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E-INVERSIVE $*$-SEMIGROUPS

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Abstract. $(S,*)$ is a semigroup S equipped with a unary operation "$*$". This work is devoted to a class of unary semigroups, namely *E-inversive $*$-semigroups.* A unary semigroup $(S,*)$ is called an E-inversive $*$-semigroup if the following identities hold:

$$x^*xx^* = x^*, (x^*)^* = xx^*x, (xy)^* = y^*x^*.$$

In this paper, E-inversive $*$-semigroups are characterized by several methods. Furthermore, congruences on these semigroups are also studied.

1. Introduction and preliminaries

A *semigroup* is a nonempty set S with an associative binary operation "$\cdot$" on S. A *unary semigroup* $(S,*)$ is a semigroup S equipped with a unary operation "$*$" on S. A class of unary semigroups $\mathcal{U}$ is called a *variety of unary semigroups* if there exists a family $\mathcal{J}$ of identities such that $\mathcal{U}$ consists of all semigroups which satisfy each identities in $\mathcal{J}$. A variety of unary semigroups $\mathcal{V}$ is *a subvariety* of the variety $\mathcal{U}$ of unary semigroups if $\mathcal{U}$ and $\mathcal{V}$ are varieties of unary semigroups of the same type unary semigroups and $\mathcal{V} \subseteq \mathcal{U}$. In this case, we denote $\mathcal{V} \leq \mathcal{U}$.

An element e in a semigroup S is called *idempotent* if $e^2 = e$, and the set of idempotents in S is denoted by $E(S)$ as usual. For a semigroup S and an element x in S,

$$V(x) = \{a \in S \mid axa = a, xax = x\}$$

and

$$W(x) = \{a \in S \mid axa = a\},$$

are called the *set of weak inverses* and the *set of inverses* of x, respectively. A semigroup S is *regular* if $V(x) \neq \emptyset$ for all x in S. On the other hand, from Weipoltshammer [10], a semigroup S is called an *E-inversive semigroup* if $W(x) \neq \emptyset$ for all x in S. Thus, regular semigroups are E-inversive semigroups.

Some important classes of regular semigroups, such as *inverse semigroups* and *completely regular semigroups*, can be regarded as varieties of regular unary

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semigroups. Inverse semigroups and completely regular semigroups are investigated extensively and a lot of remarkable results are obtained, see the books Howie [4], Petrich [7] and Petrich-Reilly [8].

As a generalization of inverse semigroups, Nordahl-Scheiblich [6] introduced *regular $*$-semigroups* in 1978. From Nordahl-Scheiblich [6], a unary semigroup $(S, *)$ is called *a regular $*$-semigroup* if the following identities hold:

$$x^*xx^* = x^*,\ (x^*)^* = x,\ (xy)^* = y^*x^*.$$

Thus, regular $*$-semigroups are regular and the class of regular $*$-semigroups forms a variety of unary semigroups, denoted by $\mathbb{R}^*$.

Regular $*$-semigroups are studied by several researchers. In particular, Yamada [9] characterized regular $*$-semigroups by using $\mathbf{p}$-*systems*, and Chae-Lee-Park [1] and Imaoka [5] considered the congruences on regular $*$-semigroups.

On the other hand, many papers are also devoted to E-inversive semigroups, such as Fan-Chen [2], Gao-Yu [3] and Weipoltshammer [10]. Observe that E-inversive semigroups are generalizations of regular semigroups. Naturally, one would ask: can we define a class of unary semigroups in the class of E-inversive semigroups and establish some analogous results of regular $*$-semigroups? The present work is an attempt in this line.

In this paper, we introduce *E-inversive $*$-semigroups* in E-inversive semigroups which are unary semigroups analogous to regular $*$-semigroups in regular semigroups, and characterize these semigroups by several methods. Furthermore, the lattices of congruences on E-inversive $*$-semigroups are also investigated. Our results enrich some results of regular $*$-semigroups in Chae-Lee-Park [1], Imaoka [5], Nordahl-Scheiblich [6] and Yamada [9].

2. Some characterizations of E-inversive $*$-semigroups

In this section, we introduce *E-inversive $*$-semigroups* and give some characterizations of these semigroups.

