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BIMINIMAL CURVES IN 2-DIMENSIONAL SPACE FORMS

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Abstract. We study biminimal curves in 2-dimensional Riemannian manifolds of constant curvature.

Introduction

Elastic curves provide examples of classically known geometric variational problem. A plane curve is said to be an *elastic curve* if it is a critical point of the elastic energy, or equivalently a critical point of the total squared curvature [9].

In this paper, we study another geometric variational problem of curves in Riemannian 2-manifolds of constant curvature. The Euler-Lagrange equation studied in this paper is derived from the theory of biharmonic maps in Riemannian geometry.

A smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_M |\tau(\phi)|^2 \, dv_g,$$

where $\tau(\phi) = \mathrm{tr}\, \nabla d\phi$ is the tension field of ϕ. Clearly, if ϕ is harmonic, *i.e.*, $\tau(\phi) = 0$, then ϕ is biharmonic. A biharmonic map is said to be *proper* if it is not harmonic.

Chen and Ishikawa [3] studied biharmonic curves and surfaces in semi-Euclidean space (see also [6]). Caddeo, Montaldo and Piu [1] studied biharmonic curves on Riemannian 2-manifolds. They showed that biharmonic curves on Riemannian 2-manifolds of non-positive curvature are geodesics. Proper biharmonic curves on the unit 2-sphere are small circles of radius $1/\sqrt{2}$.

Next, Loubeau and Montaldo introduced the notion of biminimal immersion [10].

An isometric immersion $\phi : (M, g) \to (N, h)$ is said to be *biminimal* if it is a critical point of the bienergy functional under all normal variations. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general.

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In this paper we study biminimal curves on Riemannian 2-manifolds of constant curvature. We shall give natural equations for biminimal curves explicitly in terms of Jacobi's elliptic functions.

1. Preliminaries

1.1. Let (M^m, g) and (N^n, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map. Then ϕ induces a vector bundle ϕ^*TN over M by

$$\phi^*TN = \bigcup_{p\in M} T_{\phi(p)}N,$$

where TN is the tangent bundle of N. The space of all smooth sections of ϕ^*TN is denoted by $\Gamma(\phi^*TN)$. A section of ϕ^*TN is called a *vector field along* ϕ.

The Levi-Civita connection ∇^h of (N, h) induces a unique connection ∇^ϕ of ϕ^*TN which satisfies the condition

$$\nabla^\phi_X(V \circ \phi) = (\nabla^h_{\mathrm{d}\phi(X)}V) \circ \phi$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(\phi^*TN)$ (see [4, p. 4]).

The *second fundamental form* $\nabla\mathrm{d}\phi$ is defined by

$$(\nabla\mathrm{d}\phi)(X, Y) = \nabla^\phi_X \mathrm{d}\phi(Y) - \mathrm{d}\phi(\nabla_X Y), \quad X, Y \in \Gamma(TM).$$

Here ∇ is the Levi-Civita connection of (M, g). The *tension field* $\tau(\phi)$ is a section of ϕ^*TN defined by

$$\tau(\phi) = \mathrm{tr}\,\nabla\mathrm{d}\phi.$$

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \frac{1}{2}\int |\mathrm{d}\phi|^2 \,\mathrm{d}v_g$$

with respect to all compactly supported variations.

Now let $\phi : M \to N$ be a harmonic map. Then the *Hessian* $\mathcal{H}_\phi$ of E is given by

$$\mathcal{H}_\phi(V, W) = \int h(J_\phi(V), W)\,\mathrm{d}v_g, \quad V, W \in \Gamma(\phi^*TN).$$

Here the *Jacobi operator* $\mathcal{J}_\phi$ is defined by

$$J_\phi(V) := \bar{\triangle}_\phi V - \mathcal{R}_\phi(V), \; V \in \Gamma(\phi^*TN),$$

$$\bar{\triangle}_\phi := -\sum_{i=1}^m (\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i}e_i}), \quad \mathcal{R}_\phi(V) = \sum_{i=1}^m R^N(V, \mathrm{d}\phi(e_i))\mathrm{d}\phi(e_i),$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N and a local orthonormal frame field of M, respectively. For general theory of harmonic maps, we refer to Urakawa's monograph [12].

