HOMOLOGY 3-SPHERES OBTAINED BY SURGERY ON EVEN NET DIAGRAMS

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ABSTRACT. In this paper, we characterize surgery presentations for $\mathbb{Z}$-homology 3-spheres and $\mathbb{Z}/2\mathbb{Z}$-homology 3-spheres obtained from S^3 by Dehn surgery along a knot or link which admits an even net diagram and show that the Casson invariant for $\mathbb{Z}$-homology spheres and the μ-invariant for $\mathbb{Z}/2\mathbb{Z}$-homology spheres can be directly read from the net diagram. We also construct oriented 4-manifolds bounding such homology spheres and find their some properties.

1. Introduction

A framed link in S^3 of r components is a disjoint collection of r smoothly imbedded circles $K_1, \ldots, K_r$ in S^3 with rational numbers $\frac{p_i}{q_i}$ or $\infty = \frac{1}{0}$ associated with each imbedded circle K_i. In [8] and [15], Lickorish and Wallace showed that any orientable 3-manifold can be obtained from S^3 by Dehn surgery along a framed link with integral framings. Any presentation of a 3-manifold M by surgery on a framed link is called a surgery presentation of the 3-manifold M. It is well known that a closed oriented 3-manifold is an integral homology 3-sphere if and only if it can be obtained by Dehn surgery on an algebraically split link in S^3 for which all components are framed by ± 1 (cf. [4, 9]).

In 1952, Rokhlin introduced a theorem asserting that if M is a smooth closed oriented spin 4-manifold, then the signature of M is divisible by 16 [10]. This theorem is now known as the Rokhlin Theorem, which has played a significant role in the study of 4-dimensional topology and also gives rise the μ-invariant for $\mathbb{Z}/2\mathbb{Z}$-homology, for short, $\mathbb{Z}/2$-homology 3-spheres whose properties are related to the most fundamental questions of the manifold theory such as the triangulability of topological manifolds [2, 5].

In 1984-85, Casson introduced an integral valued invariant $\lambda_c : \mathcal{S} \to \mathbb{Z}$ from the set $\mathcal{S}$ of oriented $\mathbb{Z}$-homology 3-spheres to the set $\mathbb{Z}$ of integers, which reduces modulo 2 to the μ-invariant, by using representations from their fundamental groups into $SU(2)$ [1].

In this paper, we characterize surgery presentations for $\mathbb{Z}$-homology 3-spheres and $\mathbb{Z}/2$-homology 3-spheres obtained from S^3 by Dehn surgery along a knot or link which admits an even net diagram D and show that the Casson invariant for $\mathbb{Z}$-homology spheres and the μ-invariant for $\mathbb{Z}/2$-homology spheres can be directly calculated from the given net diagram D. We also construct oriented 4-manifolds bounding such homology spheres and give some properties.

This paper is organized as follows. In Section 2, we review some fundamental notions for surgery presentations for closed 3-manifolds. In Section 3, we characterize surgery presentations for $\mathbb{Z}$- and $\mathbb{Z}/2$-homology 3-spheres obtained by Dehn surgery along even net diagrams. We also construct 4-manifolds bounding the $\mathbb{Z}$- or $\mathbb{Z}/2$-homology spheres obtained from an even net diagram and give some properties of their intersection forms. In Section 4, we give the formulas for the Casson invariant for $\mathbb{Z}$-homology 3-spheres and the μ-invariant for $\mathbb{Z}/2$-homology 3-spheres obtained by Dehn surgery along even net diagrams, which show that the invariants can be directly calculated from the given net diagrams. In Section 5, we give an infinite family of $\mathbb{Z}$- and $\mathbb{Z}/2$-homology 3-spheres whose Casson invariants and the μ-invariants are vanishing, and are not vanishing.

2. Surgery presentation

Let Σ be an integral homology 3-sphere. A *framed link L* in Σ of r components is a disjoint collection $K_1 \cup \cdots \cup K_r$ of r smoothly imbedded circles, $K_1, \ldots, K_r$, in Σ with rational numbers $\frac{p_i}{q_i}$ or $\infty = \frac{1}{0}$, called the *framing*, associated with each imbedded circle K_i. Throughout this paper, we denote the oriented closed 3-manifold obtained from Σ by a Dehn surgery along a framed link L with a framing $(\frac{p_1}{q_1}, \ldots, \frac{p_r}{q_r})$ by $M^3(K_1 \cup \cdots \cup K_r; \frac{p_1}{q_1}, \ldots, \frac{p_r}{q_r}; \Sigma)$, or simply $M^3(L; \Sigma)$, otherwise specified. When $\Sigma = S^3$, the 3-sphere, we shall delete S^3 from the notations for short. If all the framings $\frac{p_i}{q_i}$ are integers or ∞, a Dehn surgery on Σ along a framed link L is called an *integral surgery*, otherwise it is called a *rational surgery*. Lickorish [8] and Wallace [15] showed that any closed orientable 3-manifold can be obtained by an integral surgery on S^3 along a framed link in S^3.

A framed link L in S^3 with integral framings determines an orientable 4-manifold $M^4(L)$ obtained by adding 2-handles to the 4-ball D^4 along the circles in L via the framings. Note that the resulting 4-manifold makes no difference how we orient the circles, and an orientation for $M^4(L)$, and so $\partial M^4(L)$, is determined by extending a fixed orientation on D^4 over $M^4(L)$.

Any presentation of a 3-manifold M by surgery on a framed link is called a *surgery presentation* of M. There are many surgery presentations for the same

manifold. Any two integral surgery presentations of the same manifold can be related by a finite sequence of Kirby moves [6]. Any two rational surgery presentations of the same manifold can be related by a finite sequence of generalized Kirby moves [3, 11, 12]. For our convenience, we review the Kirby moves on a framed link $L = K_1 \cup \cdots \cup K_r$ in S^3 which do not change the 3-manifold $M^3(L)$ obtained by Dehn surgery on S^3 along the framed link L:

Kirby Move K1. Add or delete an unknotted circle with framing 1 or -1, which belongs to an imbedded 3-ball D^3 in S^3 that does not intersect the other components of L, see Figure 1.

$$L \longleftrightarrow L' = L \; \bigsqcup \; \bigcirc^{\pm 1}$$

<div align="center">FIGURE 1</div>

Kirby Move K2. Add one component of the link L to another as follows. Let K_i and K_j be the two components of L with framings n_i and n_j, respectively, and let $\tilde{K}_j$ be a longitude defining the framing n_j of the component K_j, that is, $lk(K_j, \tilde{K}_j) = n_j$, where lk denotes the linking number. Now, replace $K_i \cup K_j$ with $K_i' \cup K_j$, where $K_i' = K_i \,\natural_b\, \tilde{K}_j$, the band connected sum of K_i and $\tilde{K}_j$, and b is any band missing the other components of L. The rest of the link L remains unchanged, see Figure 2.
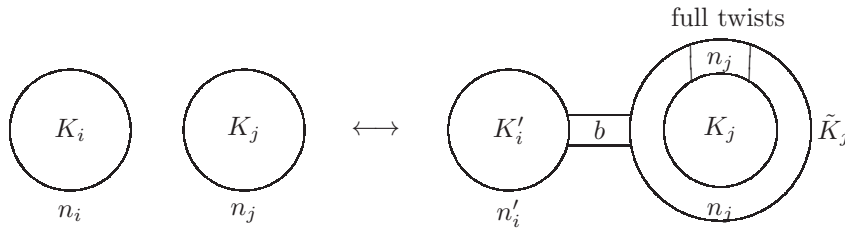

<div align="center">FIGURE 2</div>

The framings of all components but K_i' in L' are preserved, while the framing n_i' of the new component K_i' is given by the formula:

$$(2.1) \qquad n_i' = n_i + n_j + 2lk(K_i, K_j).$$

Note that the Kirby move K1 corresponds in the 4-manifold $M^4(L)$ to taking connected sum with or splitting off a copy of the complex projective plane $\mathbb{C}P^2$ with a canonical orientation or $\overline{\mathbb{C}P}^2$ with reversed orientation, depending on the framing $+1$ or -1. The Kirby move K2 corresponds in $M^4(L)$ to sliding the i-th 2-handle over the j-th 2-handle via the band b.

