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ON SUMS OF CERTAIN CLASSES OF SERIES

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Abstract. The aim of this research note is to provide the sums of the series

$$\sum_{k=0}^{\infty}(-1)^k \binom{a-i}{k} \frac{1}{2^k(a+k+1)}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The results are obtained with the help of generalization of Bailey's summation theorem on the sum of a ${}_2F_1$ obtained earlier by Lavoie et al.. Several interesting results including those obtained earlier by Srivastava, Vowe and Seiffert, follow special cases of our main findings. The results derived in this research note are simple, interesting, easily established and (potentially) useful.

1. Introduction and results required

We start with an interesting series due to Vowe and Seiffert [6]:

$$\begin{aligned}(1.1)\qquad &\sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k} \frac{1}{2^k(n+k+1)}\\ &= \frac{2^n(n-1)!n!}{(2n)!} - \frac{1}{n2^n}; (n \in \mathbb{N} = \{1, 2, 3, \ldots\}).\end{aligned}$$

Vowe and Seiffert [6] obtained this interesting series by identifying the sum related to the following Eulerian integral

$$(1.2)\qquad \int_0^1 \left(1 - \frac{t}{2}\right)^{n-1} t^n dt.$$

Later on, Srivastava [5] obtained the generalized (1.1) in the following form

$$(1.3)\qquad \sum_{k=0}^{\infty}(-1)^k \binom{a-1}{k} \frac{1}{2^k(a+k+1)} = \frac{2^a\Gamma(a)\Gamma(a+1)}{\Gamma(2a+1)} - \frac{1}{a2^a},$$

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with the help of the following Bailey's summation theorem on the sum of a ${}_2F_1$ [4],

$$(1.4)\qquad {}_2F_1\left[\begin{matrix} a, & 1-a \\ b & \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{b}{2})\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+b}{2})\Gamma\frac{(b-a+1)}{2}}; (b \neq 0, -1, -2, \ldots)$$

where the so called hypergeometric function ${}_2F_1$ [2] is defined by

$$(1.5)\qquad {}_2F_1\left[\begin{matrix} a, & b \\ c & \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^n,$$

where a, b and c are arbitrary real or complex constants and $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$(1.6)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

where Γ is the well-known gamma function [4].

In 1996, Lavoie et al. [3] have generalizaed the Bailey's summation theorem as follows:

(1.7)

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} A, & 1-A+i \\ B & \end{matrix}; \frac{1}{2}\right] \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(B)\Gamma(1-A)}{2^{B-i-1}\Gamma(1-A+\frac{1}{2}(i+|i|))} \times \left[\frac{A_i}{\Gamma(\frac{1}{2}B-\frac{1}{2}A+\frac{1}{2})\Gamma(\frac{1}{2}B+\frac{1}{2}A-[\frac{1}{2}+\frac{1}{2}i])}\right. \\ &\quad \left. + \frac{B_i}{\Gamma(\frac{1}{2}B-\frac{1}{2}A)\Gamma(\frac{1}{2}B+\frac{1}{2}A-\frac{1}{2}-[\frac{1}{2}i])}\right] \end{aligned}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

As usual, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$.

The coefficients A_i and B_i are given in the following table:

i	A_i	B_i
5	$-(4B^2-2AB-A^2-22B+13A+20)$	$4B^2+2AB-A^2-34B-A+62$
4	$2(B-2)(B-4)-(A-1)(A-4)$	$-4(B-3)$
3	$A-2B+3$	$A+2B-7$
2	$B-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	B	2
-3	$2B-A$	$A+2B+2$
-4	$2B(B+2)-A(A+3)$	$4(B+1)$
-5	$(4B^2-2AB-A^2+8B-7A)$	$(4B^2+2AB-A^2+16B-A+12)$

Remark 1.1. In (1.7), if we put $i=0$, we recover Bailey's summation theorem (1.4).

