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AUTOMORPHISM GROUP OF THE TERNARY SELF-DUAL CODE OF LENGTH 8

Hyun Jin Kim and June Bok Lee

Abstract. We study the abstract structure of the automorphism group of the ternary self-dual code of length 8 and give its convenient presentation by generators.

1. Introduction

An $[n, k]$ linear code $\mathcal{C}$ over the finite field $\mathbb{F}_3$ is a k-dimensional subspace of $\mathbb{F}_3^n$. The Hamming weight of a vector in $\mathbb{F}_3^n$ is the number of its nonzero coordinates. The minimum weight d of $\mathcal{C}$ is the minimum weight of its nonzero codewords and in this case $\mathcal{C}$ is called an $[n, k, d]$ code. For every $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ from $\mathbb{F}_3^n$, $u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$ defines the *Euclidean inner product* in $\mathbb{F}_3^n$. The dual code of $\mathcal{C}$ is defined by $\mathcal{C}^\perp = \{u \in \mathbb{F}_2^n : u \cdot v = 0 \text{ for all } v \in \mathcal{C}\}$, and $\mathcal{C}^\perp$ is a linear $[n, n-k]$ code. If $\mathcal{C} \subseteq \mathcal{C}^\perp$, then $\mathcal{C}$ is called *self-orthogonal*, and if $\mathcal{C} = \mathcal{C}^\perp$, then we call it *self-dual*. Self-dual codes over $\mathbb{F}_3$ exist only for lengths a multiple of 4 and have codewords of Hamming weight a multiple of 3. Self-dual codes with the largest minimum weight of given length are called *optimal*. *Extremal* self-dual codes have minimum distance $3[n/12]+3$ ([3]). The largest possible minimum weight of the self-dual codes of lengths $n = 4$ and 8 is 3.

When we consider code classification, a notion of equivalence is necessary. An $n \times n$ matrix with coefficients in $\mathbb{F}_3$ is called *monomial* if there is exactly one nonzero entry in each row and column. Such a matrix is invertible since all nonzero elements of $\mathbb{F}_3$ are invertible. If all nonzero entries of the monomial are 1, then it is said to be a *permutation matrix*. Any monomial matrix can be uniquely written as the product of a permutation matrix and diagonal matrix. A monomial matrix M acts on the elements $x \in \mathbb{F}_3^n$ as $M \cdot x = xM$. Two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are *permutation equivalent* if there exists a permutation matrix P

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such that $\mathcal{C}_1 = \mathcal{C}_2 P$. More generally, if there is a monomial matrix M such that $\mathcal{C}_1 = \mathcal{C}_2 M$, the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent*. The automorphism group of a ternary code $\mathcal{C}$ is the set of all monomial matrices M such that $\mathcal{C} = \mathcal{C}M$. Characteristics for ternary self-dual codes were given in [2, 5].

Classification of self-dual codes over a ring requires not only the size of automorphism groups of codes over a field as well as properties of their subgroups and their relationshilp. Recently, the automorphism group of the ternary tetracode is presented in detail by the author in [4]. In this present paper, we extend the automorphism group in the case of the extremal self-dual code of length 8 over $\mathbb{F}_3$.

2. Preliminaries

The classification of self-dual codes relies on the knowledge of the so-called counting formula for self-dual codes, and the size of automorphism groups. The following counting formula for ternary codes of length n is well-known in [5].

Lemma 2.1 ([5]). *There exists a ternary self-dual code of length n if and only if n is divisible by 4. In this case, the number of self-dual code of length n is given by*

$$2\prod_{i=1}^{\frac{n}{2}-1}(3^i+1).$$

Suppose that $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ are all inequivalent ternary self-dual codes of length n. Then

$$2\prod_{i=1}^{\frac{n}{2}-1}(3^i+1) = \sum_{j=1}^{m}\frac{|M_n(\mathbb{Z}_3)|}{|\mathrm{Aut}(\mathcal{C}_j)|}. \tag{1}$$

The tetracode is a ternary code $\mathcal{T}$ with generator matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Any self-dual code of length 4 is equivalent to $\mathcal{T}$. From (1), $\mathrm{Aut}(\mathcal{T})$ has order 48.

Lemma 2.2 ([4, Theorem 2.1]). *The automorphism group of $\mathcal{T}$ can be generated by two elements* $\mathbf{b}$ *and* $\mathbf{c}$*, where*

$$\mathbf{b} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This lemma is used to identify all elements of $\mathrm{Aut}(\mathcal{T})$. We can see them on the Table 1 in [4].

Let

$$\mathbf{i} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{d} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

These generate the unique normal subgroup of $\mathrm{Aut}(\mathcal{T})$, and $\mathbf{i}, \mathbf{j}, \mathbf{c}$, and $\mathbf{d}$ give a presentation of $\mathrm{Aut}(\mathcal{T})$ [4].