Definition 2.1. A unary semigroup $(S, *)$ is called *an E-inversive $*$-semigroup* if the following identities hold:

$$x^*xx^* = x^*,\ x^{**} = xx^*x,\ (xy)^* = y^*x^*,$$

where $x^{**} = (x^*)^*$.

Remark 2.2. E-inversive $*$-semigroups are E-inversive from the first identity above. On the other hand, E-inversive $*$-semigroups form a variety of unary semigroups, denoted by $\mathbb{E}^*$. It is easy to see that $\mathbb{R}^* \leq \mathbb{E}^*$. Furthermore, if $(S, *) \in \mathbb{E}^*$ and $S^* = \{x^* \mid x \in S\}$, then $(S^*, *) \in \mathbb{R}^*$.

The following examples show that there exist E-inversive $*$-semigroups which are not regular semigroups.

Example 2.3. Let $\mathbb{N}$ be the semigroup of non-negative integers under the usual multiplication of integers. Define a unary operation "$*$" on S by $x^* = 0$ for all $x \in \mathbb{N}$. Then $(\mathbb{N}, *) \in \mathbb{E}^*$.

Example 2.4. Let S be a semigroup with *Cayley* table:

S	a	e	f	b
a	e	a	a	e
e	a	e	e	a
f	a	e	f	a
b	e	a	b	e

Define $* : \ S \to S, \ a \mapsto a, \ b \mapsto a, \ e \mapsto e, \ f \mapsto f$. Then $(S, *) \in \mathbb{E}^*$ and S is non-regular.

The example below shows that there exists an E-inversive (regular) semigroup S such that $(S, *) \notin \mathbb{E}^*$ for any unary operation "$*$" on S.

Example 2.5. Let $S = \{e, f\}$ be a left zero semigroup, i.e., $ab = a$ for any a, b in S. If $(S, *) \in \mathbb{E}^*$ for some unary operation "$*$" on S, then we have $e^* = (ef)^* = f^*e^* = f^*$. Without loss of generality, suppose that $e^* = f^* = e$. Then we have $e = (f^*)^* = ff^*f = f$, which is a contradiction.

The following basic facts will be used frequently without mention in the sequel.

Proposition 2.6. *Let $(S, *) \in \mathbb{E}^*$. Then $x^{***} = x^*$, $xx^* = x^{**}x^*$ and $x^*x = x^*x^{**}$ for any x in S.*

Proof. For $x \in S$, we have

$$x^{***} = (xx^*x)^* = x^*x^{**}x^* = x^*(xx^*x)x^* = (x^*xx^*)xx^* = x^*xx^* = x^*.$$

On the other hand,

$$x^{**}x^* = (xx^*x)x^* = x(x^*xx^*) = xx^*, x^*x^{**} = x^*(xx^*x) = (x^*xx^*)x = x^*x,$$

as required. □

We now give a characterization of E-inversive $*$-semigroups by using regular $*$-semigroups. Recall that an equivalence relation ρ on a semigroup S is called a *congruence* on S if ρ is compatible with the multiplication of S. In such a case, $x\rho$ denotes the ρ-class containing x for any x in S.

Theorem 2.7. *Let S be a semigroup. Then $(S, *) \in \mathbb{E}^*$ for some unary operation "$*$" on S if and only if there exist a subsemigroup T of S and a congruence ρ on S such that $(T, \dagger) \in \mathbb{R}^*$ for some unary operation "$\dagger$" on T and*
(1) *there is exactly one element x° in $x\rho \cap T$ for any x in S;*
(2) *$x(x^\circ)^\dagger, (x^\circ)^\dagger x \in T$ for any x in S.*

Proof. Let $(S, *) \in \mathbb{E}^*$ and $T = S^*$. Then $(T, *) \in \mathbb{R}^*$. Define $\rho = \{(x, y) \in S \times S \mid x^* = y^*\}$. It is easy to see that ρ is a congruence on S and $x\rho \cap T = \{x^{**}\}$ for every x in S. Furthermore, it follows that

$$x(x^{**})^* = xx^* = x^{**}x^* \in T, \ (x^{**})^*x = x^*x^{**} \in T$$

for any x in S.

