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FUZZY p-IDEALS OF BCI-ALGEBRAS WITH DEGREES IN THE INTERVAL $(0, 1]$

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Abstract. The notion of an enlarged p-ideal and a fuzzy p-ideal in BCI-algebras with degree are introduced. Related properties of them are investigated.

1. Introduction

The concept of a fuzzy set is applied to generalize some of the basic concepts of general topology ([1]). Rosenfeld ([6]) constituted a similar application to the elementary theory of groupoids and groups. Xi ([7]) applied to the concept of fuzzy set to BCK-algebras. Y. B. Jun and J. Meng ([4]) introduced of fuzzy p-ideals in BCI-algebras and studied their properties.

In this paper, we introduce the notion of an enlarged p-ideal and a fuzzy p-ideal in BCI-algebras with degree. We study related properties of them.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)(x * x = 0)$,
(a4) $(\forall x, y \in X)(x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)(0 * x = 0)$,

then X is called a BCK-algebra.

In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

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(b1) $(\forall x \in X)(x * 0 = x)$.
(b2) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$.
(b3) $(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y))$.
(b4) $(\forall x, y \in X)(x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)(x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.
(b6) $(\forall x, y, z \in X)((x * z) * (y * z) \leq x * y)$.
(b7) $(\forall x, y, z \in X)(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$.
(b8) $(\forall x, y \in X)(0 * (0 * (x * y)) = (0 * y) * (0 * x))$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in A)(\forall y \in X)(x * y \in A \Rightarrow x \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in A)\,(\forall y \in X)\,(x \leq y \Rightarrow x \in A).$$

A non-empty subset A of a BCI-algebra X is called a *p-ideal* ([9]) of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)((x * z) * (y * z) \in A$ and $y \in A \Rightarrow x \in A)$.

Note that any p-ideal is an ideal, but the converse is not true in general.

We refer the reader to the book [2] for further information regarding BCI-algebras.

A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy ideal* ([4]) of X if it satisfies:

(d1) $(\forall x \in X)(\mu(0) \geq \mu(x))$,
(d2) $(\forall x, y \in X)(\mu(x) \geq \min\{\mu(x * y), \mu(y)\})$.

A fuzzy subset μ of a BCI-algebra X is called a *fuzzy p-ideal* ([4]) of X if it satisfies (d1) and

(d3) $(\forall x, y, z \in X)(\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$.

3. Fuzzy p-ideals of BCI-algebras with degrees in the interval $(0, 1]$

In what follows let X denote a BCI-algebra unless specified otherwise.

Definition 3.1 ([5])**.** Let I be a non-empty subset of a BCK/BCI-algebra X which is not necessary an ideal of X. We say that a subset J of X is an *enlarged ideal* of X related to I if it satisfies:

(1) I is a subset of J,
(2) $0 \in J$,
(3) $(\forall x \in X)(\forall y \in I)(x * y \in I \Rightarrow x \in J)$.

Definition 3.2. Let I be a non-empty subset of a BCI-algebra X which is not necessary a p-ideal of X. We say that a subset J of X is an *enlarged p-ideal* of X related to I if it satisfies:

(1) I is a subset of J,
(2) $0 \in J$,
(3) $(\forall x, y, z \in X)((x * z) * (y * z) \in I$ and $y \in I \Rightarrow x \in J)$.

Obviously, every p-ideal is an enlarged p-ideal of X related to itself. Note that there exists an enlarged p-ideal of X related to any non-empty subset I of a BCI-algebra X.

Example 3.3. Let $X := \{0, a, b, c\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $\{0, a, b\}$ is an enlarged p-ideal of X related to $\{0\}$. But $\{0, a, b\}$ is not a p-ideal since $(c * a) * (b * a) = b * c = a, b \in \{0, a, b\}$, and $c \notin \{0, a, b\}$.

Theorem 3.4. *Let I be a non-empty subset of a BCI-algebra X. Every enlarged p-ideal of X related to I is an enlarged ideal of X related to I.*

Proof. Let J be an enlarged p-ideal of X related to I. Putting $z := 0$ in Definition 3.2(3), we have

$$(\forall x, y \in X)((x * 0) * (y * 0) = x * y \in I \text{ and } y \in I \Rightarrow x \in J).$$

Hence J is an enlarged ideal of X related to I. □

The converse of Theorem 3.4 does not true in general as seen in the following example.

Example 3.5. Let $X := \{0, 1, 2, 3, 4\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	2	1	0	2
4	4	4	4	4	0

Note that $\{0, 1, 3\}$ is not both an ideal and a p-ideal of X. Then $\{0, 1, 2, 3\}$ is an enlarged ideal of X related to $\{0, 1, 3\}$ but not an enlarged p-ideal of X related to $\{0, 1, 3\}$ since $(4 * 4) * (3 * 4) = 0, 3 \in \{0, 1, 3\}$ and $4 \notin \{0, 1, 3\}$.