Eells and Sampson [5] suggested to study *polyharmonic maps.* Polyharmonic maps of order 2 are frequently called *biharmonic maps.*

Definition 1.1. A smooth map $\phi : (M, g) \to (N, h)$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, \mathrm{d}v_g,$$

with respect to all compactly supported variation.

The Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -J_\phi(\tau(\phi)) = 0.$$

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ. For more informations on biharmonic maps, we refer to a survey [11] by Montaldo and Oniciuc.

If ϕ is an isometric immersion, then $\tau(\phi) = m\mathbb{H}$, where $\mathbb{H}$ is the mean curvature vector field. Hence ϕ is harmonic if and only if ϕ is a minimal immersion. As is well known, an isometric immersion $\phi : M \to N$ is minimal if and only if it is a critical point of the volume functional $\mathcal{V}$. The Euler-Lagrange equation of $\mathcal{V}$ is $\mathbb{H} = 0$.

Motivated by this coincidence, the following notion was introduced by Loubeau and Montaldo:

Definition 1.2 ([10])**.** An isometric immersion $\phi : (M^m, g) \to (N^n, h)$ is called a *biminimal immersion* if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{\phi_t\}$ through $\phi = \phi_0$ such that the variational vector field $V = \mathrm{d}\phi_t/\mathrm{d}t|_{t=0}$ is normal to M.

The Euler-Lagrange equation of this variational problem is $\tau_2(\phi)^\perp = 0$. Here $\tau_2(\phi)^\perp$ is the normal component of $\tau_2(\phi)$. Since $\tau(\phi) = m\mathbb{H}$, the Euler-Lagrange equation is given explicitly by

$$\left\{ \bar{\triangle}_\phi \mathbb{H} - \mathcal{R}_\phi(\mathbb{H}) \right\}^\perp = 0. \tag{1.1}$$

Obviously, every biharmonic immersion is biminimal, but the converse is not always true.

2. Biminimal curves

From now on we restrict our attention to unit speed curves in Riemannian 2-manifolds.

For a unit speed curve $\gamma(s)$ in a Riemannian 2-manifold M, its tension field is given by $\tau(\gamma) = \nabla_{\gamma'}\gamma'$. Thus the bienergy of γ is the elastic energy

$$E_2(\gamma) = \frac{1}{2} \int \kappa(s)^2 \, \mathrm{d}s,$$

where $\kappa(s)$ is the signed curvature of γ.

Here we recall the following fundamental result.

Lemma 2.1 ([10]). *A unit speed curve $\gamma(s)$ in a Riemannian 2-manifold of Gaussian curvature K is biminimal if and only if its signed curvature $\kappa(s)$ satisfies*:

$$\kappa'' - \kappa^3 + \kappa K = 0. \tag{2.1}$$

Note that γ is biharmonic if and only if γ is biminimal and additionally satisfies $\kappa\kappa' = 0$. Thus non-geodesic biharmonic curves have constant curvature.

Corollary 2.1. *A non-geodesic curve in a Riemannian 2-manifold is biharmonic if and only if γ is a Riemannian circle of signed curvature κ satisfying $K = \kappa^2 > 0$. Thus proper biharmonic curves can exist only in constant positive curvature 2-manifolds.*

Remark 1. Let γ be a unit speed curve in Euclidean plane $\mathbb{R}^2$. Then γ is an elastic curve if and only if its signed curvature satisfies

$$\kappa'' + \frac{1}{2}(\kappa^3 - \lambda\kappa) = 0$$

for some constant λ [9]. Thus the Euler-Lagrange equation of the biminimal curve is different from the elastic curve equation.

3. Biminimal curves on Euclidean plane

First, we investigate biminimal curves on the Euclidean plane $\mathbb{R}^2$. In this case, the signed curvature $\kappa(s)$ is a solution to

$$\kappa''(s) - \kappa(s)^3 = 0.$$

Multiplying $2\kappa'(s)$ to both hand sides of this ordinary differential equation, we get

$$(\kappa')^2 = \frac{1}{2}(\kappa^4 + A)$$

for some constant A. Thus we obtain

$$\int \frac{\mathrm{d}\kappa}{\sqrt{\kappa^2 + A}} = \pm\frac{1}{\sqrt{2}}(s - s_0).$$

The left hand side of this equation is an elliptic integral of the first kind. Hence the signed curvature $\kappa(s)$ can be represented by Jacobi's elliptic functions.