Conversely, suppose that the given conditions are satisfied. We define a unary operation "$*$" on S as follows:

$$*: \quad S \to S, \;\; x \mapsto x^* = (x^\circ)^\dagger.$$

We assert that $(S, *) \in \mathbb{E}^*$. In fact, if $x \in S$, then $xx^* \in T$ by (2), whence $x^*xx^* \in T$. This implies that

$$(x^*xx^*)\rho = (x^*x^\circ x^*)\rho = ((x^\circ)^\dagger x^\circ (x^\circ)^\dagger)\rho = (x^\circ)^\dagger \rho = x^*\rho.$$

Observe that $x^*, x^*xx^* \in T$, $x^*xx^* = x^*$ by (1). On the other hand, let $x \in S$. Since $(x^\circ)^\dagger \in T$, we have $((x^\circ)^\dagger)^\circ = (x^\circ)^\dagger$ by (1). This implies that

$$(x^*)^* = ((x^\circ)^\dagger)^* = (((x^\circ)^\dagger)^\circ)^\dagger = ((x^\circ)^\dagger)^\dagger = x^\circ.$$

Moreover, $xx^*x = x(x^*x^\circ x^*)x = (xx^*)x^\circ(x^*x) \in T$ by (2). Observe that $x\rho = x^\circ\rho$, it follows that $(xx^*x)\rho = (x^\circ x^* x^\circ)\rho = x^\circ\rho$. Since $x^\circ, xx^*x \in T$, $x^\circ = xx^*x$ by (1). This yields that $(x^*)^* = x^\circ = xx^*x$. Finally, let $x, y \in S$. Then $xy\rho x^\circ y^\circ$, whence $(xy)^\circ = x^\circ y^\circ$. Thus, $(xy)^* = ((xy)^\circ)^\dagger = (x^\circ y^\circ)^\dagger = (y^\circ)^\dagger (x^\circ)^\dagger = y^*x^*$. □

In the following, we characterize E-inversive $*$-semigroup by so-called **wp**-*systems*. To this aim, we need some basic concepts and results. Recall from Weipoltshammer [10] that the *natural partial order* "$\leq$" on a semigroup S is defined by

$$a \leq b \text{ if } a = xb = by, \; xa = a = ay \text{ for some } x, y \in S^1,$$

the restriction of which to $E(S)$ is the usual order on $E(S)$, where S^1 is the semigroup obtained from S by adjoining an identity if necessary. In particular, if a, b are *regular elements* (i.e., both $V(a)$ and $V(b)$ are nonempty) of S, $a \leq b$ if and only if $a = eb = bf$ for some e and f in $E(S)$.

Let S be an E-inversive semigroup. From Fan-Chen [2] and Gao-Yu [3], a subset P of $E(S)$ is called a *characteristic set* of S if

(1) $P^2 \subseteq E(S)$;

(2) $(\forall q \in P)\; qPq \subseteq P$;

(3) $(\forall a \in S)(\exists a^+ \in W(a)) \;\; aP^1a^+ \subseteq P, \; a^+P^1a \subseteq P$,

where P^1 is the semigroup obtained from P by adjoining an identity if necessary. In such a case, a^+ is called a *weakly P-inverse* of a in S and the set of all weakly P-inverses of a is denoted by $W_P(a)$. Observe that $W_P(a)W_P(b) \subseteq W_P(ba)$ for all $a, b \in S$. In fact, for $a, b \in S$ and $a^+ \in W_P(a), b^+ \in W_P(b)$, we have

$$a^+b^+baa^+b^+ = a^+(aa^+b^+baa^+b^+b)b^+ = a^+aa^+b^+bb^+ = a^+b^+,$$

and

$$\begin{aligned} a^+b^+P^1ba &= a^+(b^+P^1b)a \subseteq a^+Pa \subseteq P, \\ baP^1a^+b^+ &= b(aP^1a^+)b^+ \subseteq bPb^+ \subseteq P. \end{aligned}$$

Definition 2.8. A characteristic set P of an E-inversive semigroup S is called a **wp**-system if

(1) $W_P(x)$ contains the greatest element x^* for every $x \in S$;

(2) $((xy)^*)^* = (xy)^{**} \in W_P(y^*x^*)$ for all $x, y \in S$.