Lemma 2.3 ([4, Theorem 3.6])**.** *The automorphism group of $\mathcal{T}$ can be expressed by*:

$$\mathrm{Aut}(\mathcal{T}) = \{\mathbf{i}^i\mathbf{j}^j\mathbf{c}^c\mathbf{d}^d \mid 0 \le i \le 3,\, 0 \le j \le 1,\, 0 \le c \le 2,\, 0 \le d \le 1\}.$$

Let G be a group and A be a subgroup of G. The normalizer of A in G is denoted by $N_G(A)$. In [4], we see three Sylow 2-subgroups of order 16

$$P_1 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right)\right),$$

$$P_2 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right)\right),$$

$$P_3 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right)\right),$$

and four Sylow 3-subgroups of order 3

$$Q_1 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right)\right), Q_2 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)\right),$$

$$Q_3 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right)\right), Q_4 = N_{\mathrm{Aut}(\mathcal{T})}\left(\left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right)\right).$$

2.1. Automorphism on ternary [8,4,3] code

Let $\mathcal{C}$ be a ternary $[8,4,3]$ code with the following generator matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix},$$

and let us define $\mathbf{G} = \mathrm{Aut}(\mathcal{C})$. Then this is a unique self-dual code of length 8 up to equivalence. From (1), we know that

$$2 \times (3+1) \times (9+1) \times (27+1) = \frac{2^8 \cdot 8!}{|\mathbf{G}|}.$$

In this case, the order of $\mathbf{G}$ is 4608. We define the maps

$$\phi_1 : \mathrm{Aut}(\mathcal{T}) \to \mathbf{G} \quad \text{and} \quad \phi_2 : \mathrm{Aut}(\mathcal{T}) \to \mathbf{G}$$

by $\phi_1(\sigma) = \left(\begin{smallmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & I_4 \end{smallmatrix}\right)$ and $\phi_2(\sigma) = \left(\begin{smallmatrix} I_4 & \mathbf{0} \\ \mathbf{0} & \sigma \end{smallmatrix}\right)$ for $\sigma \in \mathrm{Aut}(\mathcal{T})$, where I_4 is the 4×4 identity matrix and $\mathbf{0}$ is the 4×4 zero matrix. Obviously, $\phi_1(\mathrm{Aut}(\mathcal{T}))$ and $\phi_2(\mathrm{Aut}(\mathcal{T}))$ are subgroups of $\mathbf{G}$. For any subgroup A of $\mathbf{G}$, the centralizer of A in $\mathbf{G}$ is denoted by $C_{\mathbf{G}}(A)$. Let $\mathbf{G}_1 = \phi_1(\mathrm{Aut}(\mathcal{T}))$ and $\mathbf{G}_2 = \phi_2(\mathrm{Aut}(\mathcal{T}))$. Then $\mathbf{G}_1 \leq C_{\mathbf{G}}(\mathbf{G}_2)$ and $\mathbf{G}_2 \leq C_{\mathbf{G}}(\mathbf{G}_1)$, that is, $\overline{\sigma}_1\overline{\sigma}_2 = \overline{\sigma}_2\overline{\sigma}_1$ for any $\overline{\sigma}_1 \in \mathbf{G}_1$, $\overline{\sigma}_2 \in \mathbf{G}_2$.

Lemma 2.4. *Let Z be a group. If X and Y are subgroups of Z, then XY is a subgroup of Z if and only if $XY = YX$.*

Proof. Suppose XY is a subgroup of Z. Then for any $x \in X, y \in Y$,

$$yxXY = yXY = xx^{-1}yXY = xXY = XY \Rightarrow yx \in XY.$$

Since the order of XY is equal to the number of elements of YX, we have $XY = YX$.

For the other direction, now suppose $XY = YX$. For any $z, z' \in YX$, $z = x_1y_1$ and $z' = x_2y_2$ for some $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

$$z^{-1}z' = (x_1y_1)^{-1}x_2y_2 = y_1^{-1}x_1^{-1}x_2y_2.$$

Since X is a group, $x_1^{-1}x_2 \in X$. Let $x_3 = x_1^{-1}x_2$. Then we have

$$y_1^{-1}x_1^{-1}x_2y_2 = y_1^{-1}x_3y_2.$$

From the hypothesis, $y_1^{-1}x_3 = x_4y_3$ for some $x_4 \in X, y_3 \in Y$, and so

$$y_1^{-1}x_3y_2 = x_4y_3y_2.$$

Since Y is a group, $y_3y_2 \in Y$. Let $y_4 = y_3y_2$. Then we have

$$x_4y_3y_2 = x_4y_4 \in XY.$$

Therefore XY is a subgroup of Z. □

The following theorem shows the abstract structure of $\mathbf{G}$.