The aim of this research note is to provide the sums of the series

$$\sum_{k=0}^{\infty}(-1)^k \begin{pmatrix} a-i \\ k \end{pmatrix} \frac{1}{2^k(a+k+1)}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

The results are obtained with the help of generalization of Bailey's summation theorem on the sum of a ${}_2F_1$ obtained earlier by Lavoie et al. [3]. Several interesting results including those obtained earlier by Srivastava [5], Vowe and Seiffert[6], follow as special cases of our main findings. The results derived in this research note are simple, interesting, easily established and (potentially) useful.

2. Main results

The sums of the certain classes of series to be established are

$$\begin{aligned} &\sum_{k=0}^{\infty}(-1)^k \begin{pmatrix} a-i \\ k \end{pmatrix} \frac{1}{2^k(a+k+1)} \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(1+a-i)\Gamma(1+a)}{2^{a+1-i}\Gamma(1+a-i+\frac{1}{2}(i+|i|))} \\ &\times \left[\frac{A_i}{\Gamma(a+\frac{3}{2}-\frac{i}{2})\Gamma(1+\frac{i}{2}-[\frac{1+i}{2}])} + \frac{B_i}{\Gamma(a+1-\frac{i}{2})\Gamma(\frac{1}{2}+\frac{i}{2}-[\frac{i}{2}])}\right] \end{aligned} \tag{2.1}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

The coefficient A_i and B_i can be obtained from the table of A_i and B_i by replacing A by $-a+i$ and B by $a+2$, respectively.

3. Derivation of (2.1)

In order to derive our mail result (2.1), we proceed as follows. Denoting the left hand side of (2.1) by S, we have

$$S = \sum_{k=0}^{\infty}(-1)^k \binom{a-i}{k} \frac{1}{2^k(a+k+1)}$$

and using well known identities:

$$\binom{a-i}{k} = \frac{(a-i)!}{k!(a-i-k)!}, (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \text{ and } \Gamma(\alpha-n) = \frac{(-1)^n\Gamma(\alpha)}{(1-\alpha)_n},$$

after a little simplification, we have

$$S = \frac{1}{a+1}\sum_{k=o}^{\infty}\frac{(-a+i)_k(a+1)_k}{(a+2)_k}\frac{1}{2^k k!}.$$

Finally, summing up the series, we have

$$S = \frac{1}{a+1}\ {}_2F_1\left[\begin{matrix} -a+i, a+1 \\ a+2 \end{matrix}; \frac{1}{2}\right].$$

Now, it is easy to see that the ${}_2F_1$ can be evaluated with the help of generalized Bailey's summation theorem (1.7) by taking $A = -a+i$ and $B = a+2$. We, after a little simplification, easily arrive at the right hand side of (2.1). This completes the proof of (2.1).

4. Special cases

In (2.1), if we take $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, we get the following interesting identities.

For $i = 0$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a}{k} \frac{1}{2^k(a+k+1)} = \frac{\Gamma(\frac{1}{2})\Gamma(1+a)}{2^{a+1}\Gamma(a+\frac{3}{2})}. \tag{4.1}$$

For $i = 1$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a-1}{k} \frac{1}{2^k(a+k+1)} = \frac{2^a\Gamma(a)\Gamma(a+1)}{\Gamma(2a+1)} - \frac{1}{a2^a}. \tag{4.2}$$

For $i = -1$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a+1}{k} \frac{1}{2^k(a+k+1)} = \frac{1}{2^{a+2}(1+a)} + \frac{2^{a-2}\Gamma(a)\Gamma(1+a)}{(2a+1)\Gamma(2a)}. \tag{4.3}$$

For $i = 2$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a-2}{k} \frac{1}{2^k(a+k+1)} = \frac{2^a\Gamma(a+1)\Gamma(a-1)}{\Gamma(2a)} - \frac{2^{2-a}}{(a-1)}. \tag{4.4}$$

For $i = -2$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a+2}{k} \frac{1}{2^k(a+k+1)} = \frac{\Gamma(\frac{1}{2})(a+2)\Gamma(a+1)}{2^{a+3}\Gamma(a+\frac{5}{2})} + \frac{1}{2^{a+2}(a+1)}. \tag{4.5}$$