Proposition 2.9. *In Definition 2.8, $W_P(x)$ contains the greatest element if and only if there is $x^* \in W_P(x)$ such that $x^+ = x^+xx^* = x^*xx^+$ for every $x^+ \in W_P(x)$.*

Proof. Let x^* be the greatest element in $W_P(x)$. Then for every $x^+ \in W_P(x)$, $x^+ \leq x^*$. Since x^+ and x^* are regular, there exist two idempotents e and f such that $x^+ = ex^* = x^*f$. This implies that

$$x^*xx^+x = x^*x(x^*f)x = (x^*xx^*)fx = x^*fx = x^+x$$

and

$$xx^+xx^* = x(ex^*)xx^* = xe(x^*xx^*) = xex^* = xx^+.$$

Thus, $x^+ = (x^+x)x^+ = (x^*xx^+x)x^+ = x^*xx^+$. Dually, $x^+ = x^+xx^*$.

Conversely, let $x^* \in W_P(x)$ such that $x^+ = x^+xx^* = x^*xx^+$ for every $x^+ \in W_P(x)$. Observe that $xx^+, x^+x \in E(S)$ and x^+, x^* are regular, $x^+ \leq x^*$. This implies that x^* is the greatest element in $W_P(x)$. □

Theorem 2.10. *Let S be an E-inversive semigroup. Then $(S, *) \in \mathbb{E}^*$ for some unary operation "$*$" on S if and only if it contains a* **wp***-system.*

Proof. Let $(S, *) \in \mathbb{E}^*$ and $P = \{xx^* \mid x \in S\}$. We assert that P is a characteristic set of S. Observe that $p^* = p$ for any $p \in P$ since $(xx^*)^* = x^{**}x^* = xx^*$ for any x in S. Now, let $p, q \in P$. Then

$$pq = p^*q^* = (qp)^* = (qp)^*qp(qp)^* = p^*q^*qpp^*q^* = pqqppq = pqpq,$$

which implies that $pq \in E(S)$. Furthermore,

$$pqp = pqqp = pqq^*p^* = (pq)(pq)^* \in P.$$

Finally, for any $x \in S$, we have $xx^* = x^{**}x^* \in P$ and $x^*x = x^*x^{**} \in P$. Moreover, for any $p \in P$,

$$xpx^* = xppx^* = xpp^*x^* = xp(xp)^* \in P$$

and

$$x^*px = x^*ppx = x^*p^*px = (px)^*px = (px)^*(px)^{**} \in P,$$

which yields that $xpx^*, x^*px \in P$. Thus, P is a characteristic set of S and $x^* \in W_P(x)$ for any $x \in S$.

Now, for any $x^+ \in W_P(x)$, we have $xx^+, x^+x \in P$. This implies that $(xx^*xx^+)^* = (xx^+)^*(xx^*)^* = xx^+xx^*$ and

$$(xx^*xx^+)^* = (x^+)^*x^*x^{**}x^* = (x^+)^*x^*xx^*xx^* = (x^+)^*x^* = (xx^+)^* = xx^+.$$

Therefore, $xx^+xx^* = xx^+$. Dually, $x^*xx^+x = x^+x$. Hence,

$$x^+ = x^+(xx^+) = x^+(xx^+xx^*) = (x^+xx^+)xx^* = x^+xx^*.$$

Similarly, $x^+ = x^*xx^+$. By Proposition 2.9, x^* is the greatest element in $W_P(x)$ for every $x \in S$. On the other hand, for any $x, y \in S$, we have $(xy)^{**} = (y^*x^*)^*$. This shows that

$$(xy)^{**}y^*x^*(xy)^{**} = (y^*x^*)^*y^*x^*(y^*x^*)^* = (y^*x^*)^* = (xy)^{**}.$$

Therefore, the condition (2) of Definition 2.8 holds. Thus, P is a **wp**-system of S.

