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COINCIDENCES AND FIXED POINT THEOREMS FOR MAPPINGS SATISFYING CONTRACTIVE CONDITION OF INTEGRAL TYPE ON d-COMPLETE TOPOLOGICAL SPACES

Ramesh Chandra Dimri and Amit Singh

Abstract. In this paper, we prove some fixed point theorems for some weaker forms of compatibility satisfying a contractive condition of integral type on d-complete Hausdorff topological spaces. Our results extend and generalize some well known previous results.

1. Introduction

Branciari [7] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. The authors in [3], [4], [6], [22], [28] and [30] proved some fixed point theorems involving more general contractive conditions. Recently ([10]) some fixed point theorems have been proved in non-metric setting wherein the distance function used need not satisfy triangle inequality. The purpose of this paper is to investigate some new result of fixed points in non-metric settings. In the sequel, we use contractive condition of integral type on d-complete Hausdorff topological spaces.

Sessa [24] generalized the concept of commuting mappings by calling self-mappings A and S on metric space (X, d) a weakly commuting pair if and only if $d(ASx, SAx) \leq d(Ax, Sx)$ for all $x \in X$. He and others proved some common fixed point theorems of weakly commuting mappings [24, 25, 26]. Then, Jungck [13] introduced the concept of compatibility and he and others proved some common fixed point theorems using this concept [13, 14, 15, 29]. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible. Examples in [13, 24] show that neither converse is true. Recently, Jungck and Rhoades [15] defined the concept of weak compatibility.

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Definition 1.1 (see [15, 27])**.** Two maps $A, S : X \to X$ are said to be weakly compatible if they commute at their coincidence points.

Again, it is obvious that compatible mappings are weakly compatible. Examples in [15, 27] show that the converse is not true. Many fixed point results have been obtained for weakly compatible mappings (see [1], [9], [8], [15], [21], [27]).

Let (X, τ) be a topological space and $d : X \times X \to [0, \infty)$ be such that $d(x, y) = 0$ if and only if $x = y$. Then X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}$ is convergent in X. A mapping $T : X \to X$ is w-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$. For details on d-complete topological spaces, we refer to Iseki [11] and Kasahara [17]-[19].

In the sequel, we shall use the following:

A symmetric function on a set X is a real valued d on $X \times X$ such that for all $x, y \in X$,

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric function on a set X, and for any $\epsilon > 0$ and any $x \in X$, let $S(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. From [10], we can define a topology τ_d on X by $U \in \tau_d$ if and only if for each $x \in U$, some $S(x, \epsilon) \subset U$. A symmetric function d is a semi-metric if for each $x \in X$ and for each $\epsilon > 0$, $S(x, \epsilon)$ is a neighborhood of x in the topology τ_d. A topological space X is said to be symmetrizable (resp. semi-metrizable) if its topology is induced by a symmetric function (resp. semi-metric) on X. The d-complete symmetrizable spaces form an important class of d-complete topological spaces. Other examples of d-complete topological spaces may be found in Hicks and Rhoades [10].

Hicks and Rhoades [10] proved the following theorem.

Theorem 1.1. *Let (X, τ) be a Hausdorff d-complete topological space and f, h be w-continuous self mappings on X satisfying*

$$d(hx, hy) \leq G(M^*(x, y))$$

for x, $y \in X$, where

$$M^*(x, y) = \max\{d(fx, fy), d(fx, hx), d(fy, hy)\}$$

and G is a real-valued function satisfying the following:

(a) $0 < G(y) < y$ *for each* $y > 0$; $G(0) = 0$,

(b) $g(y) = \frac{y}{y - G(y)}$ *is a non-increasing function on* $(0, \infty)$,

(c) $\int_0^{y_1} g(y)dy < \infty$ *for each* $y_1 > 0$,

(d) $G(y)$ *is non-decreasing.*

Suppose also that

(i) *f and h commute,*

(ii) $h(X) \subseteq f(X)$. *Then f and h have a unique common fixed point in X.*

2. Main results

Theorem 2.1. *Let A, B, S and T be w-continuous self-maps defined on a Hausdorff topological space* (X,τ) *satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{1}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{2}$$

for all x, $y \in X$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lebesgue integrable mapping which is summable on each compact subset of* $\mathbb{R}^+$, *non-negative and such that*

$$\epsilon \leq \int_0^{\epsilon} \varphi(t)dt \text{ for each } \epsilon > 0, \tag{3}$$

$$M(x,y) = \max\{d(Ax,By), d(Sx,Ax), d(Ty,By)\} \tag{4}$$

and G is a real valued function satisfying the condition (a)-(d). *If one of* $A(X)$, $B(X)$, $S(X)$ *and* $T(X)$ *is a d-complete topological subspace of X, then*