In our previous paper [8], we have solved the ordinary differential equation $\kappa'' = \kappa^3$. For our purpose, we recall the integration procedure given in [8].

Definition 3.1. For a positive constant k such that $0 < k < 1$, the Jacobi's sn-function sn of modulus k is defined by

$$\mathrm{sn}^{-1}(x; k) = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1 - x^2)(1 - k^2x^2)}}, \quad -1 \le x \le 1.$$

The sn-function is defined on the interval $-K(k) \leq x \leq K(k)$, where $K(k)$ is the *complete elliptic integral of the first kind* defined by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The sn function is extended to the whole line $\mathbb{R}$ as a periodic function with fundamental period $4K(k)$. The cn function is defined by

$$\mathrm{cn}(x; k) = \sqrt{1 - \mathrm{sn}\,(x; k)^2}.$$

One can check the following integral formulas.

$$\int_1^u \frac{\mathrm{d}u}{\sqrt{u^4 - 1}} = \frac{1}{\sqrt{2}} \mathrm{cn}^{-1} \left(\frac{1}{u}; \frac{1}{\sqrt{2}} \right), \tag{3.1}$$

$$\int_1^u \frac{\mathrm{d}u}{\sqrt{u^4 + 1}} = K\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \mathrm{cn}^{-1} \left(\frac{u^2 - 1}{u^2 + 1}; \frac{1}{\sqrt{2}} \right). \tag{3.2}$$

3.1. $A = 0$. A simple and particular case is $A = 0$. In this case, κ is an elementary function given explicitly by

$$\kappa(s) = \mp \frac{\sqrt{2}}{s - s_0}. \tag{3.3}$$

The plane curve determined by this signed curvature is a logarithmic spiral. This case was discussed in [10].

3.1.1. $A > 0$. In this case we express $A = a^2$ with $a > 0$. Put $\kappa = \sqrt{a}u$, then by (3.2), we have

$$\begin{aligned} \int_{\sqrt{a}}^{\kappa} \frac{\mathrm{d}\kappa}{\sqrt{\kappa^4 + a^2}} &= \frac{1}{\sqrt{a}} \int_1^u \frac{\mathrm{d}u}{\sqrt{u^4 + 1}} \\ &= \frac{1}{\sqrt{a}} \left\{ K\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \mathrm{cn}^{-1} \left(\frac{u^2 - 1}{u^2 + 1}; \frac{1}{\sqrt{2}} \right) \right\}. \end{aligned}$$

Thus we obtain

$$\kappa(s) = \pm \sqrt{a} \left(\frac{1 + \mathrm{cn}(\nu(s); 1/\sqrt{2})}{1 - \mathrm{cn}(\nu(s); 1/\sqrt{2})} \right)^{\frac{1}{2}}, \tag{3.4}$$

where

$$\nu(s) = \mp \sqrt{2a}(s - s_0) + 2K(1/\sqrt{2}).$$

3.1.2. $A < 0$. In this case we express $A = -a^2$ with $a > 0$. Put $\kappa = \sqrt{a}u$ as before, then by (3.1) we get

$$\begin{aligned} \int_{\sqrt{a}}^{\kappa} \frac{\mathrm{d}\kappa}{\sqrt{\kappa^4 - a^2}} &= \frac{1}{\sqrt{a}} \int_1^u \frac{du}{\sqrt{u^4 - 1}} \\ &= \frac{1}{\sqrt{a}} \left\{ \frac{1}{\sqrt{2}} \mathrm{cn}^{-1} \left(\frac{1}{u}; \frac{1}{\sqrt{2}} \right) \right\}. \end{aligned}$$

From we get the following formula:

$$\kappa(s) = \frac{\sqrt{a}}{\operatorname{cn}\left(\sqrt{a}(s-s_0); \frac{1}{\sqrt{2}}\right)}. \tag{3.5}$$

Note that cn is an even function.

Theorem 3.1. *Let $\gamma(s)$ be a Frenet curve in Euclidean plane $\mathbb{R}^2$. Then γ is biminimal if and only if it is determined by one of the following natural equations.*

(1)
$$\kappa(s) = \mp\frac{\sqrt{2}}{s-s_0}.$$
In this case γ is a logarithmic spiral.