Two framed links L and L' in S^3 are said to be ∂-*equivalent* if the link L' can be obtained from L by a finite sequence of Kirby moves of types K1 and K2; denoted by $L \underset{\partial}{\sim} L'$. In [6], Kirby proved that for given two framed links L and L' in S^3, $L \underset{\partial}{\sim} L'$ if and only if $\partial M^4(L)$ is diffeomorphic to $\partial M^4(L')$ by an orientation preserving diffeomorphism.

On the other hand, Fenn and Rourke showed [3] that the Kirby moves K1 and K2 are equivalent to the *K-move* as shown in Figure 3.
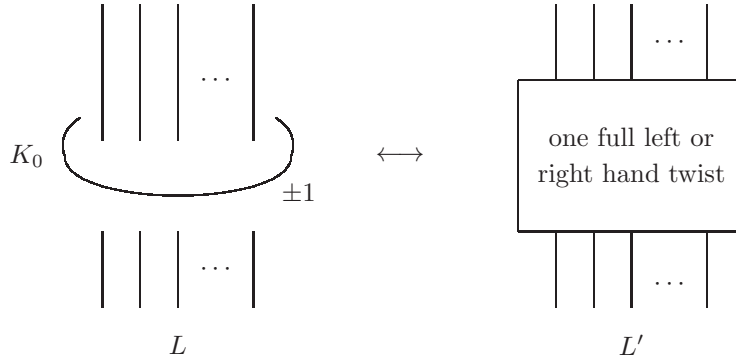

FIGURE 3

Here, if n_i is the framing of the component K_i in L, then the framing n'_i of the corresponding component K'_i in L' is given by the formula:

$$n'_i = n_i \mp lk(K_0, K_i)^2.$$

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented framed link in S^3 whose i-th component K_i is framed by $\frac{p_i}{q_i}$. Then the symmetric rational matrix $\Lambda(L) = (\ell_{ij})_{i,j=1,2,\ldots,r}$ with the entries

$$\ell_{ij} = \begin{cases} \frac{p_i}{q_i}, & \text{if } i = j; \\ lk(K_i, K_j), & \text{if } i \neq j, \end{cases}$$

is called the *linking matrix* for L. It is well known that the symmetric integral matrix $\overline{\Lambda}(L) = (q_i \ell_{ij})_{i,j=1,2,\ldots,r}$ is a presentation matrix for the homology $H_1(M^3(L); \mathbb{Z})$. We remark that the 3-manifold $M^3(L)$ is an integral homology 3-sphere if and only if the determinant of $\overline{\Lambda}(L)$ is equal to ± 1, namely, $\det(\overline{\Lambda}(L)) = \pm 1$ or, equivalently, $\det(\Lambda(L)) = \pm \frac{1}{q_1 q_2 \cdots q_r}$.

Let M be a compact oriented connected and simply connected 4-manifold with $\partial M \neq \emptyset$. The intersection number $a \cdot b$ of 2-cycles a and b induces a symmetric bilinear form

(2.2) $Q_M : H_2(M; \mathbb{Z}) \otimes H_2(M; \mathbb{Z}) \to \mathbb{Z},$

which is called the *intersection form* of M. Its signature, that is the number of positive eigenvalues minus the number of negative eigenvalues, is called the *signature* of M and is denoted by $\text{sign}(M)$. It is well known that the intersection

form Q_M in (2.2) is unimodular if and only if ∂M is an integral homology 3-sphere. Furthermore, let L be an integral framed link in S^3 and let $M = M^4(L)$ be the 4-manifold with boundary $\partial M^4(L) = M^3(L)$ obtained by Dehn surgery on S^3 along L. Then Q_M is isomorphic to the linking matrix $\Lambda(L)$ for L. Hence the 3-manifold $\partial M^4(L) = M^3(L)$ is an integral homology 3-sphere if and only if the determinant of $\Lambda(L)$ is equal to ± 1, namely, $\det(\Lambda(L)) = \pm 1$, and $\text{sign}(M^4(L)) = \text{sign}(\Lambda(L))$. For more details, see [13, 14].

3. Dehn surgery on even net diagrams

In [7], Seo and the author showed that any (oriented) link L in S^3 can be represented by a link diagram of the form as shown in Figure 4 or Figure 5. In the figures, each tangle labeled $a_j^i (1 \leq i \leq m, 1 \leq j \leq n)$ denotes a 2-tangle as shown in Figure 6. Given a link L in S^3, such a link diagram D in Figure 4 or Figure 5 representing L according as m is even or m is odd is called a *net presentation* of L or a *net diagram* of L, and is denoted by $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ in a matrix notation.
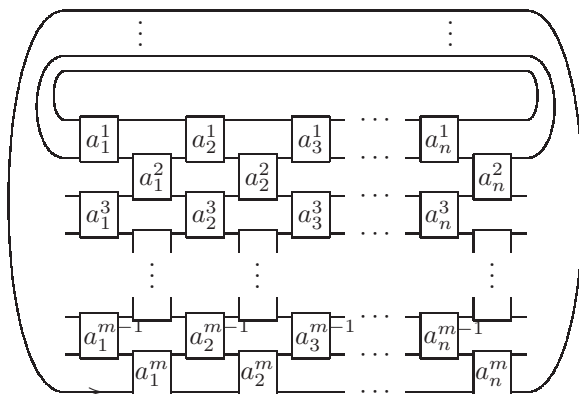

FIGURE 4. D; m is even

Definition 3.1. Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be a net diagram of a link in S^3. Then D is said to be even if all a_j^i's are even, and D is said to be odd if all a_j^i's are odd.

Remark 3.2. (1) Any link L in S^3 can always be represented by an odd net diagram $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ with $a_j^i = \pm 1$ [7]. It is worth mentioning that a link in S^3 may be represented by a net diagram which is of the even type.