In what follows let λ and κ be members of $(0, 1]$, and let n and k denote a natural number and a real number, respectively, such that $k < n$ unless otherwise specified.

Definition 3.6 ([5])**.** A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy ideal* of X with degree (λ, κ) if it satisfies:

(1) $(\forall x \in X)(\mu(0) \geq \lambda\mu(x))$,
(2) $(\forall x, y \in X)(\mu(x) \geq \kappa\min\{\mu(x * y), \mu(y)\})$.

Definition 3.7. A fuzzy subset μ of a BCI-algebra X is called a *fuzzy p-ideal* of X with degree (λ, κ) if it satisfies:

(1) $(\forall x \in X)(\mu(0) \geq \lambda\mu(x))$,
(2) $(\forall x, y \in X)(\mu(x) \geq \kappa\min\{\mu((x * z) * (y * z)), \mu(y)\})$.

Note that if $\lambda \neq \kappa$, then a fuzzy p-ideal with degree (λ, κ) may not be a fuzzy p-ideal with degree (κ, λ), and vice versa.

Example 3.8. Let $X := \{0, a, 1, 2, 3\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 0 & a & 1 & 2 & 3 \\ 0.8 & 0.6 & 0.5 & 0.5 & 0.5 \end{pmatrix}.$$

Then μ is a fuzzy p-ideal of X with degree $(\frac{4}{7}, \frac{4}{7})$, but it is neither a fuzzy p-ideal of X nor a fuzzy p-ideal of X with degree $(\frac{4}{5}, \frac{4}{5})$ since

$$\mu(a) = 0.6 \ngeq \min\{\mu((a * 1) * (0 * 1)), \mu(0)\}$$

and

$$\mu(a) = 0.6 \ngeq \frac{4}{5} \times 0.8 = \frac{4}{5}\min\{\mu((a * 1) * (0 * 1)), \mu(0)\}.$$

Obviously, every fuzzy p-ideal is a fuzzy p-ideal with degree (λ, κ), but the converse may not be true. In fact, the fuzzy p-ideal μ with degree $(\frac{5}{7}, \frac{4}{7})$ in Example 3.8 is not a fuzzy p-ideal of X. Note that a fuzzy p-ideal with degree (λ, κ) is a fuzzy p-ideal if and only if $(\lambda, \kappa) = (1, 1)$. If $\lambda_1 \geq \lambda_2$ and $\kappa_1 \geq \kappa_2$, then every fuzzy p-ideal with degree (λ_1, κ_1) is a fuzzy p-ideal with (λ_2, κ_2), but the converse is not true as shown by Example 3.8.

Proposition 3.9. *If μ is a fuzzy p-ideal of a BCI-algebra X with degree (λ, κ), then μ is a fuzzy ideal of X with degree (λ, κ).*

Proof. Put $z := 0$ in Definition 3.7(2). □

The converse of Proposition 3.9 is not true in general as seen the following example.

Example 3.10. Consider a BCI-algebra X and a fuzzy subset μ as in Example 3.8. It is routine to check that μ is a fuzzy ideal of X with degree $(\frac{4}{7}, \frac{4}{5})$. But it is not a fuzzy p-ideal of X with degree $(\frac{4}{7}, \frac{4}{5})$ since

$$\mu(a) = 0.6 \ngeq 0.64 = \frac{4}{5} \times 0.8 = \frac{4}{5}\min\{\mu((a * 1) * (0 * 1)), \mu(0)\}.$$

Proposition 3.11. *If μ is a fuzzy p-ideal of a BCI-algebra X with degree (λ, κ), then the following hold*:

(1) $(\forall x, y \in X)(x \leq y \Rightarrow \mu(x) \geq \lambda\kappa\mu(y))$,
(2) $(\forall x \in X)(\mu(x) \geq \lambda\kappa \min\{\mu(0 * (0 * x)), \mu(x)\})$.