(2)
$$\kappa(s) = \pm\sqrt{a}\left(\frac{1+\operatorname{cn}(\nu(s);1/\sqrt{2})}{1-\operatorname{cn}(\nu(s);1/\sqrt{2})}\right)^{\frac{1}{2}},$$
with $\nu(s) = \mp\sqrt{2a}(s-s_0) + 2K(1/\sqrt{2})$, or

(3)
$$\kappa(s) = \frac{\sqrt{a}}{\operatorname{cn}\left(\sqrt{a}(s-s_0);1/\sqrt{2}\right)}.$$

4. Biminimal curves on the 2-sphere and the hyperbolic plane

In this section we study biminimal curves in space forms of curvature $c \neq 0$.

Multiplying $2\kappa'$ to the biminimal equation

$$\kappa''(s) - \kappa(s)^3 + c\kappa(s) = 0, \tag{4.1}$$

we obtain

$$(\kappa')^2 - \frac{1}{2}\kappa^4 + c\kappa^2 = d,$$

where d is a constant. From this equation, we have

$$\int \frac{\mathrm{d}\kappa}{\sqrt{\kappa^4 - 2c\kappa^2 + 2d}} = \int \frac{\mathrm{d}s}{\sqrt{2}} = \frac{1}{\sqrt{2}}(s-s_0).$$

The left hand side of this equation is an elliptic integral.

4.1. $c^2 - 2d > 0$. In this case, we can put $r = \sqrt{c^2-2d} > 0$. Then we have

$$\int \frac{\mathrm{d}\kappa}{\sqrt{\kappa^4 - 2c\kappa + 2d}} = \int \frac{\mathrm{d}\kappa}{\sqrt{(\kappa^2 - c + r)(\kappa^2 - c - r)}}.$$

In this case, the positivity of $(\kappa')^2$ implies

$$\kappa^2 > c + r \quad \text{or} \quad 0 < \kappa^2 < c - r. \tag{4.2}$$

We have three possibilities.

4.1.1. $c < 0$ and $d > 0$. Since $d > 0$, we can put

$$a^2 = -c + r > 0, \quad b^2 = -c - r > 0.$$

Equivalently, we have

$$a^2 + b^2 = -2c, \quad a^2 b^2 = 2d.$$

Hence we get

$$\kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 + a^2)(\kappa^2 + b^2).$$

Note that, in this case, the positivity condition $\kappa^2 > c + r$ is satisfied. By using the following integral formula

$$\int_0^x \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}\mathrm{cn}^{-1}\left(\frac{b}{\sqrt{b^2+x^2}}; \frac{\sqrt{a^2-b^2}}{a}\right), \quad b \le a, \tag{4.3}$$

we have

$$\kappa(s) = b\left\{\frac{1 - \mathrm{cn}^2\left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{\sqrt{a^2-b^2}}{a}\right)}{\mathrm{cn}^2\left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{\sqrt{a^2-b^2}}{a}\right)}\right\}^{\frac{1}{2}}. \tag{4.4}$$

4.1.2. $c > 0$ and $d > 0$. In this case, we can put

$$a^2 = c + r > 0, \quad b^2 = c - r > 0.$$

Equivalently, we have

$$a^2 + b^2 = 2c, \quad a^2 b^2 = 2d.$$

Hence we get

$$\kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 - a^2)(\kappa^2 - b^2).$$

The positivity condition (3.4) is rewritten as

$$\kappa^2 > a^2 \text{ or } 0 < \kappa^2 < b^2.$$

Comparing this condition with the following integral formulas.

$$\int_x^\infty \frac{\mathrm{d}x}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}\mathrm{sn}^{-1}\left(\frac{a}{x}; \frac{b}{a}\right), \quad 0 < b < a \le x, \tag{4.5}$$

$$\int_0^x \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}\mathrm{sn}^{-1}\left(\frac{x}{b}; \frac{b}{a}\right), \quad 0 \le |x| \le b < a. \tag{4.6}$$

Then we obtain

$$\kappa(s) = \frac{a}{\mathrm{sn}\left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{b}{a}\right)} \quad \text{or} \tag{4.7}$$

$$\kappa(s) = b\,\mathrm{sn}\left(\frac{a(s-s_0)}{\sqrt{2}}; \frac{b}{a}\right), \tag{4.8}$$

respectively.