For $i = 3$

$$\begin{aligned} &\sum_{k=0}^{\infty}(-1)^k \binom{a-3}{k} \frac{1}{2^k(a+k+1)} \\ &= 2^{2-a}\Gamma(a-2)\left[\frac{(2-3a)}{\Gamma(a)} + \frac{a\Gamma(\frac{1}{2})}{\Gamma(a-\frac{1}{2})}\right]. \end{aligned} \tag{4.6}$$

For $i = -3$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a+3}{k} \frac{1}{2^k(a+k+1)} = \frac{\Gamma(a+1)}{2^{a+4}}\left[\frac{1}{\Gamma(a+3)} + \frac{\Gamma(\frac{1}{2})(a+3)}{\Gamma(a+\frac{5}{2})}\right]. \tag{4.7}$$

For $i = 4$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a-4}{k} \frac{1}{2^k(a+k+1)} = \frac{(a-1)\Gamma(a-3)}{2^{a-3}}\left[\frac{a\Gamma(\frac{1}{2})}{\Gamma(a-\frac{1}{2})} - \frac{4}{\Gamma(a-1)}\right]. \tag{4.8}$$

For $i = -4$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a+4}{k} \frac{1}{2^k(a+k+1)} = \frac{(a+3)\Gamma(a+1)}{2^{a+5}}\left[\frac{\Gamma(\frac{1}{2})(a+4)}{\Gamma(a+\frac{7}{2})} + \frac{4}{\Gamma(a+3)}\right]. \tag{4.9}$$

For $i = 5$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a-5}{k} \frac{1}{2^k(a+k+1)} = \frac{\Gamma(a-4)}{2^{a-4}}\left[\frac{(3a^2+47a-20)}{\Gamma(a-1)} - \frac{\Gamma(\frac{1}{2})(a^2-41a-30)}{\Gamma(a-\frac{3}{2})}\right]. \tag{4.10}$$

For $i = -5$

$$\sum_{k=0}^{\infty}(-1)^k \binom{a+5}{k} \frac{1}{2^k(a+k+1)} = \frac{\Gamma(a+1)}{2^{a+6}}\left[\frac{(5a^2+35a+62)}{\Gamma(a+4)} + \frac{\Gamma(\frac{1}{2})(a^2+9a+15)}{\Gamma(a+\frac{7}{2})}\right]. \tag{4.11}$$

Further in (4.1) to (4.11), if we set $a = n$, we get the following formulas.
For $i = 0$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{2^k(n+k+1)} = \frac{(1)_n}{2^n(\frac{3}{2})_n}. \tag{4.12}$$

For $i = 1$

$$\sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k} \frac{1}{2^k(n+k+1)} = \frac{2^n n!(n-1)!}{2n!} - \frac{1}{n.2^n}. \tag{4.13}$$

For $i = -1$

$$\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} \frac{1}{2^k(n+k+1)} = \frac{1}{2^{n+2}(1+n)} + \frac{2^{n-2}\Gamma(n)\Gamma(1+n)}{(2n+1)\Gamma(2n)}. \tag{4.14}$$

For $i = 2$

$$\sum_{k=0}^{n-2}(-1)^k \binom{a-2}{k} \frac{1}{2^k(n+k+1)} = \frac{2^n\Gamma(n+1)\Gamma(n-1)}{\Gamma(2n)} - \frac{2^{2-n}}{(n-1)}. \tag{4.15}$$

For $i = -2$

$$\begin{aligned}&\sum_{k=0}^{n+2}(-1)^k \binom{n+2}{k} \frac{1}{2^k(n+k+1)}\\ &= \frac{\Gamma(\frac{1}{2})(n+2)\Gamma(n+1)}{2^{n+3}\Gamma(n+\frac{5}{2})} + \frac{1}{2^{n+2}(n+1)}.\end{aligned} \tag{4.16}$$