(i) *A and S have a coincidence point,*

(ii) *B and T have a coincidence point.*

Further if the pairs $\{A,S\}$ *and* $\{B,T\}$ *are weakly compatible, then*

(iii) *A, B, S and T have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. From (1), we can construct a sequence $\{y_n\}$ in X as follows:

$$y_{2n+1} = Sx_{2n} = Bx_{2n+1}, \qquad y_{2n+2} = Tx_{2n+1} = Ax_{2n+2} \tag{5}$$

for all $n = 0, 1, 2, \ldots$. Define $d_n = d(y_n, y_{n+1})$. Suppose that $d_{2n} = 0$ for some n. Then $y_{2n} = y_{2n+1}$, i.e., $Tx_{2n-1} = Ax_{2n} = Sx_{2n} = Bx_{2n+1}$, hence A and S have a coincidence point.

Similarly if $d_{2n+1} = 0$, then B and T have a coincidence point. Assume that $d_n \neq 0$ for each n. Then by (2)

$$\int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt \leq G\left(\int_0^{M(x_{2n},x_{2n+1})} \varphi(t)dt\right), \tag{6}$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &= \max\{d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1})\} \\ &= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\} \\ &= \max\{d_{2n}, d_{2n+1}\}. \end{aligned}$$

Thus from (6), we have

$$\int_0^{d_{2n+1}} \varphi(t)dt \leq G\left(\int_0^{max\{d_{2n},d_{2n+1}\}} \varphi(t)dt\right). \tag{7}$$

Now, if $d_{2n+1} \geq d_{2n}$ for some n, then from (7), we have

$$\int_0^{d_{2n+1}} \varphi(t)dt \leq G\left(\int_0^{d_{2n+1}} \varphi(t)dt\right) < \int_0^{d_{2n+1}} \varphi(t)dt,$$

which is a contradiction. Thus $d_{2n} > d_{2n+1}$ for all n. Therefore from (7), we have

$$\int_0^{d_{2n+1}} \varphi(t)dt \leq G\left(\int_0^{d_{2n}} \varphi(t)dt\right). \tag{8}$$

Similarly

$$\int_0^{d_{2n}} \varphi(t)dt \leq G\left(\int_0^{d_{2n-1}} \varphi(t)dt\right). \tag{9}$$

In general, we have for all $n = 1, 2, 3, \ldots,$

$$\int_0^{d_n} \varphi(t)dt \leq G\left(\int_0^{d_{n-1}} \varphi(t)dt\right). \tag{10}$$

Next we define a sequence $\{S_n\}$ of real numbers by $S_{n+1} = G(S_n)$ with $S_1 = \int_0^{d(Sx_0,Tx_1)} \varphi(t)dt > 0$, then by (a), we have $0 < S_{n+1} < S_n < S_1$ for $n \geq 1$.

Moreover by (b) and (c), the series $\sum_{n=1}^{\infty} S_n$ converges (see [1]). We shall show that $\int_0^{d_n} \varphi(t)dt \leq S_{n+1}$ for $n \geq 1$.
From (10), we have

$$\int_0^{d_1} \varphi(t)dt \leq G\left(\int_0^{d(Sx_0,Tx_1)} \varphi(t)dt\right) = G(S_1) = S_2$$

and the desired inequality is valid for $n = 1$. So, assume that it is true for some $n > 1$. From (10) again, we have

$$\int_0^{d_n} \varphi(t)dt \leq G\left(\int_0^{d_{n-1}} \varphi(t)dt\right) \leq G(S_n) = S_{n+1}.$$

Since $\sum_{n=1}^{\infty} S_n$ is convergent, it follows that $\sum_{n=1}^{\infty} \int_0^{d_n} \varphi(t)dt$ is also convergent. By (3), the series $\sum_{n=1}^{\infty} d_n$ converges.

Since $A(X)$ is d-complete, then the sequence $\{y_n\}$ converges to some u in X. Hence, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to u.