Theorem 2.5. *The automorphism group of $\mathcal{C}$ can be expressed by the product of $\mathbf{G}_1, \mathbf{G}_2$, and $\mathbf{G}_3$, where $\mathbf{G}_3 = \{I_8, \gamma\}$ with $\gamma = \left(\begin{smallmatrix} \mathbf{0} & I_4 \\ I_4 & \mathbf{0} \end{smallmatrix}\right)$.*

Proof. It is known that $\mathbf{G}_1 \leq C_{\mathbf{G}}(\mathbf{G}_2)$ implies $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$. From Lemma 2.4, $\mathbf{G}_1\mathbf{G}_2$ is a subgroup of $\mathbf{G}$. Since the index of $\mathbf{G}_1\mathbf{G}_2$ in $\mathbf{G}$ is 2, $\mathbf{G}_1\mathbf{G}_2$ is a normal subgroup of $\mathbf{G}$. Let $\gamma = \left(\begin{smallmatrix} \mathbf{0} & I_4 \\ I_4 & \mathbf{0} \end{smallmatrix}\right)$. It is easy to check that $\gamma \in \mathbf{G}$. Let $\mathbf{G}_3 = \{I_8, \gamma\}$. Then $\mathbf{G}_3 \leq N_{\mathbf{G}}(\mathbf{G}_1\mathbf{G}_2)$, and by the result of Lemma 2.4, we know that $\mathbf{G}_1\mathbf{G}_2\mathbf{G}_3$ is a subgroup of $\mathbf{G}$. Each order of $\mathbf{G}_1$ and $\mathbf{G}_2$ is 48, and $\mathbf{G}_1 \cap \mathbf{G}_2 = \{I_8\}$. The order $\mathbf{G}_3$ is 2, and $\mathbf{G}_1\mathbf{G}_2 \cap \mathbf{G}_3 = \{I_8\}$. Since $|\mathbf{G}_1\mathbf{G}_2\mathbf{G}_3| = 4608$, we get the desired result: $\mathbf{G} = \mathbf{G}_1\mathbf{G}_2\mathbf{G}_3$. □

The group $\mathbf{G}_1\mathbf{G}_2$ is isomorphic to $\mathrm{Aut}(\mathcal{T}) \times \mathrm{Aut}(\mathcal{T})$. Lemma 2.2 gives the following corollary. This corollary identifies all elements of $\mathbf{G}$.

Corollary 2.6. *The automorphism group of $\mathcal{C}$ is generated by $\phi_1(\mathbf{b}), \phi_1(\mathbf{c})$, and γ.*

Proof. From Lemma 2.2 and Theorem 2.5, $\phi_1(\mathbf{b}), \phi_1(\mathbf{c}), \phi_2(\mathbf{b}), \phi_2(\mathbf{c})$, and γ generate $\mathbf{G}$. Clearly,

$$\phi_2(\mathbf{b}) = \phi_1(\mathbf{b})\gamma, \quad \phi_2(\mathbf{c}) = \phi_1(\mathbf{c})\gamma.$$

Therefore, $\phi_1(\mathbf{b}), \phi_1(\mathbf{c})$, and γ generate $\mathbf{G}$. □

From Lemma 2.3 we can obtain the following corollary.

Corollary 2.7. *The automorphism group of $\mathcal{C}$ can be presented by:*

$$\begin{aligned}\mathbf{G} = \{\phi_1(\mathbf{i}^i\mathbf{j}^j\mathbf{c}^c\mathbf{d}^d)\phi_2(\mathbf{i}^{i'}\mathbf{j}^{j'}\mathbf{c}^{c'}\mathbf{d}^{d'})\gamma^k \mid\ & 0 \leq i, i' \leq 3, 0 \leq j, j' \leq 1, 0 \leq c, c' \leq 2, \\ & 0 \leq d, d' \leq 1, 0 \leq k \leq 1, \}.\end{aligned}$$

Let $g, g' \in \mathbf{G}$ with $g = \left(\begin{smallmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \sigma_2 \end{smallmatrix}\right), g' = \left(\begin{smallmatrix} \mathbf{0} & \sigma_1 \\ \sigma_2 & \mathbf{0} \end{smallmatrix}\right)$ for $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathcal{T})$. Then

$$\gamma g \gamma^{-1} = \begin{pmatrix} \sigma_2 & \mathbf{0} \\ \mathbf{0} & \sigma_1 \end{pmatrix}, \gamma g' \gamma^{-1} = \begin{pmatrix} \mathbf{0} & \sigma_2 \\ \sigma_1 & \mathbf{0} \end{pmatrix}.$$