(2) Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of a link L in S^3 with r components. Then it follows at once from the diagrams in Figures 4 and 5
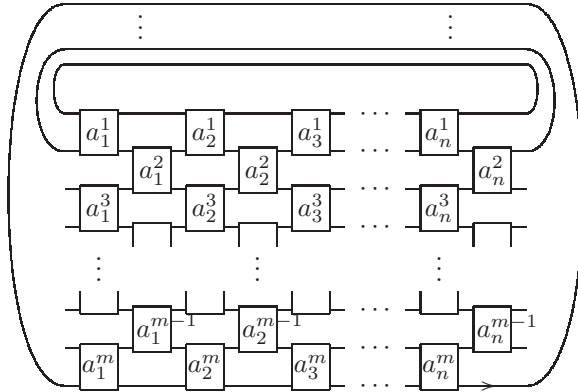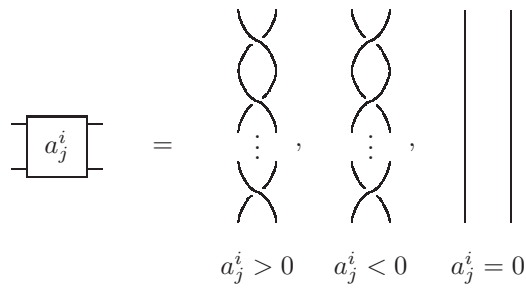
FIGURE 5. D; m is odd

$$a_j^i > 0 \qquad a_j^i < 0 \qquad a_j^i = 0$$

FIGURE 6

that m is an even integer if and only if $r = 1$, that is, D is a knot diagram, and that m is an odd integer if and only if $r = n$.

(3) Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an odd net diagram of a link L in S^3 with r components. Then it is not difficult to see from the diagrams in Figures 4 and 5 that $\gcd(m + 1, n) = r$.

Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n} = D_1 \cup \cdots \cup D_r (r \geq 1)$ be any oriented even net diagram of an integral framed link L in S^3 with framings $p_1, \ldots, p_r$. We define $\widehat{D}$ to be a framed link diagram obtained from D by replacing each 2-tangle labeled $a_j^i (\neq 0)$ with a new 2-tangle E_j^i as shown in Figure 7. The framing $\hat{p}_k$ on $\widehat{D}_k$ $(k = 1, \ldots, r)$ in $\widehat{D}$ corresponding to D_k in D is defined by the following

formula. For $k = 1, \ldots, r$,

$$(3.3) \qquad \hat{p}_k = p_k - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_j^i}{2} lk(\widehat{D}_i, \widehat{D}_1^{ij})^2.$$
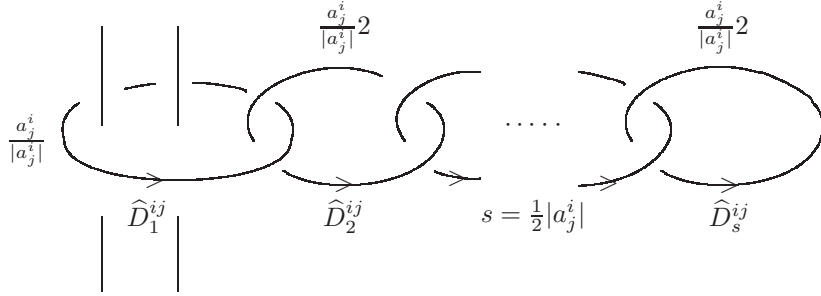

FIGURE 7. E_j^i

FIGURE 8

Theorem 3.3. *Let $L = K_1 \cup \cdots \cup K_r (r \geq 1)$ be a framed link in S^3 with integral framings $p_1, \ldots, p_r$ which has an even net diagram $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n} = D_1 \cup \cdots \cup D_r$. Let $\widehat{L}$ be the framed link in S^3 represented by $\widehat{D}$. Then $\widehat{L}$ is ∂-equivalent to L.*

Proof. We first observe that the K-move for two vertical strands as shown in Figure 8 produces the ∂-equivalences as shown in Figure 9 at the site of each tangle a_j^i. Using the Kirby move K2, we obtain the ∂-equivalence as illustrated in Figure 10 from which we can replace each tangle b_j^i in Figure 9 to the tangle E_j^i.

To compute the framings on $\widehat{D}$, we first choose an arbitrary orientation for D. Let $\overline{D}$ be the framed link diagram obtained from D by replacing each 2-tangle labeled $a_j^i \neq 0$ with new 2-tangle b_j^i in Figure 9, and let $\overline{D}_1, \ldots, \overline{D}_r$ be the components in $\overline{D}$ corresponding to $D_1, \ldots, D_r$ in D, respectively. Then the orientation for D induces an orientation for each $\overline{D}_k (1 \leq k \leq r)$ in $\overline{D}$ except new added unknotted circles $\overline{D}_u^{ij} (1 \leq u \leq s = \frac{|a_j^i|}{2})$ in b_j^i. Now we take the orientation for each component $\overline{D}_u^{ij}$ in $\overline{D}$ alternately starting $\overline{D}_1^{ij}$ as shown in

Figure 9. It follows that this orientation for $\overline{D}$ induces an orientation for $\widehat{D}$ which is the same on the components $\widehat{D}_u^{ij}(1 \leq u \leq s) \subset E_j^i$ in $\widehat{D}$ in Figure 7, after the transformations as illustrated in Figure 10.

Using Figure 9 and (2.1), it is not difficult to see that the framing $\bar{p}_k$ on $\overline{D}_k$ is given by the right hand side of the formula in (3.3). Since the linking numbers of all pairs of the unknotted circles $\overline{D}_u^{ij}(1 \leq u \leq s)$ in the tangle b_j^i are all zero, the transformation in Figure 10 gives the framing $\frac{a_j^i}{|a_j^i|}2$ on $\widehat{D}_u^{ij}$. Also, the transformation does not change the framing on $\overline{D}_k$ for each $k = 1, \ldots, r$ and hence $\hat{p}_k = \bar{p}_k$ for $k = 1, \ldots, r$. This shows that $L \underset{\partial}{\sim} \widehat{L}$, completing the proof. $\qquad \square$
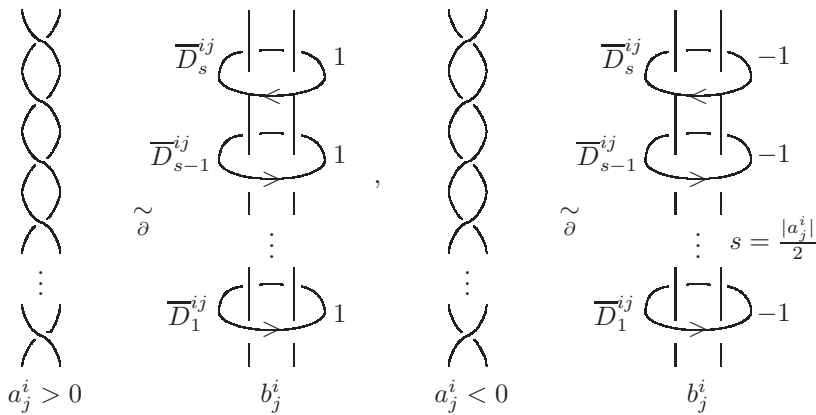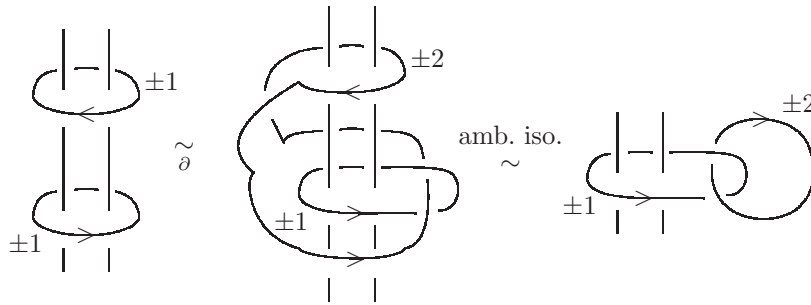

FIGURE 9

FIGURE 10

Definition 3.4. Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of an integral framed link in S^3. Then the framed link diagram $\widehat{D}$ is called the *blow-up* of D.

