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$R(g, g')$-CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. We introduce the notion of $R(g, g^{'})$-continuity on generalized topological spaces, which is a strong form of (g, g')-continuity. We investigate some properties and relationships among $R(g, g^{'})$-continuity, $(g, g^{'})$-continuity and some strong forms of (g, g')-continuity.

1. Introduction

Császár [1] introduced the notion of generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized topological spaces. Characterizations for the generalized continuous (= (g, g')-continuous) function were investigated in [1, 3]. In [5], we introduced and investigated the notions of super $(g, g^{'})$-continuous functions and strongly $\theta(g, g')$-continuous functions on generalized topological spaces. The purpose of this paper is to introduce the notion of $R(g, g^{'})$-continuity on generalized topological spaces, which is a strong form of (g, g')-continuity. We investigate some properties and relationships among $R(g, g^{'})$-continuity $(g, g^{'})$-continuity and some strong forms of (g, g')-continuity.

2. Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set and g be a collection of subsets of X. Then g is called a *generalized topology* (simply GT) on X if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in g$. We call the pair (X, g) a *generalized topological space* on X. We denote $\mathcal{M}_g = \cup\{A \subseteq X : A \in g\}$. A generalized topology g on X is called *strong* [2] if $X \in g$. The elements of g are called *g-open* sets and the complements are called *g-closed* sets. The generalized-closure of a subset S of X, denoted by $c_g(S)$, is the intersection of generalized closed sets including S.

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And the interior of S, denoted by $i_g(S)$, the union of generalized open sets included in S.

Let g and g' be generalized topologies on X and Y, respectively. Then a function $f : (X, g) \to (Y, g')$ is said to be

(1) (g, g')*-continuous* [1] if $G' \in g'$ implies that $f^{-1}(G') \in g$;

(2) *super* (g, g')*-continuous* [5] if for each $x \in X$ and each g'-open set V containing $f(x)$, there exists a g-open set U containing x such that $f(i_g(c_g(U))) \subseteq V$;

(3) *strongly* $\theta(g, g')$*-continuous* [5] if for each $x \in X$ and each g'-open set V of $f(x)$, there exists a g-open set U of x such that $f(c_g(U)) \subseteq V$.

3. $R(g, g')$-continuous functions

Definition 3.1. Let (X, g) and (Y, g') be generalized topological spaces. Then a function $f : X \to Y$ is said to be $R(g, g')$*-continuous* if for each $x \in X$ and each g'-open set V containing $f(x)$, there is a g-open set U containing x such that $c_{g'}(f(U)) \cap \mathcal{M}_{g'} \subseteq V$.

Theorem 3.2. *Let $f : X \to Y$ be a $R(g, g')$-continuous function on GTS's (X, g) and (Y, g'). Then if $f(\mathcal{M}_g) \subseteq \mathcal{M}_{g'}$, then $f(c_g(U)) \subseteq c_{g'}(f(U))$ for every g-open set $U \subseteq X$.*

Proof. Let U be a g-open set in X. For each $x \in c_g(U)$, let V be any g'-open set containing $f(x)$. Since f is $R(g, g')$-continuous, there exists a g-open set G containing x such that $c_{g'}(f(G)) \cap \mathcal{M}_{g'} \subseteq V$. Furthermore, since $x \in c_g(U)$ and a g-open set G contains x, $U \cap G \neq \emptyset$. From $f(\mathcal{M}_g) \subseteq \mathcal{M}_{g'}$, it follows

$$\begin{aligned} \emptyset &\neq f(U \cap G) \subseteq f(U) \cap f(G) \subseteq f(U) \cap c_{g'}(f(G)) \\ &= (f(U) \cap \mathcal{M}_{g'}) \cap c_{g'}(f(G)) \subseteq f(U) \cap V. \end{aligned}$$

So $f(U) \cap V \neq \emptyset$ and $f(x) \in c_{g'}(f(U))$. This implies $f(c_g(U)) \subseteq c_{g'}(f(U))$. □

Theorem 3.3. *Let $f : (X, g) \to (Y, g')$ be a function on GTS's (X, g) and (Y, g'). If $f(\mathcal{M}_g) \subseteq \mathcal{M}_{g'}$, then the following are equivalent:*

(1) f is $R(g, g')$-continuous.

(2) For each point $x \in X$ and a g'-open set V containing $f(x)$, there is a g-open set U containing x such that $c_{g'}(f(c_g(U))) \cap \mathcal{M}_{g'} \subseteq V$.

(3) For each point $x \in X$ and a g'-closed set F with $f(x) \notin F$, there is a g-open set U containing x and a g'-open set V such that $F \cap \mathcal{M}_{g'} \subseteq V$ and $f(c_g(U)) \cap V = \emptyset$.