Proposition 2.8. *Let A be a subgroup of $\mathbf{G}$ with $\gamma \in A$. If $g, g' \in A$, where $g = \left(\begin{smallmatrix} \sigma_1 & \mathbf{0} \\ \mathbf{0} & \sigma_2 \end{smallmatrix}\right)$ and $g' = \left(\begin{smallmatrix} \mathbf{0} & \sigma_1 \\ \sigma_2 & \mathbf{0} \end{smallmatrix}\right)$, then A contains $\left(\begin{smallmatrix} \sigma_2 & \mathbf{0} \\ \mathbf{0} & \sigma_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbf{0} & \sigma_2 \\ \sigma_1 & \mathbf{0} \end{smallmatrix}\right)$. In particular, if $A = A_1A_2G_3$ for some $A_1 \leq \mathbf{G}_1, A_2 \leq \mathbf{G}_2$, then $\phi_1^{-1}(A_1) = \phi_2^{-1}(A_2)$.*

We compute the conjugacy classes of $\mathbf{G}$. First we consider the conjugacy classes of elements of $\mathbf{G}_1\mathbf{G}_2$.

Proposition 2.9. *If $g \in \mathbf{G}_1\mathbf{G}_2$, then the conjugates of g belongs to $\mathbf{G}_1\mathbf{G}_2$.*

Proof. For any $a \in \mathbf{G}_1\mathbf{G}_2$, $aga^{-1} \in \mathbf{G}_1\mathbf{G}_2$ since $\mathbf{G}_1\mathbf{G}_2$ is a subgroup of $\mathbf{G}$, and in addition

$$(a\gamma)g(a\gamma)^{-1} = a\gamma g\gamma^{-1}a^{-1} = a(\gamma g\gamma^{-1})a^{-1}.$$

Since $(\gamma g\gamma^{-1}) \in \mathbf{G}_1\mathbf{G}_2$ from Proposition 2.8, $(a\gamma)g(a\gamma)^{-1}$ belongs to $\mathbf{G}_1\mathbf{G}_2$. □

This shows that the conjugacy classes of elements of $\mathbf{G}_1\mathbf{G}_2$ are closed in $\mathbf{G}_1\mathbf{G}_2$. The following proposition gives the property which the conjugacy classes of elements of $\mathbf{G}_1\mathbf{G}_2$ have.

Proposition 2.10. *For $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathcal{T})$, $\phi_1(\sigma_1)\phi_2(\sigma_2)$ and $\phi_1(\sigma_2)\phi_2(\sigma_1)$ are affiliated to same conjugacy class.*

Proof. From Proposition 2.8, we have

$$\gamma\phi_1(\sigma_1)\phi_2(\sigma_2)\gamma^{-1} = \phi_1(\sigma_2)\phi_2(\sigma_1).$$

Therefore, $\phi_1(\sigma_2)\phi_2(\sigma_1)$ is a conjugate of $\phi_1(\sigma_1)\phi_2(\sigma_2)$. □

The following theorem presents the conjugacy classes of elements of $\mathbf{G}_1\mathbf{G}_2$ clearly. We can see conjugacy classes of $\mathrm{Aut}(\mathcal{T})$ in [Table 2, [4]].

Theorem 2.11. *For $\sigma_i, \sigma_j \in \mathrm{Aut}(\mathcal{T})$, the conjugacy class of $\phi_1(\sigma_i)\phi_2(\sigma_j)$ is the union of $\phi_1(Ci)\phi_2(Cj)$ and $\phi_1(Cj)\phi_2(Ci)$ where Ci and Cj are the conjugacy classes of σ_i and σ_j in $\mathrm{Aut}(\mathcal{T})$, respectively.*

Proof. Let $a = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with $x, y \in \mathrm{Aut}(\mathcal{T})$. Then since $a^{-1} = \begin{pmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}$,

$$a\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_j \end{pmatrix} a^{-1} = \begin{pmatrix} x\sigma_i x^{-1} & 0 \\ 0 & y\sigma_j y^{-1} \end{pmatrix} \in \phi_1(Ci)\phi_2(Cj),$$

and

$$\begin{aligned} rl(a\gamma)\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_j \end{pmatrix}(a\gamma)^{-1} &= a(\gamma\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_j \end{pmatrix}\gamma^{-1})a^{-1} \\ &= a\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_i \end{pmatrix} a^{-1} \in \phi_1(Cj)\phi_2(Ci). \end{aligned}$$

Therefore, the union of $\phi_1(Ci)\phi_2(Cj)$ and $\phi_1(Cj)\phi_2(Ci)$ contains the conjugacy class of $\phi_1(\sigma_i)\phi_2(\sigma_j)$. On the other hand, since the conjugacy class of $\phi_1(\sigma_i)\phi_2(\sigma_j)$ contains $\phi_1(Ci)\phi_2(Cj)$, the conjugacy class of $\phi_1(\sigma_i)\phi_2(\sigma_j)$ includes $\phi_1(Cj)\phi_2(Ci)$ from Proposition 2.10. We get the desired result: the conjugacy class of $\phi_1(\sigma_i)\phi_2(\sigma_j)$ is equal to the union of $\phi_1(Ci)\phi_2(Cj)$ and $\phi_1(Cj)\phi_2(Ci)$. □

We consider the conjugacy class of $a\gamma$ where a is any element of $\mathbf{G}_1\mathbf{G}_2$.