Let $D = (a_j^i)_{1 \le i \le m, 1 \le j \le n}$ be any oriented even net diagram. For the rest of the paper we refer the orientation for $\overline{D}$ and $\widehat{D}$ described in the proof of Theorem 3.3 the *induced orientation* by D. Also, let $\sigma_+(D)$ denote the sum of all positive a_j^i's and let $\sigma_-(D)$ denote the sum of all the absolute values $|a_j^i|$ with $a_j^i < 0$, and thus

$$\sigma_+(D) + \sigma_-(D) = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_j^i|.$$

Lemma 3.5. *Let $L = K_1 \cup \cdots \cup K_r (r \ge 1)$ be a framed link in S^3 with integral framings $p_1, \ldots, p_r$ which admits an even net diagram $D = (a_j^i)_{1 \le i \le m, 1 \le j \le n}$ and let $\widehat{L}$ be the framed link in S^3 represented by the blow-up $\widehat{D}$ of D. Let $\Lambda(\widehat{L})$ be the linking matrix for $\widehat{L}$.*

(1) *If m is even, then there exists a unimodular integral matrix U such that*

$$U\Lambda(\widehat{L})U^T = (p_1) \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \, I_{\frac{\sigma_-(D)}{2}}.$$

(2) *If m is odd, then there exists a unimodular integral matrix U such that*

$$(3.4) \qquad U\Lambda(\widehat{L})U^T = \begin{pmatrix} \hat{p}_1 & 0 & \cdots & 0 \\ 0 & \hat{p}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{p}_n \end{pmatrix} \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \, I_{\frac{\sigma_-(D)}{2}},$$

where $I_{\frac{\sigma_\pm(D)}{2}}$ denotes the $(\frac{\sigma_\pm(D)}{2} \times \frac{\sigma_\pm(D)}{2})$ identity matrix, U^T denotes the transpose matrix of U, and

$$(3.5) \qquad \hat{p}_k = p_k - \frac{1}{2} \sum_{\ell=1}^{\frac{m+1}{2}} \left(a_k^{2\ell-1} + a_{k+1}^{2\ell-1} \right), k = 1, \ldots, n.$$

Proof. First, if m is even, then it is obvious that D is a knot diagram with framing p_1. Choose an orientation for D as shown in Figure 4. Then it is obvious from Figure 9 that $\overline{D}$ is an algebraically split link with $d = (\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|a_j^i|}{2}) + 1$ components and so it is straightforward that $\Lambda(\overline{D})$ is the $d \times d$ diagonal matrix whose diagonal entries are the framings of the components. For each pair (i, j), let I_{ij} denote the $(\frac{|a_j^i|}{2} \times \frac{|a_j^i|}{2})$ identity matrix. Then for some unimodular integral matrices U_1 and U_2, we have

$$(3.6) \qquad \begin{aligned} U_2\Lambda(\overline{D})U_2^T &= (p_1) \oplus \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \frac{a_j^i}{|a_j^i|} I_{ij} \right), \\ U_1 U_2 \Lambda(\overline{D}) U_2^T U_1^T &= (p_1) \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \, I_{\frac{\sigma_-(D)}{2}}. \end{aligned}$$

Next, suppose that m is odd. Then we know that $r = n$, that is, D is a framed link diagram with exactly n components $D_1, \ldots, D_n$ with framings $p_1, \ldots, p_n$. We choose an orientation for D as shown in Figure 11. Let
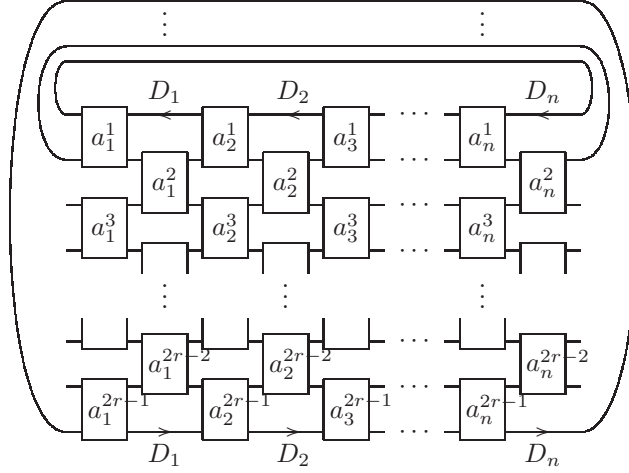

FIGURE 11

$\overline{D}_1, \ldots, \overline{D}_n$ be the components of $\overline{D}$ with the induced orientation by D, corresponding to $D_1, \ldots, D_n$ in D, and let $\overline{D}_1^{ij}, \ldots, \overline{D}_{s_{ij}}^{ij}$ $(s_{ij} = \frac{|a_j^i|}{2})$ be the components of $b_j^i \subset \overline{D}$ as shown in Figure 9. We observe that for each pair (i, j) and $u = 1, \ldots, s_{ij}$,

$$lk(\overline{D}_u^{ij}, D_k) = \begin{cases} (-1)^{u+1}, & \text{if } i \text{ is odd and } j = k; \\ (-1)^u, & \text{if } i \text{ is odd and } j = k + 1; \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$lk(\overline{D}_u^{ij}, D_k)^2 = \begin{cases} 1, & \text{if } i \text{ is odd and } j = k, k + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{p}_1, \ldots, \bar{p}_n$ denote the framings of $\overline{D}_1, \ldots, \overline{D}_n$, respectively. By (2.1), we obtain that for each $k = 1, \ldots, n$,

$$\bar{p}_k = p_k - \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{u=1}^{s_{ij}} \frac{a_j^i}{|a_j^i|} lk(\overline{D}_u^{ij}, D_k)^2$$

$$= p_k - \sum_{\ell=1}^{\frac{m+1}{2}} \left(\sum_{u=1}^{s_{(2\ell-1)k}} \frac{a_k^{2\ell-1}}{|a_k^{2\ell-1}|} + \sum_{u=1}^{s_{(2\ell-1)(k+1)}} \frac{a_{k+1}^{2\ell-1}}{|a_{k+1}^{2\ell-1}|} \right)$$

$$= p_k - \sum_{\ell=1}^{\frac{m+1}{2}} \frac{a_k^{2\ell-1} + a_{k+1}^{2\ell-1}}{2}.$$

Since the transformation in Figure 10 does not change the framing on $\overline{D}_k$ for all $k = 1, \ldots, n$, we get $\hat{p}_k = \bar{p}_k$ and hence

$$\hat{p}_k = p_k - \sum_{\ell=1}^{\frac{m+1}{2}} \frac{a_k^{2\ell-1} + a_{k+1}^{2\ell-1}}{2}, \ k = 1, \ldots, n.$$