For $i = 3$

$$\begin{aligned}&\sum_{k=0}^{n-3}(-1)^k \binom{n-3}{k} \frac{1}{2^k(n+k+1)}\\ &= 2^{2-n}\Gamma(n-2)\left[\frac{(2-3n)}{\Gamma(n)} + \frac{n\Gamma(\frac{1}{2})}{\Gamma(n-\frac{1}{2})}\right].\end{aligned} \tag{4.17}$$

For $i = -3$

$$\begin{aligned}&\sum_{k=0}^{n+3}(-1)^k \binom{n+3}{k} \frac{1}{2^k(n+k+1)}\\ &= \frac{\Gamma(n+1)}{2^{n+4}}\left[\frac{1}{\Gamma(n+3)} + \frac{\Gamma(\frac{1}{2})(n+3)}{\Gamma(n+\frac{5}{2})}\right].\end{aligned} \tag{4.18}$$

For $i = 4$

$$\begin{aligned}&\sum_{k=0}^{n-4}(-1)^k \binom{n-4}{k} \frac{1}{2^k(n+k+1)}\\ &= \frac{(n-1)\Gamma(n-3)}{2^{n-3}}\left[\frac{n\Gamma(\frac{1}{2})}{\Gamma(n-\frac{1}{2})} - \frac{4}{\Gamma(n-1)}\right].\end{aligned} \tag{4.19}$$

For $i = -4$

$$\begin{aligned}&\sum_{k=0}^{n+4}(-1)^k \binom{n+4}{k} \frac{1}{2^k(n+k+1)}\\ &= \frac{(n+3)\Gamma(n+1)}{2^{n+5}}\left[\frac{\Gamma(\frac{1}{2})(n+4)}{\Gamma(n+\frac{7}{2})} + \frac{4}{\Gamma(n+3)}\right].\end{aligned} \tag{4.20}$$

For $i = 5$

$$\begin{aligned}&\sum_{k=0}^{n-5}(-1)^k \binom{n-5}{k} \frac{1}{2^k(n+k+1)}\\ &= \frac{\Gamma(n-4)}{2^{n-4}}\left[\frac{(3n^2+47n-20)}{\Gamma(n-1)} - \frac{\Gamma(\frac{1}{2})(n^2-41n-30)}{\Gamma(n-\frac{3}{2})}\right].\end{aligned} \tag{4.21}$$

For $i = -5$

$$\begin{aligned}(4.22)\qquad &\sum_{k=0}^{n+5}(-1)^k \binom{n+5}{k} \frac{1}{2^k(n+k+1)} \\ &= \frac{\Gamma(n+1)}{2^{n+6}}\left[\frac{(5n^2+35n+62)}{\Gamma(n+4)} + \frac{\Gamma(\frac{1}{2})(n^2+9a+15)}{\Gamma(n+\frac{7}{2})}\right].\end{aligned}$$

We conclude this section by remarking that the result (4.1) has been obtained earlier by Srivatava [5] and (4.12) was obtained earlier by Vowe and Seiffert [6]. Also, the result (4.2) to (4.11) are closely related to (4.1) due to Srivastava [5], and (4.13) to (4.22) are closely related to (4.12) due to Vowe and Seiffert [6].

Note. For other interesting identities of similar type, see a paper by J. Choi *et al.* [1].

References

[1] J. Choi, Peter Zöring, and A. K. Rathie, *Sums of certain classes of series*, Comm. Korean Math. Soc. **14** (1999), no. 3, 641–647.

[2] E. E. Kummer, *Über die hypergeometrische Reihe*, J. Reine Angew. Math. **15** (1836), 39–83; 127–172.

[3] J. L. Lavoie, F. Grondin, and A. K. Rathie, *Generalizations of Whipple's theorem on the sum of a* ${}_3F_2$, J. Comput. Appl. Math. **72** (1996), no. 2, 293–300.

[4] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[5] H. M. Srivastava, *Sums of a certain family of series*, Elem. Math. **43** (1988), no. 2, 54–58.

[6] M. Vowe and H. J. Seiffert, *Aufgabe 946*, Elem. Math. **42** (1987), 111–112.

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