Theorem 2.12. *The conjugacy class of γ is the set*

$$\left\{\begin{pmatrix} 0 & \sigma \\ \sigma^{-1} & 0 \end{pmatrix} \mid \quad \sigma \in \mathrm{Aut}(\mathcal{T})\right\}. \tag{2}$$

Proof. Let $a\gamma$ be any element of $\mathbf{G}$ with $a \in \mathbf{G}_1\mathbf{G}_2$. Then one has

$$(a\gamma)\gamma(a\gamma)^{-1} = (a\gamma)\gamma(\gamma^{-1}a^{-1}) = a(\gamma\gamma\gamma^{-1})a^{-1} = a\gamma a^{-1}.$$

For any $x, y \in \mathrm{Aut}(\mathcal{T})$,

$$\begin{aligned} rl \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \gamma \begin{pmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} &= \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & xy^{-1} \\ yx^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & xy^{-1} \\ (xy^{-1})^{-1} & 0 \end{pmatrix}. \end{aligned}$$

Since all elements of $\mathrm{Aut}(\mathcal{T})$ can be presented as the form of xy^{-1}, this proof is complete. □

From this, we know that the length of the conjugacy class of γ is 48.

Proposition 2.13. *Let $\bar{\gamma} = y^{-1}\gamma y$ where y is any element of $\mathbf{G}$. Then the conjugates of $a\gamma$ are equal to the conjugates of $y^{-1}ay\bar{\gamma}$.*

Proof. For any $x \in \mathbf{G}$,

$$\begin{aligned} xa\gamma x^{-1} &= xa(y\bar{\gamma}y^{-1})x^{-1} \\ &= x(yy^{-1})a(y\bar{\gamma}y^{-1})x^{-1} \\ &= (xy)(y^{-1}ay)\bar{\gamma}(y^{-1}x^{-1}) \\ &= (xy)(y^{-1}ay)\bar{\gamma}(xy)^{-1}. \end{aligned}$$

This proves that the conjugates of $a\gamma$ and $y^{-1}ay\bar{\gamma}$ are the same. □

Proposition 2.14. *The conjugacy class of $a\gamma$ is contained in the set C_aC_γ where C_a is the conjugacy class of a and C_γ is the conjugacy class of γ.*

Proof. Let $C_{a\gamma}$ be the conjugacy class of $a\gamma$. Then any element g of $C_{a\gamma}$ is representative as $b(a\gamma)b^{-1}$ for some $b \in \mathbf{G}$.

$$\begin{aligned} g &= b(a\gamma)b^{-1} \\ &= b\{a(b^{-1}b)\gamma\}b^{-1} \\ &= (bab^{-1})(b\gamma b^{-1}). \end{aligned}$$

Consequently, $g \in C_aC_\gamma$ since $bab^{-1} \in C_a$ and $b\gamma b^{-1} \in C_\gamma$. □

These two propositions characterize conditions the conjugacy classes of elements which have γ as the factor should satisfy.

If h is an element in C_aC_γ, then there are some b and d in $\mathbf{G}$ such that $h = (bab^{-1})(d\gamma d^{-1})$.

Proposition 2.15. *If $h \in C_aC_\gamma$, then $h = d\{(d^{-1}b)a(d^{-1}b)^{-1}\}\gamma d^{-1}$. If $d^{-1}b$ belongs to centralizer of a, then the element h of C_aC_γ is contained in $C_{a\gamma}$.*

Proof. By the property of identity and the association law, one has

$$\begin{aligned} h &= (bab^{-1})(d\gamma d^{-1}) \\ &= (dd^{-1})(bab^{-1})(d\gamma d^{-1}) \\ &= d(d^{-1}bab^{-1}d)\gamma d^{-1} \\ &= d\{(d^{-1}b)a(b^{-1}d)\}\gamma d^{-1} \\ &= d\{(d^{-1}b)a(d^{-1}b)^{-1}\}\gamma d^{-1}. \end{aligned}$$

□

Let us consider the conjugacy classes of special elements of **G**.