Conversely, suppose that S contains a **wp**-system P. Define a unary operation on S by

$$* : S \to S,\ x \mapsto x^*,$$

where x^* is the greatest element of $W_P(x)$ for any x in S. We show that $(S, *) \in \mathbb{E}^*$. Evidently, $x^* = x^*xx^*$. For $x \in S$, let $t = xx^*x$. Then $tx^*t = xx^*xx^*xx^*x = t$, whence $t \in W(x^*)$. Furthermore,

$$tP^1x^* = xx^*(xP^1x^*)xx^* \subseteq xx^*Pxx^* \subseteq P.$$

Dually, $x^*P^1t \subseteq P$. Thus, $t \in W_P(x^*)$. By Definition 2.8 and Proposition 2.9, we have

$$xx^*x = t = tx^*x^{**} = (xx^*x)x^*x^{**} = x(x^*xx^*)x^{**} = xx^*x^{**}.$$

This means that

$$x^*x = (x^*xx^*)x = x^*(xx^*x) = x^*(xx^*x^{**}) = (x^*xx^*)x^{**} = x^*x^{**}.$$

Dually, $xx^* = x^{**}x^*$. Hence,

$$x^{**} = x^{**}x^*x^{**} = xx^*x^{**} = xx^*x.$$

Now, for any $x, y \in S$, we have $(xy)^{**} = xy(xy)^*xy$. By (2) of Definition 2.8, it follows that

$$xy(xy)^*xyy^*x^*(xy)(xy)^*xy = xy(xy)^*xy.$$

Noticing that $y^*x^* \in W_P(xy)$, by Proposition 2.9 and its proof, we have

$$(xy)^*xyy^*x^*xy(xy)^* = [(xy)^*xyy^*x^*]xy(xy)^* = y^*x^*xy(xy)^* = y^*x^*$$

whence $xy(xy)^*xy = xyy^*x^*xy$. This implies that

$$(xy)^* = (xy)^*xy(xy)^*xy(xy)^* = (xy)^*xyy^*x^*xy(xy)^* = y^*x^*.$$

Thus, $(S, *) \in \mathbb{E}^*$. □

In the end of this section, we characterize E-inversive semigroups with a **wp**-system P such that $|W_P(x)| = 1$ for all x in S. Such semigroups are special E-inversive $*$-semigroups by Theorem 2.10. The following result is useful.

Proposition 2.11. *Let S be an E-inversive semigroup with* **wp***-system P. Then $W_P(p)$ is a commutative sub-semigroup of S contained in P with greatest element p for each $p \in P$.*

Proof. In view of the proof of Theorem 2.10, $(S, *) \in \mathbb{E}^*$ with respect to the unary operation on S defined by

$$* : S \to S,\ x \mapsto x^*,$$

where x^* is the greatest element of $W_P(x)$ for any x in S.

We first assert that $p^* = p$ for any $p \in P$. In fact, since

$$p^* = p^*pp^* = (p^*p)(pp^*) \in PP \subseteq E(S)$$

and $p \in W_P(p)$, we have $p \leq p^*$ and so $pp^* = p^*p = p$. This implies that

$$p^* = p^*pp^* = pp^* = p.$$

Now, let $p \in P$ and $s \in W_P(p)$. Then $s \leq p^* = p$ and $s = sps = spps \in PP \subseteq E(S)$. This implies that $s = ps = sp \in P$. Thus $W_P(p) \subseteq P$. If $s_1, s_2 \in W_P(p)$, then $s_1, s_2 \in P$ and $s_1s_2 \in W_P(p)W_P(p) \subseteq W_P(p^2) = W_P(p)$ whence $s_1s_2 \in P$. This implies that

$$s_1s_2 = (s_1s_2)^* = s_2^*s_1^* = s_2s_1.$$

Thus, $W_P(p)$ is a commutative sub-semigroup of S contained in P with greatest element p for each $p \in P$. □

Example 2.12. In Example 2.4, S has a **wp**-system $P = \{e, f\}$ and $W_P(e) = \{e\}, W_P(f) = \{e, f\}$. Clearly, $W_P(e)$ and $W_P(f)$ are commutative sub-semigroups of S contained in P with greatest elements e and f, respectively.