(4) For each point $x \in X$ and a g'-closed set F with $f(x) \notin F$, there is a g-open set U containing x and a g'-open set V such that $F \cap \mathcal{M}_{g'} \subseteq V$ and $f(U) \cap V = \emptyset$.

Proof. (1) ⇒ (2) For $x \in X$, let V be a g'-open set containing $f(x)$. Then there is a g-open set U containing x such that $c_{g'}(f(U)) \cap \mathcal{M}_{g'} \subseteq V$. By

Theorem 3.2, we have $f(c_g(U)) \subseteq c_{g'}(f(U))$. It implies $c_{g'}(f(c_g(U))) \cap \mathcal{M}_{g'} \subseteq c_{g'}(f(U))) \cap \mathcal{M}_{g'} \subseteq V$.

(2) $\Rightarrow$ (3) For $x \in X$, let F be a g'-closed set with $f(x) \notin F$. Since $f(x) \in Y - F$ and $Y - F$ is g'-open, by (2), there is a g-open set U containing x such that $c_{g'}(f(c_g(U))) \cap \mathcal{M}_{g'} \subseteq Y - F$. Set $V = Y - (c_{g'}(f(c_g(U)))$. Then V is a g'-open set such that $F \cap \mathcal{M}_{g'} \subseteq V$ and $f(c_g(U)) \cap V = \emptyset$.

(3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (1) Let $x \in X$ and V a g'-open set containing $f(x)$. Then $Y - V$ is a g'-closed set and $f(x) \notin Y - V$. By (4), there is a g-open set U containing x and a g'-open set W such that $(Y - V) \cap \mathcal{M}_{g'} \subseteq W$ and $f(U) \cap W = \emptyset$. So $c_{g'}(f(U)) \cap \mathcal{M}_{g'} \subseteq c_{g'}(Y - W) \cap \mathcal{M}_{g'} = (Y - W) \cap \mathcal{M}_{g'} \subseteq V$, and hence f is $R(g, g^{'})$-continuous. □

Corollary 3.4. *Let $f : (X, g) \to (Y, g')$ be a function on GTS's (X, g) and (Y, g'). If Y is strong, then the following are equivalent*:

(1) *f is $R(g, g^{'})$-continuous.*

(2) *For each point $x \in X$ and a g'-open set V containing $f(x)$, there is a g-open set U containing x such that $c_{g'}(f(c_g(U))) \subseteq V$.*

(3) *For each point $x \in X$ and a $g^{'}$-closed set F with $f(x) \notin F$, there is a g-open set U containing x and a g'-open set V such that $F \subseteq V$ and $f(c_g(U)) \cap V = \emptyset$.*

(4) *For each point $x \in X$ and a g'-closed set F with $f(x) \notin F$, there is a g-open set U containing x and a g'-open set V such that $F \subseteq V$ and $f(U) \cap V = \emptyset$.*

Theorem 3.5. *Let $f : X \to Y$ be a function on GTS's (X, g) and (Y, g'). Then if f is $R(g, g^{'})$-continuous and Y is strong, then it is strongly $\theta(g, g')$-continuous.*

Proof. It follows from Corollary 3.4(2). □

Remark 3.6. The converse of Theorem 3.5 is not true in general as shown by the next example.

Example 3.7. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Consider generalized topologies $g = \{\emptyset, \{a\}\}$ on X and $g' = \{\emptyset, \{1\}, Y\}$ on Y. Let us define a function $f : X \to Y$ as $f(a) = f(b) = f(c) = 1$. Then f is strongly $\theta(g, g')$-continuous. But since $c_{g'} f(\{a\}) = c_{g'}(\{1\}) = Y$, f can not be $R(g, g^{'})$-continuous.

From Remark 3.8 of [5] and Theorem 3.5, we have the implications:

$R(g, g^{'})$-continuous $\Rightarrow$ strongly $\theta(g, g')$-continuous $\Rightarrow$ super (g, g')-continuous $\Rightarrow$$(g, g')$-continuous.

Definition 3.8. Let (X, g) and (Y, g') be generalized topological spaces. Then a function $f : X \to Y$ is said to be *weakly (g, g')-closed* if for each g-closed set F in X, $c_g'(f(i_g(F)) \subseteq f(F)$.