Theorem 2.16. *The conjugacy class of $\phi_1(\sigma)\phi_2(I_4)\gamma$ is equal to $C_\sigma C_\gamma$, where C_σ is the conjugacy class of $\phi_1(\sigma)\phi_2(I_4)$ and C_γ is the conjugacy class of γ.*

Proof. From Proposition 2.14 the conjugacy class of $\phi_1(\sigma)\phi_2(I_4)\gamma$ is contained in $C_\sigma C_\gamma$. Let a be an element of $C_\sigma C_\gamma$. Then

$$a = \begin{pmatrix} x\sigma x^{-1} & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_4 & 0 \\ 0 & x\sigma x^{-1} \end{pmatrix}\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix}$$

for some x and y in $\mathrm{Aut}(\mathcal{T})$ from Proposition 2.10. In the first case we can choose an element z in $\mathrm{Aut}(\mathcal{T})$ such that $z = y^{-1}x$. Then

$$\begin{aligned} rla &= \begin{pmatrix} x\sigma x^{-1} & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}\right]\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}\right]\left[\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}\right] \\ &= \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}. \end{aligned}$$

Since $\left(\begin{smallmatrix} \sigma & 0 \\ 0 & I_4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & I_4 \\ I_4 & 0 \end{smallmatrix}\right) = \phi_1(\sigma)\phi_2(I)\gamma$, a is contained in the conjugacy class of $\phi_1(\sigma)\phi_2(I)\gamma$. For the second case, we take z in $\mathrm{Aut}(\mathcal{T})$ such that $z = yx$. Then

$$\begin{aligned} rla &= \begin{pmatrix} I_4 & 0 \\ 0 & x\sigma x^{-1} \end{pmatrix}\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} I_4 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}\right]\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} I_4 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}\right]\left[\begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}\right] \\ &= \begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} I_4 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}. \end{aligned}$$

Since $\left(\begin{smallmatrix} I_4 & 0 \\ 0 & \sigma \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & I_4 \\ I_4 & 0 \end{smallmatrix}\right) = \phi_2(\sigma)\phi_1(I)\gamma$, and $\phi_2(\sigma)\phi_1(I) = \phi_1(I)\phi_2(\sigma)$, from Proposition 2.10, a is contained in the conjugacy class of $\phi_1(\sigma)\phi_2(I)\gamma$. Therefore the conjugacy class of $\phi_1(\sigma)\phi_2(I_4)\gamma$ and $C_\sigma C_\gamma$ are the same. □

If we apply the same argument in Theorem 2.16, the following corollarly is true.

Corollary 2.17. *The conjugacy class of $\phi_1(\sigma)\phi_2(-I_4)\gamma$ is equal to $C_\sigma C_\gamma$, where C_σ is the conjugacy class of $\phi_1(\sigma)\phi_2(-I_4)$ and C_γ is the conjugacy class of γ.*

The following proposition gives the number of elements of conjugacy class of $\phi_1(\sigma)\phi_2(I_4)\gamma$.

Proposition 2.18. *The number of elements of $C_\sigma C_\gamma$ is equal to $\frac{1}{2} \mid C_\sigma \mid\mid C_\gamma \mid$, where C_σ is the conjugacy class of $\phi_1(\sigma)\phi_2(I_4)$ and C_γ is the conjugacy class of γ.*

Proof. For any $\left(\begin{smallmatrix} I_4 & 0 \\ 0 & \sigma' \end{smallmatrix}\right) \in C_\sigma$ and $x \in \mathrm{Aut}(\mathcal{T})$

$$\begin{pmatrix} I_4 & 0 \\ 0 & \sigma' \end{pmatrix}\begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ \sigma' x^{-1} & 0 \end{pmatrix}.$$

We can choose $\left(\begin{smallmatrix} \sigma'' & 0 \\ 0 & I_4 \end{smallmatrix}\right) \in C_\sigma$ and $y \in \mathrm{Aut}(\mathcal{T})$ such that

$$\sigma'' = x\sigma' x^{-1} \quad \text{and} \quad y = x{\sigma'}^{-1},$$

since $x\sigma' x^{-1}$ is a conjugate of σ. Then

$$\begin{aligned} rl \begin{pmatrix} \sigma'' & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \sigma'' y \\ y^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (x\sigma' x^{-1})(x{\sigma'}^{-1}) \\ (x{\sigma'}^{-1})^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x \\ \sigma' x^{-1} & 0 \end{pmatrix}. \end{aligned}$$

Hence we obtain the products of the form of $\left(\begin{smallmatrix} I_4 & 0 \\ 0 & \overline{\sigma} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & x \\ x^{-1} & 0 \end{smallmatrix}\right)$ by the products of the form of $\left(\begin{smallmatrix} \overline{\sigma} & 0 \\ 0 & I_4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & x \\ x^{-1} & 0 \end{smallmatrix}\right)$.

Suppose that

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix}.$$

Then we should have $x = y$ and $\sigma_1 = \sigma_2$. Therefore we get the desired result $\mid C_\sigma C_\gamma \mid = \frac{1}{2} \mid C_\sigma \mid\mid C_\gamma \mid$. □

We can compute the number of elements of conjugacy class of $\phi_1(\sigma)\phi_2(-I_4)\gamma$ using the same argument in Proposition 2.18.