Let $Av = u$ for some v in X. Putting $x = v$ and $y = x_{2n-1}$ in (2), we have

$$\int_0^{d(Sv,y_{2n})} \varphi(t)dt = \int_0^{d(Sv,Tx_{2n-1})} \varphi(t)dt \leq G\left(\int_0^{M(v,x_{2n-1})} \varphi(t)dt\right), \tag{11}$$

where

$$\begin{aligned} M(v, x_{2n-1}) &= \max\{d(Av, Bx_{2n-1}), d(Sv, Av), d(Tx_{2n-1}, Bx_{2n-1})\} \\ &= \max\{d(u, y_{2n-1}), d(Sv, u), d(y_{2n}, y_{2n-1})\}. \end{aligned}$$

Using above inequality in (11) and letting $n \to \infty$, we have

$$\int_0^{d(Sv,u)} \varphi(t)dt \leq G\left(\int_0^{max\{d(u,u),d(Sv,u),d(u,u)\}} \varphi(t)dt\right)$$

which implies that

$$\int_0^{d(Sv,u)} \varphi(t)dt \leq G\left(\int_0^{d(Sv,u)} \varphi(t)dt\right) < \int_0^{d(Sv,u)} \varphi(t)dt,$$

which is a contradiction. Hence from (3), $Sv = u$. This proves (i)

Since $S(X) \subseteq B(X), Sv = u$ implies that $u \in B(X)$. Let $w \in B^{-1}u$. Then $Bw = u$. By using the argument of previous part of the proof, it can be easily verified that $Tw = u$. This proves (ii).

The same result holds if we assume that $B(X)$ is d-complete instead of $A(X)$. Now, if $T(X)$ is d-complete, then by (1), $u \in T(X) \subseteq A(X)$.

Similarly, if $S(X)$ is d-complete, then $u \in S(X) \subseteq B(X)$.

Thus (i) and (ii) are completely established.

To prove (iii), suppose that the pairs $\{A, S\}$, $\{B, T\}$ are weakly compatible and

$$u = Sv = Av = Tw = Bw, \tag{12}$$

then

$$\begin{cases} Au = ASv = SAv = Su \\ Bu = BTw = TBw = Tu. \end{cases} \tag{13}$$

If $Tw \neq w$, then from (2), (12) and (13), we have

$$\begin{aligned} \int_0^{d(u,Tu)} \varphi(t)dt &= \int_0^{d(Sv,Tu)} \varphi(t)dt \leq G\left(\int^{M(v,u)} \varphi(t)dt\right) \\ &= G\left(\int_0^{d(u,Tu)} \varphi(t)dt\right) < \int_0^{d(u,Tu)} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Hence $Tu = u$. Similarly $Su = u$. Then, evidently from (13), u is a common fixed point of A, B, S and T.

To prove its uniqueness, let us suppose that z is another common fixed point of A, B, S and T. Then by (2), we have

$$\int_0^{d(u,z)} \varphi(t)dt = \int_0^{d(Su,Tz)} \varphi(t)dt \leq G\left(\int_0^{d(u,z)} \varphi(t)dt\right)$$

which implies that $\int_0^{d(u,z)} \varphi(t)dt = 0$, which from (3) implies that $d(u, z) = 0$ or $u = z$. Therefore u is a unique common fixed point of A, B, S and T. □

Theorem 2.2. *Let (X, τ) be a Hausdorff topological space, A, B, S and T be w-continuous self-mappings of X satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{14}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{15}$$

for all x, $y \in X$, where φ and G are as in Theorem 2.1 and

(16)
$$M(x,y) = \max\left\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}.$$

Suppose if one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a d-complete topological subspace of X, then

(i) *A and S have a coincidence point,*

(ii) *B and T have a coincidence point.*

Further if the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then

(iii) *A, B, S and T have a unique common fixed point.*

Proof. It follows easily from the basis of Theorem 2.1. □

Theorem 2.3. *Let (X, τ) be a Hausdorff topological space, A, B, S and T be w-continuous self-maps defined on X satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{17}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{18}$$

for all x, $y \in X$, where φ and G as in Theorem 2.1 and

$$M(x,y) = \max\{d(Ax,By), d(Sx,Ax), d(Ty,By)\}. \tag{19}$$

Suppose if one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a d-complete topological subspace of X and the pairs $\{A,S\}$ and $\{B,T\}$ are semi-compatible, then A, B, S and T have a unique common fixed point.

Proof. It follows easily if we take semi-compatible mappings instead of weakly compatible mappings in Theorem 2.1. □

Theorem 2.4. *Let A, B, S and T be self-maps defined on a Hausdorff topological space (X, τ) satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{20}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{21}$$

for all x, $y \in X$, where φ and G are as in Theorem 2.1 and

(22)
$$M(x,y) = \max\left\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}.$$

Suppose if one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a d-complete topological subspace of X and the pairs $\{A,S\}$ and $\{B,T\}$ are semi-compatible, then A, B, S and T have a unique common fixed point.