Now, for two pairs (i, j) and (k, l) with $1 \le i, k \le m, 1 \le j, l \le n$, let

$$\Lambda_{kl}^{ij} = (\gamma_{uv})_{1 \le u \le \frac{|a_j^i|}{2}, 1 \le v \le \frac{|a_l^k|}{2}}$$

be the $(\frac{|a_j^i|}{2} \times \frac{|a_l^k|}{2})$ integral matrix defined by

$$\gamma_{uv} = \begin{cases} \frac{a_j^i}{|a_j^i|}, & \text{if } (i, j) = (k, l) \text{ and } u = v; \\ lk(\overline{D}_u^{ij}, \overline{D}_v^{kl}), & \text{otherwise.} \end{cases}$$

Observe that if either $(i, j) \ne (k, l)$ or $u \ne v$, then $lk(\overline{D}_u^{ij}, \overline{D}_v^{kl}) = 0$. This gives that for all pairs (i, j) and (k, l) with $(i, j) \ne (k, l)$, the matrix Λ_{kl}^{ij} is the $(\frac{|a_j^i|}{2} \times \frac{|a_l^k|}{2})$ zero matrix, and for $(i, j) = (k, l)$, $\Lambda_{kl}^{ij} = \Lambda_{ij}^{ij}$ is the $\frac{|a_j^i|}{2} \times \frac{|a_j^i|}{2}$ diagonal matrix given by

$$\Lambda_{ij}^{ij} = \begin{pmatrix} \frac{a_j^i}{|a_j^i|} & 0 & \cdots & 0 \\ 0 & \frac{a_j^i}{|a_j^i|} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{a_j^i}{|a_j^i|} \end{pmatrix} = \frac{a_j^i}{|a_j^i|} I_{ij}.$$

Let

$$\Lambda_0 = \begin{pmatrix} \bar{p}_1 & 0 & \cdots & 0 \\ 0 & \bar{p}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{p}_n \end{pmatrix}.$$

For each pair (i, j) with $1 \le i \le m, 1 \le j \le n$, let

$$\Lambda_j^i = (\gamma_{uv})_{1 \le u \le n, 1 \le v \le \frac{|a_j^i|}{2}}$$

be the $(n \times \frac{|a_j^i|}{2})$ matrix defined by

$$\gamma_{uv} = lk(\overline{D}_u, \overline{D}_v^{ij}).$$

Then it is straightforward from Figure 9 that for each $j = 1, \ldots, n$ and $\ell = 1, \ldots, \frac{m-1}{2}$, $\Lambda_j^{2\ell}$ is the zero matrix. For $k = 1, \ldots, \frac{m+1}{2}$, we have

$$
\Lambda_1^{2k-1} = \begin{pmatrix}
\frac{a_1^{2k-1}}{|a_1^{2k-1}|} & -\frac{a_1^{2k-1}}{|a_1^{2k-1}|} & \frac{a_1^{2k-1}}{|a_1^{2k-1}|} & \cdots & (-1)^{s_{2k-1}1}\frac{a_1^{2k-1}}{|a_1^{2k-1}|} \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
-\frac{a_1^{2k-1}}{|a_1^{2k-1}|} & \frac{a_1^{2k-1}}{|a_1^{2k-1}|} & -\frac{a_1^{2k-1}}{|a_1^{2k-1}|} & \cdots & (-1)^{s_{2k-1}1+1}\frac{a_1^{2k-1}}{|a_1^{2k-1}|}
\end{pmatrix},
$$

$$
\Lambda_2^{2k-1} = \begin{pmatrix}
\frac{a_2^{2k-1}}{|a_2^{2k-1}|} & -\frac{a_2^{2k-1}}{|a_2^{2k-1}|} & \frac{a_2^{2k-1}}{|a_2^{2k-1}|} & \cdots & (-1)^{s^{2k-1}2}\frac{a_2^{2k-1}}{|a_2^{2k-1}|} \\
-\frac{a_2^{2k-1}}{|a_2^{2k-1}|} & \frac{a_2^{2k-1}}{|a_2^{2k-1}|} & -\frac{a_2^{2k-1}}{|a_2^{2k-1}|} & \cdots & (-1)^{s^{2k-1}2+1}\frac{a_2^{2k-1}}{|a_2^{2k-1}|} \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

$$
\Lambda_3^{2k-1} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\frac{a_3^{2k-1}}{|a_3^{2k-1}|} & -\frac{a_3^{2k-1}}{|a_3^{2k-1}|} & \frac{a_3^{2k-1}}{|a_3^{2k-1}|} & \cdots & (-1)^{s^{2k-1}3}\frac{a_3^{2k-1}}{|a_3^{2k-1}|} \\
-\frac{a_3^{2k-1}}{|a_3^{2k-1}|} & \frac{a_3^{2k-1}}{|a_3^{2k-1}|} & -\frac{a_3^{2k-1}}{|a_3^{2k-1}|} & \cdots & (-1)^{s^{2k-1}3+1}\frac{a_3^{2k-1}}{|a_3^{2k-1}|} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

$$\vdots$$

$$
\Lambda_n^{2k-1} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{a_n^{2k-1}}{|a_n^{2k-1}|} & -\frac{a_n^{2k-1}}{|a_n^{2k-1}|} & \frac{a_n^{k-1}}{|a_n^{2k-1}|} & \cdots & (-1)^{s^{2k-1}n}\frac{a_n^{2k-1}}{|a_n^{2k-1}|} \\
-\frac{a_n^{2k-1}}{|a_n^{2k-1}|} & \frac{a_n^{2k-1}}{|a_n^{2k-1}|} & -\frac{a_n^{2k-1}}{|a_n^{2k-1}|} & \cdots & (-1)^{s^{2k-1}n+1}\frac{a_n^{2k-1}}{|a_n^{2k-1}|}
\end{pmatrix}.
$$

Then the linking matrix $\Lambda(\overline{D})$ of $\overline{D}$ with respect to the order for the pairs

$$(1,1), \ldots, (1,n), (2,1), \ldots, (2,n), \ldots, (m,1), \ldots, (m,n)$$

is given by the $(mn + 1) \times (mn + 1)$ block matrix:

$$\Lambda(\overline{D}) = \begin{pmatrix} \Lambda_0 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 & \cdots & \Lambda_n^m \\ (\Lambda_1^1)^T & \Lambda_{11}^{11} & O & O & \cdots & O \\ (\Lambda_2^1)^T & O & \Lambda_{12}^{12} & O & \cdots & O \\ (\Lambda_3^1)^T & O & O & \Lambda_{13}^{13} & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\Lambda_n^m)^T & O & O & O & \cdots & \Lambda_{mn}^{mn} \end{pmatrix}.$$

It is not difficult to see that for unimodular integral matrices U_1 and U_2,

$$(3.7) \quad U_2 \Lambda(\overline{D}) U_2^T = \Lambda_0 \oplus \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \Lambda_{ij}^{ij} \right) = \Lambda_0 \oplus \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \frac{a_j^i}{|a_j^i|} I_{ij} \right),$$

$$U_1 U_2 \Lambda(\overline{D}) U_2^T U_1^T = \Lambda_0 \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \, I_{\frac{\sigma_-(D)}{2}}.$$

On the other hand, the effect of the Kirby move K1 replaces the linking matrix A to $A \oplus (\pm 1)$, and the Kirby move K2 that slides K_i over K_j replaces the linking matrix A to A' obtained from A by adding (or subtracting) the j-th row to (from) the i-th row and the j-th column to (from) the i-th column. This facts, together with (3.6) and (3.7), implies the desired assertions (1) and (2). This completes the proof of Lemma 3.5. $\qquad \square$