4.1.3. $d<0$. In this case, we can put

$$a^2=-c+r,\quad b^2=c+r.$$

Equivalently,

$$a^2-b^2=-2c,\quad a^2b^2=-2d.$$

Thus we have

$$\kappa^4-2c\kappa^2+2d=(\kappa^2+a^2)(\kappa^2-b^2).$$

By using the integral formula

$$\int_b^x \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)(x^2-b^2)}}=\frac{1}{\sqrt{a^2+b^2}}\mathrm{cn}^{-1}\left(\frac{b}{x};\frac{a}{\sqrt{a^2+b^2}}\right),\quad b\leq x, \tag{4.9}$$

we get

$$\kappa(s)=\frac{b}{\mathrm{cn}\left(\frac{\sqrt{a^2+b^2}}{\sqrt{2}}(s-s_0);\frac{a}{\sqrt{a^2+b^2}}\right)}. \tag{4.10}$$

4.2. $c^2-2d=0$. In this case, the ordinary equation is reduced to

$$\int\frac{\mathrm{d}\kappa}{\kappa^2-c}=\frac{1}{\sqrt{2}}(s-s_0).$$

Thus we obtain

$$\kappa(s) = -\sqrt{c}\tanh\left\{\sqrt{c}(s-s_0)/\sqrt{2}\right\},\ c>0, \tag{4.11}$$

$$\kappa(s) = \sqrt{-c}\tan\left\{\sqrt{-c}(s-s_0)/\sqrt{2}\right\},\ c<0. \tag{4.12}$$

4.3. $c^2-2d<0$. Since $2d>c^2>0$, we may put $2d=\alpha^4$ $(\alpha>0)$. Then we can express c as

$$c=-\alpha^2\cos(2\theta).$$

Because $|c/\sqrt{2d}|=|c/\alpha^2|\leq 1$. Then by using the integral formula

$$\int_0^x\frac{\mathrm{d}x}{\sqrt{x^4+2\alpha^2\cos(2\theta)x^2+\alpha^4}}=\frac{1}{2\alpha}\mathrm{cn}^{-1}\left(\frac{\alpha^2-x^2}{\alpha^2+x^2};\sin\theta\right),$$

we obtain

$$\kappa(s)=\alpha\left(\frac{1-\mathrm{cn}\left(\sqrt{2}\alpha(s-s_0);\sin\theta\right)}{1+\mathrm{cn}\left(\sqrt{2}\alpha(s-s_0);\sin\theta\right)}\right)^{\frac{1}{2}}. \tag{4.13}$$

Theorem 4.1. *Let $\gamma(s)$ be a unit speed curve in a Riemannian 2-manifold $M^2(c)$ of constant curvature $c\neq 0$. If γ is biminimal and not a geodesic. Then the signed curvature κ of γ is given by* (4.4), (4.7), (4.8), (4.10), (4.11), (4.12), *or* (4.13).

5. Concluding remarks

Submanifolds with harmonic mean curvature ($\triangle\mathbb{H} = 0$) or normal harmonic mean curvature ($\triangle^{\perp}\mathbb{H} = 0$) have been studied extensively. Here $\Delta^{\perp}$ is the rough Laplacian of the normal bundle (and called the *normal Laplacian*). More generally, submanifolds with property $\triangle\mathbb{H} = \lambda\mathbb{H}$ or $\triangle^{\perp}\mathbb{H} = \lambda\mathbb{H}$ have been studied extensively by many authors (See references in [2], [7]). Analogously, we may generalize the notion of biminimal immersion to the following one:

Definition 5.1 ([10])**.** An isometric immersion $\phi : M \to N$ is called a λ-*biminimal immersion* if it is a critical point of the functional:

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R}.$$

The Euler-Lagrange equation for λ-biminimal immersions is

$$\tau_2(\phi)^{\perp} = \lambda\tau(\phi).$$

More explicitly,

$$\{\bar{\triangle}_{\phi}\mathbb{H} - \mathcal{R}_{\phi}(\mathbb{H})\}^{\perp} = -\lambda\mathbb{H}$$

or equivalently

$$J_{\phi}(\mathbb{H})^{\perp} = -\lambda\mathbb{H}.$$

Corollary 5.1. *A non-geodesic curve γ in a Riemannian 2-manifold is λ-biminimal if and only if*

$$\kappa'' - \kappa^3 + \kappa\,(K - \lambda) = 0.$$

Thus, by replacing K by $c - \lambda$ in (2.1), one can obtain natural equations for λ-biminimal curves in the space form of curvature c.

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