Theorem 2.13. *Let S be an E-inversive semigroup with a* **wp**-*system P. Then the followings are equivalent*:

(1) $|W_P(x)| = 1$ *for all* $x \in S$;
(2) $|W_P(p)| = 1$ *for all* $p \in P$;
(3) $W_P(p) \cap W_P(q) \neq \emptyset$ *implies that* $W_P(p) = W_P(q)$ *for all* $p, q \in P$.

Proof. In view of the proof of Theorem 2.10, $(S, *) \in \mathbb{E}^*$ with respect to the unary operation on S defined by

$$* : S \to S,\ x \mapsto x^*,$$

where x^* is the greatest element of $W_P(x)$ for any x in S.

Clearly, (1) $\Rightarrow$ (3) is obvious. We need to prove (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). Assume (3) holds. Let $p \in P$ and $x \in W_P(p)$. Then by Proposition 2.11 and its proof, we have $p^* = p \in W_P(p)$ and $x \in P$, $x = px = xp$. Since

$$(pxp)x(pxp) = pxp, pxpP^1x = pxpP^1xpx = px(pP^1x)px \subseteq pxPpx \subseteq P$$

and $xP^1pxp \subseteq P$, we have $pxp \in W_P(x)$. Moreover,

$$pxp \in W_P(p)W_P(p)W_P(p) \subseteq W_P(ppp) = W_P(p).$$

Thus, $pxp \in W_P(x) \cap W_P(p)$. It follows that $W_P(x) = W_P(p)$ from (3) whence $x = x^* = p^* = p$ by the proof of Proposition 2.11. This shows that $W_P(p) = \{p\}$, (2) holds.

Suppose that (2) holds and $x^+ \in W_P(x)$. By Proposition 2.9, we have

$$xx^+(xx^*)xx^+ = xx^+x(x^*xx^+) = x(x^+xx^+) = xx^+,$$

which shows that $xx^+ \in W(xx^*)$. Moreover, $xx^+P^1xx^*, xx^*P^1xx^+ \subseteq P$ whence $xx^+ \in W_P(xx^*)$. Observe that $xx^* \in P$ and $xx^+, xx^* \in W_P(xx^*)$, it follows that $xx^* = xx^+$ from (2). Similarly, we obtain $x^+x = x^*x$. Thus, $x^+ = x^+xx^+ = x^*xx^+ = x^*xx^* = x^*$. □

Example 2.14. The semigroup in Example 2.3 is an E-inversive semigroup with a **wp**-system $P = \{0\}$. Clearly, this semigroup satisfies the conditions in Theorem 2.13.

3. Congruences on E-inversive $*$-semigroups

In this section, we consider congruences on E-inversive $*$-semigroups. The following definition is fundamental.

Definition 3.1. Let $(S, *) \in \mathbb{E}^*$ and ρ a congruence on the semigroup S. Then ρ is called a *unary congruence* on S if $a\rho b$ implies that $a^*\rho b^*$ for all $a, b \in$ S. A unary congruence on S is called a *strongly unary congruence* on S if $a\rho\, aa^*a$ for all $a \in S$.

Remark 3.2. If $(S, *) \in \mathbb{R}^*$, then a unary congruence on S is always a strongly unary congruence since in this case, the identity $aa^*a = a$ always holds. However, in the case of E-inversive $*$-semigroups, the situation is different. For example, let $S = \{e, f\}$ be a chain such that $e \leq f$. Define a unary operation "$*$" by $e^* = f^* = e$. Then $(S, *) \in \mathbb{E}^*$. Obviously, the equality relation on S is a unary congruence on S. However, the equality relation on S is not a strongly unary congruence, since $f = f$ and $f \neq fe^*f = e$.

Let $(S, *) \in \mathbb{E}^*$. Then $(S^*, *) \in \mathbb{R}^*$. In the sequel, we denote the set of strongly unary congruences on S and the set of (strongly) unary congruences on S^* by $\mathcal{SC}^*(S)$ and $\mathcal{C}^*(S^*)$, respectively. Clearly, for $\rho \in \mathcal{SC}^*(S)$, the restriction $\rho|_{S^*}$ of ρ to S^* is a (strongly) unary congruences on S^*.