For a given integer p, we define

$$\epsilon(p) = \begin{cases} \frac{p}{|p|}, & \text{if } p \neq 0; \\ 0, & \text{if } p = 0. \end{cases}$$

Theorem 3.6. *Let $L = K_1 \cup \cdots \cup K_r (r \geq 1)$ be a framed link in S^3 with integer framings $p_1, \ldots, p_r$ which admits an even net diagram $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ and let $\widehat{L}$ be the framed link in S^3 represented by the blow-up $\widehat{D}$ of D. Let $\Lambda(L)$ and $\Lambda(\widehat{L})$ be the linking matrices for L and $\widehat{L}$, respectively.*

(1) *If m is even, then $r = 1$ and*

$$|\det(\Lambda(L))| = |\det(\Lambda(\widehat{L}))| = |p_1|,$$

$$\text{sign}(\Lambda(\widehat{L})) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_j^i}{2} + \epsilon(p_1).$$

(2) *If m is odd, then $r = n$ and*

$$|\det(\Lambda(L))| = |\det(\Lambda(\widehat{L}))| = \left| \prod_{k=1}^{n} \hat{p}_k \right|,$$

$$\text{sign}(\Lambda(\widehat{L})) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_j^i}{2} + \sum_{k=1}^{n} \epsilon(\hat{p}_k),$$

where $\hat{p}_1, \ldots, \hat{p}_n$ are the integers in (3.5).

Proof. (1) Suppose that m is even. By Lemma 3.5(1), we obtain that $r = 1$ and

$$|\det(\Lambda(\widehat{L}))| = |p_1|,$$

$$\text{sign}(\Lambda(\widehat{L})) = \epsilon(p_1) + \frac{\sigma_+(D) - \sigma_-(D)}{2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_j^i}{2} + \epsilon(p_1).$$

(2) Suppose that m is odd. By Lemma 3.5(2), we have that $r = n$ and

$$|\det(\Lambda(\widehat{L}))| = |\prod_{k=1}^{n} \hat{p}_k|,$$

$$\text{sign}(\Lambda(\widehat{L})) = \sum_{k=1}^{n} \epsilon(\hat{p}_k) + \frac{\sigma_+(D) - \sigma_-(D)}{2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_j^i}{2} + \sum_{k=1}^{n} \epsilon(\hat{p}_k).$$

Finally, it follows from Theorem 3.3 that L and $\widehat{L}$ are ∂-equivalent and hence $|\det(\Lambda(L))| = |\det(\Lambda(\widehat{L}))|$. This completes the proof. □

Theorem 3.7. *Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of a link L in S^3 of components $K_1, \ldots, K_r$ with integer framings $p_1, \ldots, p_r$. Then $M^3(L)$ is a $\mathbb{Z}$-homology 3-sphere if and only if either $r = 1$ and $p_1 = \pm 1$ or $r = n$ and*

$$(3.8) \qquad \prod_{k=1}^{n} \left(p_k - \frac{1}{2} \sum_{\ell=1}^{\frac{m+1}{2}} (a_k^{2\ell-1} + a_{k+1}^{2\ell-1}) \right) = \pm 1.$$

Proof. Let $D = (a_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of a link L in S^3 of components $K_1, \ldots, K_r$ with integer framings $p_1, \ldots, p_r$. and let $\Lambda(L)$ and $\Lambda(\widehat{L})$ be the linking matrices for L and $\widehat{L}$, respectively.

Suppose that $M^3(L)$ is a $\mathbb{Z}$-homology 3-sphere. Then $\det(\Lambda(L)) = \pm 1$. Note that m is either even or odd. If m is even, then it follows from Theorem 3.6 (1) that $r = 1$ and $\Lambda(L) = (p_1)$. This gives $p_1 = \pm 1$. Now if m is odd, then it follows from Theorem 3.6(2) that $r = n$ and $|\det(\Lambda(L))| = |\prod_{k=1}^{n} \hat{p}_k| = \pm 1$. This gives the equality (3.8).

Conversely, we first suppose that m is even and $p_1 = \pm 1$. By Lemma 3.5, the matrix

$$\Lambda(\widehat{L}) = (\pm 1) \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \ I_{\frac{\sigma_-(D)}{2}}$$

is a presentation matrix for $H_1(\partial M^4(\widehat{L}); \mathbb{Z})$. This implies that $H_1(\partial M^4(\widehat{L}); \mathbb{Z}) = 0$. By Theorem 3.3, L is ∂-equivalent to $\widehat{L}$, $\partial M^4(\widehat{L})$ and $\partial M^4(L)$ are diffeomorphic, and hence $H_1(M^3(L); \mathbb{Z}) = H_1(\partial M^4(L); \mathbb{Z}) = H_1(\partial M^4(\widehat{L}); \mathbb{Z}) = 0$.

Now we suppose that m is odd and the equality (3.8) holds. Then for $k = 1, \ldots, n$, we have

$$\hat{p}_k = p_k - \frac{1}{2} \sum_{\ell=1}^{\frac{m+1}{2}} (a_k^{2\ell-1} + a_{k+1}^{2\ell-1}) = \pm 1.$$

By Lemma 3.5, the matrix

$$\Lambda(\widehat{L}) = (\pm 1)I_n \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \ I_{\frac{\sigma_-(D)}{2}}$$

is a presentation matrix for $H_1(\partial M^4(\widehat{L}); \mathbb{Z})$. This implies that $H_1(\partial M^4(\widehat{L}); \mathbb{Z}) = 0$. By a similar argument as above, we obtain $H_1(M^3(L); \mathbb{Z}) = 0$. This completes the proof. $\qquad\square$

Theorem 3.8. *Let $D = (a_j^i)_{1 \le i \le m, 1 \le j \le n}$ be an even net diagram of a link L in S^3 of components $K_1, \ldots, K_r$ with integer framings $p_1, \ldots, p_r$. Then $M^3(L)$ is a $\mathbb{Z}/2$-homology 3-sphere if and only if either $r = 1$ and p_1 is odd or $r = n$ and*

$$(3.9) \qquad \prod_{k=1}^{n} \left(p_k - \frac{1}{2} \sum_{\ell=1}^{\frac{m+1}{2}} (a_k^{2\ell-1} + a_{k+1}^{2\ell-1}) \right)$$

is an odd integer.

Proof. Let $D = (a_j^i)_{1 \le i \le m, 1 \le j \le n}$ be an even net diagram of a link L in S^3 of components $K_1, \ldots, K_r$ with integer framings $p_1, \ldots, p_r$ and let $\Lambda(L)$ and $\Lambda(\widehat{L})$ be the linking matrices for L and $\widehat{L}$, respectively.

Suppose that $M^3(L)$ is a $\mathbb{Z}/2$-homology 3-sphere. Then $\det(\Lambda(L))$ must be an odd integer. Note that m is either even or odd. If m is even, then it follows from Theorem 3.6(1) that $r = 1$ and $\Lambda(L) = (p_1)$. This gives p_1 is odd. Now if m is odd, then it follows from Theorem 3.6(2) that $r = n$ and $|\det(\Lambda(L))| = |\prod_{k=1}^{n} \hat{p}_k|$ is odd.