Proof. It follows easily if we take semi-compatible mappings instead of weakly compatible mappings in Theorem 2.2. □

Corollary 2.1. *Let A and S be w-continuous self-maps defined on a Hausdorff topological space (X, τ) satisfying the following conditions*:

$$S(X) \subseteq A(X), \tag{23}$$

$$\int_0^{d(Sx,Sy)} \varphi(t)dt \le G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{24}$$

for all x, $y \in X$, where φ and G are as in Theorem 2.1 and

(25)
$$M(x,y) = \max\left\{d(Ax,Ay), d(Ax,Sx), d(Ay,Sy), \frac{d(Ax,Sy)+d(Ay,Sx)}{2}\right\}.$$

Suppose if $A(X)$ or $S(X)$ is a d-complete topological subspace of X, then

(i) *A and S have a coincidence point.*

Further if the pair $\{A,S\}$ is weakly compatible, then

(iii) *A and S have a unique common fixed point.*

Proof. It follows from Theorem 2.2 when B and T are identity maps on X. □

Corollary 2.2. *Let A and S be w-continuous self-maps defined on a Hausdorff topological space (X, τ) satisfying the following conditions*:

$$S(X) \subseteq A(X), \tag{26}$$

$$\int_0^{d(Sx,Sy)} \varphi(t)dt \le G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{27}$$

for all x, $y \in X$, where φ and G are as in Theorem 2.1 and

(28)
$$M(x,y) = \max\left\{d(Ax,Ay), d(Ax,Sx), d(Ay,Sy), \frac{d(Ax,Sy)+d(Ay,Sx)}{2}\right\}.$$

Suppose if $A(X)$ or $S(X)$ is a d-complete topological subspace of X and the pair $\{A,S\}$ is semi-compatible, then A and S have a unique common fixed point.

Proof. It follows from Theorem 2.4 when B and T are identity maps on X. □

Remark 2.1. If we take $S = T$ and $A = B$ in Theorem 2.1, then we have Theorem 3 of [5].

Remark 2.2. If we take $A = B = S = T$ in Theorem 2.1, then we have Theorem 2 of [5].

Remark 2.3. If we take $\varphi(t) = 1$ and $A = B = S = T$ in Theorem 2.1, then we have Theorem 1.1.

Remark 2.4. If we take a complete metric space instead of Hausdorff d-complete topological space in Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4, we have the following theorems. Note that the condition (3) has been weakened in these theorems, but we have changed the conditions of the function G.

We need the following lemma for the proofs of these theorems.

Lemma 2.1. *Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be a right continuous function such that $G(t) < t$ for every $t > 0$. Then* $\lim_{n\to\infty} G^n(t) = 0$.

Theorem 2.5. *Let A, B, S and T be self-maps defined on a metric space (X, d) satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{29}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{30}$$

for all x, y $\in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon > 0, \tag{31}$$

$$M(x, y) = \max\{d(Ax, By), d(Sx, Ax), d(Ty, By)\} \tag{32}$$

and $G : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous and nondecreasing function such that $G(0) = 0$ and $G(t) < t$ for each $t > 0$. If one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then

(i) *A and S have a coincidence point,*

(ii) *B and T have a coincidence point.*

Further if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then

(iii) *A, B, S and T have a unique common fixed point.*

Theorem 2.6. *Let (X, d) be a metric space, A, B, S and T be self-mappings of X satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{33}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{34}$$

for all $x,\ y \in X$, where φ and G are as in Theorem 2.5 and

(35)

$$M(x,y) = \max\left\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}.$$

Suppose if one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then

(i) *A and S have a coincidence point,*

(ii) *B and T have a coincidence point.*

Further if the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then

(iii) *A, B, S and T have a unique common fixed point.*

Theorem 2.7. *Let (X,d) be a metric space, A, B, S and T be self-maps defined on X satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{36}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{37}$$

for all $x,\ y \in X$, where φ and G are as in Theorem 2.5 and

$$M(x,y) = \max\{d(Ax,By), d(Sx,Ax), d(Ty,By)\}. \tag{38}$$

Suppose if one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a d-complete topological subspace of X and the pairs $\{A,S\}$ and $\{B,T\}$ are semi-compatible, then A, B, S and T have a unique common fixed point.