Corollary 2.19. *The number of elements of $C_\sigma C_\gamma$ is equal to $\frac{1}{2} \mid C_\sigma \mid\mid C_\gamma \mid$, where C_σ is the conjugacy class of $\phi_1(\sigma)\phi_2(-I_4)$ and C_γ is the conjugacy class of γ.*

Proof. For any $\begin{pmatrix} -I_4 & 0 \\ 0 & \sigma' \end{pmatrix} \in C_\sigma$ and $x \in \mathrm{Aut}(\mathcal{T})$, we have

$$\begin{pmatrix} -I_4 & 0 \\ 0 & \sigma' \end{pmatrix} \begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x \\ \sigma' x^{-1} & 0 \end{pmatrix}.$$

We can choose $\begin{pmatrix} \sigma'' & 0 \\ 0 & -I_4 \end{pmatrix} \in C_\sigma$ and $y \in \mathrm{Aut}(\mathcal{T})$ such that

$$\sigma'' = -x\sigma' x^{-1} \quad \text{and} \quad y = x{\sigma'}^{-1},$$

since $x\sigma' x^{-1}$ is a conjugate of σ. Then we have

$$rl \begin{pmatrix} \sigma'' & 0 \\ 0 & I_4 \end{pmatrix} \begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma'' y \\ y^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (-x\sigma' x^{-1})(x{\sigma'}^{-1}) \\ (x{\sigma'}^{-1})^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -x \\ \sigma' x^{-1} & 0 \end{pmatrix}.$$

This completes the statement. □

Let us show the result in Table 1, where C_i is in [Table 2, [4]]. From this Table 1, we obtain the class equation for $\mathbf{G}$:

$$\sum_{i=1}^{44} C_i = 4608.$$

Lemma 2.20 ([4, Theorems 3.3 and 3.4])**.** *The automorphism group of $\mathcal{T}$ has four Sylow 3-subgroups of order 3 and three Sylow 2-subgroups of order 16. Thus no Sylow subgroups are normal.*

We consider the Sylow subgroups of $\mathcal{C}$. The results about Sylow subgroups are the following.

Theorem 2.21. *The automorphism group of $\mathcal{C}$ has 9 Sylow 2-subgroups of order 512 and 16 Sylow 3-subgroups of order 9.*

Proof. Let S be a Sylow 2-subgroup of $\mathbf{G}$. It is obvious that the order of S is 512. Since 512 does not divide the order of $\mathbf{G}_1\mathbf{G}_2$, S can not be the form of XY for some $X \leq \mathbf{G}_1, Y \leq \mathbf{G}_2$. Now, we consider the normalizer of $\phi_1(P_i)\phi_2(P_j)$ for $i, j = 1, 2, 3$, where P_1, P_2, and P_3 are Sylow 2-subgroups of $\mathrm{Aut}(\mathcal{T})$ in [4]. Since $|\phi_1(P_i)\phi_2(P_j)|$ is 256, its normalizer must contain some Sylow 2-subgroup from the first Sylow theorem in [3]. Since P_1, P_2 and P_3 are distinct Sylow 2-subgroups of $\mathrm{Aut}(\mathcal{T})$ and $|\mathrm{Aut}(\mathcal{T})| = 48$, one should have $N_{\mathrm{Aut}(\mathcal{T})}(P_i) = P_i$ for $i = 1, 2, 3$. Therefore, $\phi_1(P_i)\phi_2(P_j)$ is not a normal subgroup of $\mathbf{G}_1\mathbf{G}_2$, and hence its normalizer does't have an element of order 3. Consequently, the normalizer of $\phi_1(P_i)\phi_2(P_j)$ has order 512 from the first Sylow theorem in [3].

Since $N_{\mathrm{Aut}(\mathcal{T})}(P_i)$ is not equal to $N_{\mathrm{Aut}(\mathcal{T})}(P_j)$ where i and j are distinct, we obtain distinct 9 subgroups of order 512. They are Sylow 2-subgroups of $\mathbf{G}$. Let N_2 be the number of Sylow 2-subgroups of $\mathbf{G}$. By Sylow theorems, we should have $N_2 \equiv 1 (\mathrm{mod}\ 2)$ and $N_2 \mid 9$. However, the possible numbers for N_2 are 1, 3, or 9 and we already show there exist only 9 Sylow 2-subgroups. Hence $\mathbf{G}$ has only 9 Sylow 2-subgroups.