Lemma 3.9. *Let (X, g) and (Y, g') be GTS's. Then if a function $f : X \to Y$ is weakly (g, g')-closed, then $c_g'(f(U)) \subseteq f(c_g(U))$ for every g-open set U in X.*

Proof. For any g-open set $U \subseteq X$, since $c_g(U)$ is g-closed and $U \subseteq i_g(c_g(U))$, it is obtained. □

Theorem 3.10. *Let (X, g) and (Y, g') be GTS's. Then if a function $f : X \to Y$ is weakly (g, g')-closed and strongly $\theta(g, g')$-continuous, then it is $R(g, g^{'})$-continuous.*

Proof. For $x \in X$, let V be a g'-open set containing $f(x)$. Then from the strong $\theta(g, g')$-continuity of f, there exists a g-open set U of x such that $f(c_g(U)) \subseteq V$. From Lemma 3.9, it follows $c'_g(f(U)) \cap \mathcal{M}_{g'} \subseteq f(c_g(U)) \cap \mathcal{M}_{g'} \subseteq V$. Hence by Theorem 3.3(2), f is $R(g, g^{'})$-continuous. □

Definition 3.11. Let (X, g) be a generalized topological space. Then X is said to be *relative G-regular* (simply, *G-regular*) [4] on $\mathcal{M}_g$ if for $x \in \mathcal{M}_g$ and a g-closed set F with $x \notin F$, there exist $U, V \in g$ such that $x \in U$, $F \cap \mathcal{M}_g \subseteq V$ and $U \cap V = \emptyset$.

Theorem 3.12 ([4])**.** *Let (X, g) be a GTS. Then X is G-regular if and only if for $x \in \mathcal{M}_g$ and a g-open set U containing x, there is a g-open set V containing x such that $x \in V \subseteq c_g V \cap \mathcal{M}_g \subseteq U$.*

Theorem 3.13. *Let (X, g) and (Y, g') be GTS's. Then a function $f : X \to Y$ is strongly $\theta(g, g')$-continuous and Y is G-regular, then it is $R(g, g^{'})$-continuous.*

Proof. For $x \in X$, let V be a g'-open set containing $f(x)$. Since Y is G-regular, for the g'-open set V containing $f(x)$, there is a g'-open set W containing $f(x)$ such that $f(x) \in W \subseteq c'_g W \cap \mathcal{M}_{g'} \subseteq V$. For the g'-open set W containing $f(x)$, from the strong $\theta(g, g')$-continuity of f, there exists a g-open set U of x such that $f(c_g(U)) \subseteq W$. This implies $c'_g(f(c_g(U))) \cap \mathcal{M}_{g'} \subseteq c'_g(W) \cap \mathcal{M}_{g'} \subseteq V$. By Theorem 3.3(2), $R(g, g^{'})$-continuous. □

From Corollary 3.13 of [5], Lemma 3.5 and Theorem 3.13, the following corollary is easily obtained:

Corollary 3.14. *Let $f : X \to Y$ be a function between two GTS's (X, g) and (Y, g'). Then if Y is G-regular and strong, then the following things are equivalent*:
(1) *$R(g, g^{'})$-continuity.*
(2) *strongly $\theta(g, g')$-continuity.*
(3) *(g, g')-continuity.*

Let (X, g) and (Y, g') be GTS's. Then a function $f : X \to Y$ is said to be *(g, g')-open* [3] if for every g-open set G in X, $f(G)$ is g'-open in Y.

Theorem 3.15. *Let (X, g) and (Y, g') be GTS's and $f(\mathcal{M}_g) = \mathcal{M}_{g'}$. Then if a function $f : X \to Y$ is (g, g')-open and $R(g, g^{'})$-continuous, then Y is G-regular.*

Proof. Let $y \in \mathcal{M}'_g$ and V any g'-open set containing y. Let $f(x) = y$ for $x \in X$. Then since f is $R(g, g^{'})$-continuous, there exists a g-open set U containing x such that $c'_g(f(U)) \cap \mathcal{M}_{g'} \subseteq V$. Since f is (g, g')-open, $f(U)$ is a g'-open set containing y, and so $f(U) = f(U) \cap \mathcal{M}_{g'} \subseteq c'_g(f(U)) \cap \mathcal{M}_{g'} \subseteq V$. Therefore, since $f(U)$ is a g'-open set containing y, by Theorem 3.12, Y is G-regular. □

References

[1] Á. Császár, *Generalized topology, generalized continuity,* Acta Math. Hungar. **96** (2002), no. 4, 351–357.

[2] ———, *δ-and θ-modificatons of generalized topologies*, Acta Math. Hungar. **120** (2008), no. 3, 275–279.

[3] W. K. Min, *Some results on generalized topological spaces and generalized systems*, Acta Math. Hungar. **108** (2005), no. 1-2, 171–181.

[4] ———, *(δ, δ')-continuity on generalized topological spaces*, Acta Math. Hungar. **129** (2010), no. 4, 350–356.

[5] ———, *Some strong forms of (g, g')-continuity on generalized topological spaces*, submitted.

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