Proposition 3.3. *Let $(S, *) \in \mathbb{E}^*$. Then $\mathcal{SC}^*(S)$ is a complete sublattice of the lattice of congruences on S, and $\mathcal{C}^*(S^*)$ is a complete sublattice of the lattice of congruences on S^*.*

Proof. It is well known that the set of congruences on every semigroup forms a complete lattice under set inclusion. Now, let $I \subseteq \mathcal{SC}^*(S)$. Clearly, $\bigwedge_{\rho \in I} \rho = \bigcap_{\rho \in I} \rho \in \mathcal{SC}^*(S)$. On the other hand, if $a, b \in S$ and $a \bigvee_{\rho \in I} \rho b$, then there exist a positive integer n and

$$\rho_1, \rho_2, \ldots, \rho_{n+1} \in \mathcal{SC}^*(S), \quad x_1, x_2, \ldots, x_n \in S$$

such that

$$a\rho_1 x_1, x_1\rho_2 x_2, \ldots, x_{n-1}\rho_n x_n, x_n\rho_{n+1} b.$$

Observe that $\rho_1, \rho_2, \ldots, \rho_{n+1} \in \mathcal{SC}^*(S)$, it follows that

$$a^* \rho_1 x_1^*, x_1^* \rho_2 x_2^*, \ldots, x_{n-1}^* \rho_n x_n^*, x_n^* \rho_{n+1} b^*,$$

whence $a^* \bigvee_{\rho \in I} \rho b^*$. Since $a\rho aa^*a$ for any $\rho \in \mathcal{SC}^*(S)$, we have $a \bigvee_{\rho \in I} \rho (aa^*a)$. Thus, $\bigvee_{\rho \in I} \rho \in \mathcal{SC}^*(S)$. □

The following result explores the relationship between $\mathcal{SC}^*(S)$ and $\mathcal{C}^*(S^*)$ for $(S,*) \in \mathbb{E}^*$.

Theorem 3.4. *Let $(S,*) \in \mathbb{E}^*$. Then $\mathcal{SC}^*(S)$ is isomorphic to $\mathcal{C}^*(S^*)$ as complete lattice.*

Proof. Let

$$\varphi: \ \ \mathcal{C}^*(S^*) \to \mathcal{SC}^*(S), \ \ \sigma \mapsto \ \varphi(\sigma),$$

where $\varphi(\sigma) = \{(a,b) \mid a^* \sigma b^*\}$.

(i) $\varphi(\sigma) \in \mathcal{SC}^*(S)$ for each $\sigma \in \mathcal{C}^*(S^*)$. In fact, if $a\varphi(\sigma)b$, then $a^*\sigma b^*$ and so $(a^*)^* \sigma (b^*)^*$. This shows that $a^* \varphi(\sigma) b^*$. Moreover, observe that $(aa^*a)^* = a^*a^{**}a^* = a^*$, $a\varphi(\sigma)aa^*a$.

(ii) φ is injective. In fact, let $\sigma, \tau \in \mathcal{C}^*(S^*)$ and $\varphi(\sigma) = \varphi(\tau)$. If $a, b \in S^*$ and $a\sigma b$, then $(a^*)^* = a\sigma b = (b^*)^*$. This implies that $a^*\varphi(\sigma)b^*$ and so $a^*\varphi(\tau)b^*$, whence $a = (a^*)^* \tau (b^*)^* = b$. Therefore $\sigma \subseteq \tau$. Dually, $\tau \subseteq \sigma$.

(iii) φ is surjective. In fact, let $\rho \in \mathcal{SC}^*(S)$. Clearly, $\rho|_{S^*} \in \mathcal{C}^*(S^*)$. We assert that $\varphi(\rho|_{S^*}) = \rho$. To see this, let $a, b \in S$. If $a\rho b$, then $a^* \rho|_{S^*} b^*$ and so $a\varphi(\rho|_{S^*})b$. On the other hand, if $a\varphi(\rho|_{S^*})b$, then $a^*\rho|_{S^*} b^*$ and so $a^{**} \rho|_{S^*} b^{**}$. This implies $a\rho aa^*a = a^{**}\rho|_{S^*} b^{**} \rho bb^*b \rho b$ whence $a\rho b$.