Conversely, we first suppose that m is even and p_1 is odd. By Lemma 3.5, the matrix

$$\Lambda(\widehat{L}) = (p_1) \oplus I_{\frac{\sigma_+(D)}{2}} \oplus (-1) \ I_{\frac{\sigma_-(D)}{2}}$$

is a presentation matrix for $H_1(\partial M^4(\widehat{L}); \mathbb{Z})$. This implies that $H_1(\partial M^4(\widehat{L}); \mathbb{Z}/2)$ vanishes. By Theorem 3.3, L is ∂-equivalent to $\widehat{L}$, $\partial M^4(\widehat{L})$ and $\partial M^4(L)$ are diffeomorphic, and hence

$$H_1(M^3(L); \mathbb{Z}/2) = H_1(\partial M^4(L); \mathbb{Z}/2) = H_1(\partial M^4(\widehat{L}); \mathbb{Z}/2) = 0.$$

Now we suppose that m is odd and the integer in (3.9) is odd. Then for $k = 1, \ldots, n$, the integer

$$\hat{p}_k = p_k - \frac{1}{2} \sum_{\ell=1}^{\frac{m+1}{2}} (a_k^{2\ell-1} + a_{k+1}^{2\ell-1})$$

must be odd. The matrix (3.4) in Lemma 3.5 is a presentation matrix for $H_1(\partial M^4(\widehat{L}); \mathbb{Z})$. Hence $H_1(\partial M^4(\widehat{L}); \mathbb{Z}/2) = 0$, and thus $H_1(M^3(L); \mathbb{Z}/2) = 0$, completing the proof. $\qquad\square$

Corollary 3.9. *Let $L = K_1 \cup \cdots \cup K_r (r \geq 1)$ be a framed link in S^3 with integer framings $p_1, \ldots, p_r$ which admits an even net diagram $D = (a^i_j)_{1 \leq i \leq m, 1 \leq j \leq n}$ and let $\widehat{L}$ be the framed link in S^3 represented by the blow-up $\widehat{D}$ of D. Then*

$$
\text{sign}(M^4(\widehat{L}))) = \begin{cases} \displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a^i_j}{2} + \epsilon(p_1), & \text{if } m \text{ is even;} \\ \displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a^i_j}{2} + \sum_{k=1}^{n} \epsilon(\hat{p}_k), & \text{if } m \text{ is odd,} \end{cases}
$$

where $\hat{p}_1, \ldots, \hat{p}_r$ are the integers in (3.5).

In particular, if $M^3(L)$ is a $\mathbb{Z}$-homology or a $\mathbb{Z}/2$-homology 3-sphere, then the rank of the intersection form $Q_{\widehat{L}} = Q_{M^4(\widehat{L})}$ of $M^4(\widehat{L})$ is given by

$$
\text{rank } Q_{\widehat{L}} = r + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|a^i_j|}{2}.
$$

Proof. By Theorems 3.6, 3.7 and 3.8, the assertion follows immediately. □

4. The μ-invariant and the Casson invariant

Let M be an oriented $\mathbb{Z}/2$-homology 3-sphere. Then it has a unique spin structure. Choose a smooth compact oriented spin 4-manifold X such that $\partial X = M$. The μ-*invariant*, $\mu(M)$, of M is a rational residue modulo 2 defined by

$$
\mu(M) = \frac{\text{sign}(X)}{8} \pmod 2.
$$

We note that if M is a $\mathbb{Z}$-homology sphere, then $\text{sign}(X)$ is divisible by 8 so that $\mu(M) = 0$ or 1 modulo 2. For more details, see [13, 14]. For a given a knot K in S^3 and a reduced fraction $\frac{p}{q}$, it is well known that the manifold $M^3(K; \frac{p}{q})$ is a $\mathbb{Z}/2$-homology 3-sphere if and only if p is odd. Let $\Delta_K(t) = \Delta_{K;\Sigma}(t)$ be the Alexander polynomial of K normalized so that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$. Then it follows from [14, Theorem 2.13] that for every odd p,

$$
(4.10) \qquad \mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{q}{2p} \Delta''_K(1) \pmod 2,
$$

where $L(p, q)$ is the lens space obtained by $\frac{-p}{q}$-surgery on an unknot in S^3 and $\Delta''_K(1)$ is the second derivative of $\Delta_K(t)$ evaluated at $t = 1$. For the μ-invariant $\mu(L(p, q))$ of the lens space $L(p, q)$, we refer to [14, Corollary 2.24].

Theorem 4.1. *Let $D = (a^i_j)_{1 \leq i \leq 2r, 1 \leq j \leq n}$ be an even net diagram of a knot K in S^3. For any odd p,*

$$
\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{q}{4p} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a^{2\ell}_j \left(a^{2k-1}_j + a^{2k-1}_{j+1} \right) \pmod 2
$$

where $a_{n+1}^{2k-1} = a_1^{2k-1}$ *for each* $k = 1, 2, \ldots, \ell$. *In particular,*

$$(4.11) \qquad \mu(M^3(K; \frac{1}{q})) = \frac{q}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a_j^{2\ell} \left(a_j^{2k-1} + a_{j+1}^{2k-1} \right) \quad (\text{mod } 2).$$

Proof. Since p is odd, $M^3(K; \frac{p}{q})$ is a $\mathbb{Z}/2$-homology 3-sphere. Hence we obtain from (4.10) that

$$
\begin{aligned}
&\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{q}{2p} \Delta''_K(1) \quad (\text{mod } 2), \\
(4.12) \quad & \\
&\mu(M^3(K; \frac{1}{q})) = \frac{q}{2} \Delta''_K(1) \quad (\text{mod } 2)
\end{aligned}
$$

because $\mu(L(1, q)) = 0$. But it follows from [7, Theorem 3.5] that

$$(4.13) \qquad \Delta''_K(1) = \frac{1}{2} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a_j^{2\ell} \left(a_j^{2k-1} + a_{j+1}^{2k-1} \right),$$

where $a_{n+1}^{2k-1} = a_1^{2k-1}$, $k = 1, 2, \ldots, \ell$. Substituting (4.13) to (4.12), we obtain the desired formula. $\qquad \square$

Let $\mathcal{S}$ be the class of oriented $\mathbb{Z}$-homology 3-spheres modulo orientation preserving homeomorphism. A *Casson invariant* [1] is a map $\lambda_c : \mathcal{S} \longrightarrow \mathbb{Z}$ such that

(1) $\lambda_c(S^3) = 0$,
(2) For any $\mathbb{Z}$-homology 3-sphere Σ, a knot $K \subset \Sigma$, and an integer $m = 0, \pm 1, \pm 2, \ldots$,

$$\lambda_c \left(M^3(K; \frac{1}{m+1}; \Sigma) \right) - \lambda_c \left(M^3(K; \frac{1}{m}; \Sigma) \right)$$

is independent of m.
(3) For any boundary link $K \cup K'$ in a $\mathbb{Z}$-homology 3-sphere Σ, that is, K and K' bound disjoint Seifert surfaces in Σ, and for any two integers m and n,

$$\lambda_c \left(M^3(K \cup K'; \frac{1}{m+1}, \frac{1}{n+1}; \Sigma) \right) - \lambda_c \left(M^3((K \cup K'; \frac{1}{m}, \frac{1}{n+1}; \Sigma) \right)$$
$$- \lambda_c \left(M^3((K \cup K'; \frac{1}{m+1}, \frac{1}{n}; \Sigma) \right) + \lambda_c \left(M^3(K \cup K'; \frac{1}{m}, \frac{1}{n}; \Sigma) \right) = 0.$$