Proof. (1) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. Putting $z := 0$ in Definition 3.7(2) and using (b1), we have

$$\begin{aligned}\mu(x) &\geq \kappa\min\{\mu((x * 0) * (y * 0)), \mu(y)\}\\ &= \kappa\min\{\mu(x * y), \mu(y)\}\\ &= \kappa\min\{\mu(0), \mu(y)\}\\ &\geq \kappa\min\{\lambda\mu(y), \mu(y)\}\\ &= \lambda\kappa\mu(y).\end{aligned}$$

(2) For any $x \in X$, we have

$$\begin{aligned}\mu(x) &\geq \kappa\min\{\mu((x * x) * (0 * x)), \mu(0)\}\\ &= \kappa\min\{\mu((0 * (0 * x)), \mu(0)\}\\ &\geq \kappa\min\{\mu((0 * (0 * x)), \lambda\mu(x)\}\\ &\geq \kappa\min\{\lambda\mu((0 * (0 * x)), \lambda\mu(x)\}\\ &= \lambda\kappa\min\{\mu((0 * (0 * x)), \mu(x)\}.\end{aligned}$$ □

Theorem 3.12. *If μ is a fuzzy p-ideal of a BCI-algebra X with degree (λ, κ), then the following hold*:

$$(\forall x, y, z \in X)(\mu((x * z) * (y * z)) \geq \kappa\lambda\mu(x * y)).$$

Proof. Let $x, y, z \in X$. Using (a1) and (b2), the inequality $(x*z)*(y*z) \leq x*y$ holds. By Proposition 3.11(1), we have $\mu((x * z) * (y * z)) \geq \lambda\kappa\mu(x * y)$. This completes the proof. □

Theorem 3.13. *Let μ be a fuzzy ideal of a BCI-algebra X with degree (λ, κ). If μ satisfies $\mu(x * y) \geq \mu((x * z) * (y * z))$ for all $x, y, z \in X$, then μ is a fuzzy p-ideal of X with degree (λ, κ).*

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned}\mu(x) &\geq \kappa\min\{\mu(x * y), \mu(y)\}\\ &\geq \kappa\min\{\mu((x * z) * (y * z)), \mu(y)\}.\end{aligned}$$

Thus μ is a fuzzy p-ideal of X with degree (λ, κ). □

Lemma 3.14. *Let μ be a fuzzy ideal of a BCI-algebra X with degree (λ, κ). Then*

$$\mu(0 * (0 * x)) \geq \kappa\lambda\mu(x), \forall x \in X.$$

Proof. For any $x \in X$, we have

$$\begin{aligned}
\mu(0 * (0 * x)) &\geq \kappa\min\{\mu((0 * (0 * x)) * x), \mu(x)\} \\
&= \kappa\min\{\mu(0), \mu(x)\} \\
&\geq \kappa\min\{\lambda\mu(x), \mu(x)\} \\
&\geq \kappa\lambda\min\{\mu(x), \mu(x)\} \\
&= \kappa\lambda\mu(x).
\end{aligned}$$

□

Theorem 3.15. *Let μ be a fuzzy ideal of a BCI-algebra X with degree (λ, κ) satisfying*

$$\mu(0 * (0 * x)) \leq \kappa\lambda\mu(x) \text{ for all } x \in X.$$

Then μ is a fuzzy p-ideal of a BCI-algebra X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using Lemma 3.14, (b7) and (b8), we have

$$\begin{aligned}
\mu((x * z) * (y * z)) &\leq \frac{1}{\kappa\lambda}\mu(0 * (0 * ((x * z) * (y * z)))) \\
&= \frac{1}{\kappa\lambda}\mu((0 * y) * (0 * x)) \\
&= \frac{1}{\kappa\lambda}\mu(0 * (0 * (x * y))) \\
&\leq \frac{1}{\kappa\lambda}\kappa\lambda\mu(x * y) \\
&= \mu(x * y).
\end{aligned}$$

By Theorem 3.13, μ is a fuzzy p-ideal of X with degree (λ, κ). □

Denote by $\mathcal{I}(X)$ and $\mathcal{I}_p(X)$ the set of all ideals and p-ideals of a BCI-algebra X, respectively. Note that a fuzzy subset μ of a BCI-algebra X is a fuzzy p-ideal of X if and only if

$$(\forall t \in [0, 1])(U(\mu; t) \in \mathcal{I}_p(X) \cup \{\emptyset\}).$$

But we know that for any fuzzy subset μ of a BCI-algebra X there exist $\lambda, \kappa \in (0, 1)$ and $t \in [0, 1]$ such that

(1) μ is a fuzzy p-ideal of X with degree (λ, κ),
(2) $U(\mu; t) \notin \mathcal{I}_p(X) \cup \{\emptyset\}$.

Example 3.16. Consider a BCI-algebra $X = \{0, a, 1, 2, 3\}$ as in Example 3.8. Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 0 & a & 1 & 2 & 3 \\ 0.7 & 0.6 & 0.7 & 0.5 & 0.5 \end{pmatrix}.$$

Then μ is a fuzzy p-ideal of X with degree $(0.6, 0.7)$. If $t \in (0.6, 0.7]$, then $U(\mu; t) = \{0, 1\}$ is not a p-ideal of X since $(a * 1) * (0 * 1) = 0 \in \{0, 1\}$ and $a \notin \{0, 1\}$.

