambda_i$, $f_i(x) < N/\lambda^*$ for $\forall x \ge \mu_i$. Based on these facts, we can rewrite the optimality conditions in equation (2) and (3) as follows :

$$x_i^* = \begin{cases} \mu_i + \sigma_i \sqrt{2\ln\frac{\lambda^*}{\lambda_i}} & when\ \lambda^* > \lambda_i \\ \mu_i & when\ \lambda^* \le \lambda_i \end{cases} \quad (5)$$

$$\alpha - \sum_{i=1}^{N} \frac{F_i(x_i^*)}{N} = 0 \quad (6)$$

where $\lambda_i = N\sqrt{2\pi\sigma_i}$, x_i^* is an optimal initial inventory level for retailer store i and λ^* is the optimal Lagrangian multiplier for ISR constraints.

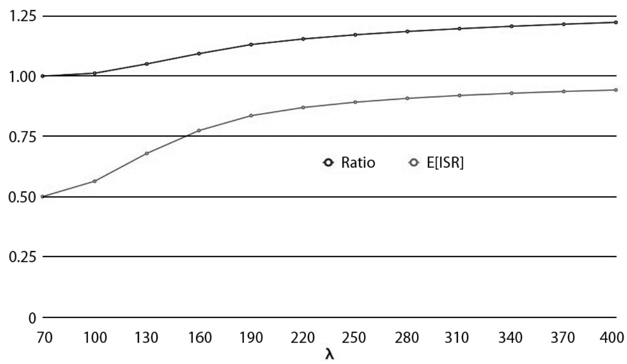
Though it is not easy to calculate x_i^* and λ^* from equation (5) and (6) explicitly, trial-and-error methods such as the bi-section algorithm can be used to derive the solution. For example, λ^* value turned out to converge pretty fast using the bi-section algorithm on a spreadsheet simulation as illustrated in the following example.

Example 3.1 : We consider an illustrative case with 10 retail stores in <Table 1>. The demand of each retail store is normally distributed with mean μ_i and standard deviation σ_i. In addition, μ_i and σ_i are generated in $U[50, 300]$ and $U[5, 50]$, respectively. For simple comparison, we set $\sum_{i=1}^{10}\mu_i = 1500$and define $ratio = \sum_{i=1}^{10}\frac{x_i}{1500}$.

<Table 1> Sample Data

store	μ_i	σ_i	λ_i
1	248	12	87
2	199	14	94
3	144	20	112
4	215	26	128
5	83	18	106
6	65	22	118
7	137	24	123
8	98	30	137
9	186	45	168
10	125	8	71

In order to show the relationship among λ, α, and $ratio$, we obtained the value of α and $ratio$ changing the value of λ shown in <Figure 2>. In <Figure 2>, the change of α is stable when $\lambda \in [\max_{i \le 10}\lambda_i, \infty)$ and if λ is less than the minimum value of λ_i, we can verify $\alpha = 0.5$ since all $x_i^* = \mu_i$.



<Figure 2> $D_i \sim N(\mu_i, \sigma_i)$

4. Conclusion

In this paper, we reviewed the nationwide retailer's problem of seeking the optimal inventory level for each retail store having stochastic demand while fulfilling target customer service level. For this purpose, we first defined a new metric which is called as ISR for the product availability. This new metric is appropriate to be used by nationwide retailers such as Best Buy, Walmart, and Sears that are selling products at their stores located across the entire nation. Then, using this new metric we introduced an optimization model where the objective is to minimize the inventory level at each individual store of the retailers while the average ISR, or target customer service level, is kept above the target level. From our study we showed that there always exist an optimal solution to this problem and the generic expression for the optimal inventory level has been derived for the general case which is independent of customer demand distribution. We also considered cases where the customer demand at each store follow a specific probability distribution such as the normal distribution. In this case, we derived a general expression for the optimal inventory level. Finally, we demonstrated a numerical example showing relationship among the optimal inventory level and distribution parameters. It will be an interesting future research topic to consider cases where the weekly demands follow some other probability distributions (other than the normal distribution discussed in this paper) as well as cases with multiple items.

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