Theorem 2.8. *Let A, B, S and T be self-maps defined on a metric space (X,d) satisfying the following conditions*:

$$S(X) \subseteq B(X), \quad T(X) \subseteq A(X), \tag{39}$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le G\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{40}$$

for all $x,\ y \in X$, where φ and G are as in Theorem 2.5 and

(41)

$$M(x,y) = \max\left\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}.$$

Suppose if one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X and the pairs $\{A,S\}$ and $\{B,T\}$ are semi-compatible, then A, B, S and T have a unique common fixed point.

Remark 2.5. If we take $S = T$ and $A = B$ in Theorem 2.5, then we have Theorem 5 of [5].

Remark 2.6. If we take $A = B = S = T$ in Theorem 2.5, then we have Theorem 4 of [5].

Remark 2.7. If we take $\varphi(t) = 1$ and $A = B = S = T$ in Theorem 2.5, then we have a generalisation of the main theorem of [12].

Remark 2.8. Theorem 2.6 is a generalisation of the main theorem of [7], Theorem 2 of [22] and Theorem 2 of [30].

Remark 2.9. If $\varphi(t) \equiv 1$, then Theorem 2.6 of this paper reduces to Theorem 2.1 of [27].

Remark 2.10. If $\varphi(t) \equiv 1$ and $G = ht$, $0 \leq h < 1$, then Theorem 2.6 of this paper reduces to Corollary 3.1 of [8].

The following example shows that Theorem 2.6 is a generalisation of Corollary 3.1 of [8].

Example 2.1. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with Euclidean metric and A, B, S and T are self mappings on X defined by

$$S(\frac{1}{n}) = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd,} \\ \frac{1}{n+2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n = \infty, \end{cases} \quad T(\frac{1}{n}) = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is even,} \\ \frac{1}{n+2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n = \infty, \end{cases}$$

$$A(\frac{1}{n}) = B(\frac{1}{n}) = \frac{1}{n} \ \forall n \in \mathbb{N} \cup \{\infty\}.$$

Clearly $S(X) \subseteq B(X), T(X) \subseteq A(X), A(X)$ is a complete subspace of X and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

Now suppose that the contractive condition of Corollary 3.1 of [8] is satisfying, that is, there exists $h \in [0, 1)$ such that

$$d(Sx, Ty) \leq hM(x, y) \tag{42}$$

for all $x, y \in X$. Therefore, for $x \neq y$, we have

$$\frac{d(Sx, Ty)}{M(x, y)} \leq h < 1,$$

but since $\sup_{x \neq y}(d(Sx, Ty)/M(x, y)) = 1$, one has a contradiction. Thus the condition (42) is not satisfied.

Now we define $\varphi(t) = \max\{0, t^{\frac{1}{t-2}}[1 - \log t]\}$ for $t > 0$, $\varphi(0) = 0$. Then for any $\tau \in (0, e)$,

$$\int_0^\tau \varphi(t)dt = \tau^{\frac{1}{\tau}}.$$

Thus we must show that there exists a right continuous function $G : \mathbb{R}^+ \to \mathbb{R}^+, G(s) < s$ for $s > 0, G(0) = 0$ such that

$$(d(Sx, Ty))^{\frac{1}{d(Sx,Ty)}} \leq G(M(x, y)^{\frac{1}{M(x,y)}}) \tag{43}$$

for all $x, y \in X$. Now we claim that (43) is satisfying with $G(s) = \frac{s}{2}$, that is,

$$(d(Sx, Ty))^{\frac{1}{d(Sx,Ty)}} \leq \frac{1}{2}(M(x, y)^{\frac{1}{M(x,y)}}) \tag{44}$$

for all $x, y \in X$. Since the function $\tau \to \tau^{\frac{1}{\tau}}$ is nondecreasing, we show sufficiently that

$$(d(Sx,Ty))^{\frac{1}{d(Sx,Ty)}} \le \frac{1}{2}(d(x,y)^{\frac{1}{d(x,y)}}) \tag{45}$$

instead of (44). Now using Example 4 of [30], we have (45), thus the condition (43) is satisfied.

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Ramesh Chandra Dimri
Department of Mathematics
H.N.B. Garhwal University
Srinagar (Garhwal), Uttarakhand 246174, India
E-mail address: dimrirc@gmail.com

Amit Singh
Department of Mathematics
H.N.B. Garhwal University
Srinagar (Garhwal), Uttarakhand 246174, India
E-mail address: singhamit841@gmail.com