Theorem 3.17. *Let μ be a fuzzy subset of a BCI-algebra X. For any $t \in [0, 1]$ with $t \leq \max\{\lambda, \kappa\}$, if $U(\mu; t)$ is an enlarged p-ideal of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, then μ is a fuzzy p-ideal of X with degree (λ, κ).*

Proof. Assume that $\mu(0) < t \leq \lambda\mu(x)$ for some $x \in X$ and $t \in (0, \lambda]$. Then $\mu(x) \geq \frac{t}{\lambda} \geq \frac{t}{\max\{\lambda,\kappa\}}$ and so $x \in U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, i.e., $U(\mu; \frac{t}{\max\{\lambda,\kappa\}}) \neq \emptyset$. Since $U(\mu; t)$ is an enlarged p-ideal of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, we have $0 \in U(\mu; t)$, i.e., $\mu(0) \geq t$. This is a contradiction, and thus $\mu(0) \geq \lambda\mu(x)$ for all $x \in X$.

Now suppose that there exist $a, b, c \in X$ such that $\mu(a) < \kappa\min\{\mu((a * c) * (b * c)), \mu(b)\}$. If we take $t := \kappa\min\{\mu((a * c) * (b * c)), \mu(b)\}$, then $t \in (0, \kappa] \subseteq (0, \max\{\lambda, \kappa\}]$. Hence $(a * c) * (b * c) \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$ and $b \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$. It follows from Definition 3.2(3) that $a \in U(\mu; t)$ so that $\mu(a) \geq t$, which is impossible. Therefore

$$\mu(x) \geq \kappa\min\{\mu((x * z) * (y * z)), \mu(y)\}$$

for all $x, y, z \in X$. Thus μ is a fuzzy p-ideal of X with degree (λ, κ). □

Corollary 3.18. *Let μ be a fuzzy subset of a BCI-algebra X. For any $t \in [0, 1]$ with $t \leq \frac{k}{n}$, if $U(\mu; t)$ is an enlarged p-ideal of X related to $U(\mu; \frac{n}{k}t)$, then μ is a fuzzy p-ideal of X with degree $(\frac{k}{n}, \frac{k}{n})$.*

Theorem 3.19. *Let $t \in [0, 1]$ be such that $U(\mu; t)(\neq \emptyset)$ is not necessary a p-ideal of a BCI-algebra X. If μ is a fuzzy p-ideal of X with degree (λ, κ), then $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged p-ideal of X related to $U(\mu; t)$.*

Proof. Since $t\min\{\lambda, \kappa\} \leq t$, we have $U(\mu; t) \subseteq U(\mu; t\min\{\lambda, \kappa\})$. Since $U(\mu; t) \neq \emptyset$, there exists $x \in U(\mu; t)$ and so $\mu(x) \geq t$. By Definition 3.7(1), we obtain $\mu(0) \geq \lambda\mu(x) \geq \lambda t \geq t\min\{\lambda, \kappa\}$. Therefore $0 \in U(\mu; t\min\{\lambda, \kappa\})$.

Let $x, y, z \in X$ be such that $(x * z) * (y * z) \in U(\mu; t)$ and $y \in U(\mu; t)$. Then $\mu((x * z) * (y * z)) \geq t$ and $\mu(y) \geq t$. It follows from Definition 3.7(2) that

$$\begin{aligned}\mu(x) &\geq \kappa\min\{\mu((x * z) * (y * z)), \mu(y)\} \\ &\geq \kappa t \geq t\min\{\lambda, \kappa\},\end{aligned}$$

so that $x \in U(\mu; t\min\{\lambda, \kappa\})$. Thus $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged p-ideal of X related to $U(\mu; t)$. □

References

[1] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
[2] Y. Huang, *BCI-Algebras*, Science Press, Beijing, 2006.
[3] K. Iśeki, *On BCI-algebras*, Math. Math. Sem. Notes Kobe Univ. **8** (1980), 125–130.

[4] Y. B. Jun and J. Meng, *Fuzzy p-ideals in BCI-algebras*, Math. Japon. **40** (1994), no. 2, 271–282.
[5] Y. B. Jun, E. H. Roh, and K. J. Lee, *Fuzzy subalgebras and ideals of BCK/BCI-algebras with degree in the interval* $(0, 1]$, Fuzzy Sets and Systems, submitted.
[6] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
[7] O. G. Xi, *Fuzzy BCK-algebras*, Math. Japon. **36** (1991), no. 5, 935–942.
[8] L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.
[9] X. Zhang, J. Hao, and S. A. Bhatti, *On p-ideals of a BCI-algebra*, Punjab Univ. J. Math. **27** (1994), 121–128.

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