Note that for any integral homology sphere Σ, it is well known [1, 13, 14] that

$$(4.14) \qquad\qquad\qquad \lambda_c(\Sigma) = \mu(\Sigma) \quad (\text{mod } 2).$$

In fact, λ_c is unique up to sign ± 1. Normalizing λ_c by setting $\lambda_c(\text{trefoil}) = +1$, Seo and the author gave the following Theorems 4.2 and 4.3 in [7]:

Theorem 4.2. *Let* $D = (a^i_j)_{1 \le i \le 2r, 1 \le j \le n}$ *be an even net diagram* $(r \ge 1, n \ge 2)$ *of a knot* K *in* S^3. *Then*

$$(4.15) \qquad \lambda_c(M^3(K; \frac{1}{q})) = \frac{q}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a^{2\ell}_j \left(a^{2k-1}_j + a^{2k-1}_{j+1} \right),$$

where $a^{2k-1}_{n+1} = a^{2k-1}_1$ *for each* $k = 1, 2, \ldots, \ell$. $\qquad\qquad\square$

Theorem 4.3. *Let* $D = (a^i_j)_{1 \le i \le 2r-1, 1 \le j \le n}$ *be an even net diagram* $(r, n \ge 2)$ *of a link* $L = K_1 \cup K_2 \cup \cdots \cup K_n$ *in* S^3 *such that for each* $j = 1, 2, \ldots, n$,

$$\sum_{k=1}^{r} a^{2k-1}_j = 0.$$

Then for any nonzero integers $q_1, q_2, \ldots, q_n$,

$$(4.16) \qquad \begin{aligned} &\lambda_c \left(M^3 \left(L; \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n} \right) \right) \\ &= -\frac{1}{8} \sum_{k=1}^{n} \sum_{\ell=1}^{r-1} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{r} q_k a^{2\ell}_k \left(q_{k-1} a^{2i-1}_k a^{2j-1}_k + q_{k+1} a^{2i-1}_{k+1} a^{2j-1}_{k+1} \right), \end{aligned}$$

where $q_0 = q_n, q_{n+1} = q_1$ *and* $a^{2k-1}_{n+1} = a^{2k-1}_1$ *for all* $k = 1, 2, \ldots, r$. $\qquad\square$

We remark that (4.14) and (4.15) imply the formula in (4.11). Furthermore, we obtain the following immediate corollary from (4.16).

Corollary 4.4. *Under the same assumption as in Theorem 4.3, if follows that for any nonzero integers* $q_1, q_2, \ldots, q_n$,

$$\begin{aligned} &\mu(M^3(L; \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n})) \\ &= -\frac{1}{8} \sum_{k=1}^{n} \sum_{\ell=1}^{r-1} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{r} q_k a^{2\ell}_k \left(q_{k-1} a^{2i-1}_k a^{2j-1}_k + q_{k+1} a^{2i-1}_{k+1} a^{2j-1}_{k+1} \right) \quad (\text{mod } 2). \end{aligned}$$
$\qquad\qquad\square$

5. Examples

In this section, we give an infinite family of $\mathbb{Z}$- and $\mathbb{Z}/2$-homology 3-spheres whose Casson invariants and the μ-invariants vanish. We also give an infinite family of homology 3-spheres whose invariants do not vanish.

Example 5.1. Let K be a knot in S^3 which has an even net diagram $D = (a^i_j)_{1 \leq i \leq 2r, 1 \leq j \leq 2n}(r, n \geq 1)$ given by

$$
D = \begin{pmatrix}
a_1 & -a_1 & a_1 & -a_1 & \cdots & a_1 & -a_1 \\
* & * & * & * & \cdots & * & * \\
a_2 & -a_2 & a_2 & -a_2 & \cdots & a_2 & -a_2 \\
* & * & * & * & \cdots & * & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_r & -a_r & a_r & -a_r & \cdots & a_r & -a_r \\
* & * & * & * & \cdots & * & *
\end{pmatrix},
$$

where $a_i \in 2\mathbb{Z}(1 \leq i \leq r)$ and each $*$ denotes an arbitrary even (possibly distinct) integer. By Theorem 4.1, $\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q))$ for any odd integer p. By Theorem 4.2, $\lambda_c(M^3(K; \frac{1}{q})) = 0$. In particular, if the sum of all entries labeled $*$ are equal to ± 1, then it follows from Theorem 3.6(1) that $\lambda_c(M^3(K; \mp 1)) = \text{sign}(M^4(\widehat{K})) = 0$.

Example 5.2. Let K be a knot in S^3 which has an even net diagram $D = (a^i_j)_{1 \leq i \leq 2r, 1 \leq j \leq n}(r \geq 1, n \geq 2)$ given by

$$
D = \begin{pmatrix}
2m_1 & 2m_1 & \cdots & 2m_1 \\
2m_2 & 2m_2 & \cdots & 2m_2 \\
\vdots & \vdots & \ddots & \vdots \\
2m_1 & 2m_1 & \cdots & 2m_1 \\
2m_2 & 2m_2 & \cdots & 2m_2
\end{pmatrix}, \quad m_1, m_2 \in \mathbb{Z}.
$$

By Theorem 4.1, we obtain that for any odd integer p,

$$
\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{qnr(r+1)m_1 m_2}{p} \quad (\text{mod } 2).
$$

By Theorem 4.2, $\lambda_c(M^3(K; \frac{1}{q})) = qnr(r + 1)m_1 m_2$. Let $M^4(\widehat{L}; \pm 1)$ be the 4-manifold with $\partial M^4(\widehat{L}; \pm 1) = M^3(L; \pm 1)$. Then, by Theorem 3.6(1), we have $\text{sign}(M^4(\widehat{L}; \pm 1)) = \pm rn(m_1 + m_2) \pm 1$.

Example 5.3. Let L be a link in S^3 which has an even net diagram $D = (a^i_j)_{1 \leq i \leq 2m-1, 1 \leq j \leq 4n}(m \geq 2, n \geq 1)$ given by

$$
D = \begin{pmatrix}
a_1 & a_1 & a_1 & a_1 & \cdots & a_1 \\
* & * & * & * & \cdots & * \\
a_2 & a_2 & a_2 & a_2 & \cdots & a_2 \\
* & * & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{r-1} & a_{r-1} & a_{r-1} & a_{r-1} & \cdots & a_{r-1} \\
* & * & * & * & \cdots & * \\
a_r & a_r & a_r & a_r & \cdots & a_r
\end{pmatrix},
$$

where $a_i \in 2\mathbb{Z}(1 \leq i \leq r)$ such that $\sum_{i=1}^{r} a_i = 0$, and each $*$ denotes an arbitrary even (possibly distinct) integer. Let $q_i(1 \leq i \leq 4n)$ be $4n$ integers with $q_1 = q_{4i+1} = -q_{4i+3}$ and $q_2 = q_{4i+2} = -q_{4(i+1)}$ for all $i = 0, 1, 2, \ldots, n-1$. By Theorem 4.3, $\lambda_c(M^3(L; \frac{1}{q_1}, \ldots, \frac{1}{q_{4n}})) = 0$, and so $\mu(M^3(L; \frac{1}{q_1}, \ldots, \frac{1}{q_{4n}})) = 0$.

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