(iv) φ is a complete lattice isomorphism. In fact, for any $J \subseteq \mathcal{C}^*(S^*)$ and $a, b \in S$, we have

$$a\varphi\left(\bigcap_{\rho \in J} \rho\right) b \text{ if and only if } a^* \left(\bigcap_{\rho \in J} \rho\right) b^* \text{ if and only if } a \left(\bigcap_{\rho \in J} \varphi(\rho)\right) b.$$

Moreover, $a\varphi(\bigvee_{\rho \in J} \rho)b$ if and only if $a^*(\bigvee_{\rho \in J} \rho)b^*$ if and only if

$$\begin{aligned}&(\exists n \in \mathbb{N})(\exists \rho_1, \ldots, \rho_{n+1} \in J)(\exists x_1^*, \ldots, x_n^* \in S^*)\\ &a^* \rho_1 x_1^*, \ldots, x_{n-1}^* \rho_n x_n^*, x_n^* \rho_{n+1} b^*\end{aligned}$$

if and only if

$$\begin{aligned}&(\exists n \in \mathbb{N})(\exists \rho_1, \ldots, \rho_{n+1} \in J)(\exists x_1, \ldots, x_n \in S)\\ &a\varphi(\rho_1)x_1, \ldots, x_{n-1}\varphi(\rho_n)x_n, x_n\varphi(\rho_{n+1})b\end{aligned}$$

if and only if $a(\bigvee_{\rho \in J} \varphi(\rho))b$. □

Remark 3.5. For any $(S,*) \in \mathbb{R}^*$, the unary congruences on S are extensively studied by Chae-Lee-Park [1], Imaoka [5], Nordahl-Scheiblich [6] and Yamada [9]. Thus, the above Theorem 3.4 provides a characterization of the strongly unary congruences on a member in $\mathbb{E}^*$.

Example 3.6. In Example 2.4, $S^* = \{a, e, f\}$ and $(S^*, *) \in \mathbb{R}^*$. Observe that S^* has the following five partitions:

$$\{\{a, e, f\}\}, \{\{a\}, \{e\}, \{f\}\}, \{\{a\}, \{e, f\}\}, \{\{f\}, \{a, e\}\}, \{\{e\}, \{a, f\}\},$$

whose corresponding equivalences are:

- $\omega_{S^*} = S^* \times S^*$;
- $\epsilon_{S^*} = \{(a, a), (e, e), (f, f)\}$;
- $\rho = \{(a, a), (e, e), (f, f), (e, f), (f, e)\}$;
- $\sigma = \{(a, a), (e, e), (f, f), (e, a), (a, e)\}$;
- $\sigma' = \{(a, a), (e, e), (f, f), (f, a), (a, f)\}$.

It is routine to check that $\omega_{S^*}, \epsilon_{S^*}, \rho, \sigma \in \mathcal{C}^*(S^*)$ and σ' is not a congruence on S. By Theorem 3.4, the strong unary congruences are:

- $\varphi(\omega_{S^*}) = S \times S$;
- $\varphi(\epsilon_{S^*}) = \{(a, a), (e, e), (f, f), (b, b), (a, b), (b, a)\}$;
- $\varphi(\rho) = \{(a, a), (e, e), (f, f), (b, b), (a, b), (b, a), (e, f), (f, e)\}$;
- $\varphi(\sigma) = \{(a, a), (e, e), (f, f), (b, b), (a, b), (b, a), (e, a), (a, e), (e, b), (b, e)\}$.

On the other hand,

$$\delta_1 = \{(a, a), (e, e), (f, f), (b, b)\}$$

and

$$\delta_2 = \{(a, a), (e, e), (f, f), (b, b), (e, a), (a, e)\}$$

are unary congruences but not strongly unary congruences on S, since

$$(b, bb^*b) = (b, bab) = (b, a) \notin \delta_1 \cup \delta_2.$$

In fact,

$$\varphi(\omega_{S^*}), \varphi(\epsilon_{S^*}), \varphi(\rho), \varphi(\sigma), \delta_1, \delta_2$$

are the whole congruences on S.

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