Each element of $\mathbf{G}$ can be written uniquely as a product $g_1g_2g_3$ for some $g_1 \in \mathbf{G}_1, g_2 \in \mathbf{G}_2$, and $g_3 \in \mathbf{G}_3$. The fact that the order of $g_1g_2g_3$ divides 9 implies $g_3 = I_8$. Hence $\phi_1(Q_i)\phi_2(Q_j)$ are Sylow 3-subgroups of $\mathbf{G}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$, where Q_1, Q_2, Q_3, and Q_4 in [4]. Since there is no element of $\mathbf{G}_1\mathbf{G}_2$ whose its order is 9, only these 16 subgroups are Sylow 3-subgroups. □

TABLE 1. Conjugacy classes of $\mathbf{G}$

Class	Representative	Elements	Length	Order
C_1	$\phi_1(1)\phi_2(1)$	$\phi_1(C1)\phi_2(C1)$	1	1
C_2	$\phi_1(48)\phi_2(48)$	$\phi_1(C8)\phi_2(C8)$	1	2
C_3	$\phi_1(48)\phi_2(1)$	$\phi_1(C8)\phi_2(C1) \cup \phi_1(C1)\phi_2(C8)$	2	2
C_4	$\phi_1(48)\phi_2(6)$	$\phi_1(C8)\phi_2(C3) \cup \phi_1(C3)\phi_2(C8)$	24	2
C_5	$\phi_1(1)\phi_2(43)$	$\phi_1(C1)\phi_2(C3) \cup \phi_1(C3)\phi_2(C1)$	24	2
C_6	γ	(2) in Proposition 2.12	48	2
C_7	$\phi_1(3)\phi_2(3)$	$\phi_1(C3)\phi_2(C3)$	144	2
C_8	$\phi_1(1)\phi_2(4)$	$\phi_1(C1)\phi_2(C2) \cup \phi_1(C2)\phi_2(C1)$	16	3
C_9	$\phi_1(2)\phi_2(4)$	$\phi_1(C2)\phi_2(C2)$	64	3
C_{10}	$\phi_1(1)\phi_2(27)$	$\phi_1(C1)\phi_2(C7) \cup \phi_1(C7)\phi_2(C1)$	12	4
C_{11}	$\phi_1(48)\phi_2(22)$	$\phi_1(C8)\phi_2(C7) \cup \phi_1(C7)\phi_2(C8)$	12	4
C_{12}	$\phi_1(37)\phi_2(27)$	$\phi_1(C7)\phi_2(C7)$	36	4
C_{13}	$\phi_1(48)\phi_2(1)\gamma$	C_3C_6	48	4
C_{14}	$\phi_1(37)\phi_2(6)$	$\phi_1(C7)\phi_2(C3) \cup \phi_1(C3)\phi_2(C7)$	144	4
C_{15}	$\phi_1(1)\phi_2(43)\gamma$	C_5C_6	576	4
C_{16}	$\phi_1(48)\phi_2(4)$	$\phi_1(C8)\phi_2(C2) \cup \phi_1(C2)\phi_2(C8)$	16	6
C_{17}	$\phi_1(48)\phi_2(45)$	$\phi_1(C8)\phi_2(C6) \cup \phi_1(C6)\phi_2(C8)$	16	6
C_{18}	$\phi_1(1)\phi_2(45)$	$\phi_1(C1)\phi_2(C6) \cup \phi_1(C6)\phi_2(C1)$	16	6
C_{19}	$\phi_1(47)\phi_2(45)$	$\phi_1(C6)\phi_2(C6)$	64	6
C_{20}	$\phi_1(47)\phi_2(4)$	$\phi_1(C6)\phi_2(C2) \cup \phi_1(C2)\phi_2(C6)$	128	6
C_{21}	$\phi_1(2)\phi_2(44)$	$\phi_1(C2)\phi_2(C3) \cup \phi_1(C3)\phi_2(C2)$	192	6
C_{22}	$\phi_1(47)\phi_2(5)$	$\phi_1(C6)\phi_2(C3) \cup \phi_1(C3)\phi_2(C6)$	192	6
C_{23}	$\phi_1(1)\phi_2(4)\gamma$	C_8C_6	384	6
C_{24}	$\phi_1(1)\phi_2(33)$	$\phi_1(C1)\phi_2(C4) \cup \phi_1(C4)\phi_2(C1)$	12	8
C_{25}	$\phi_1(48)\phi_2(16)$	$\phi_1(C8)\phi_2(C5) \cup \phi_1(C5)\phi_2(C8)$	12	8
C_{26}	$\phi_1(1)\phi_2(16)$	$\phi_1(C1)\phi_2(C5) \cup \phi_1(C5)\phi_2(C1)$	12	8
C_{27}	$\phi_1(48)\phi_2(33)$	$\phi_1(C8)\phi_2(C4) \cup \phi_1(C4)\phi_2(C8)$	12	8
C_{28}	$\phi_1(25)\phi_2(41)$	$\phi_1(C5)\phi_2(C5)$	36	8

TABLE 1. continued

C_{29}	$\phi_1(24)\phi_2(8)$	$\phi_1(C4)\phi_2(C4)$	36	8
C_{30}	$\phi_1(12)\phi_2(33)$	$\phi_1(C7)\phi_2(C4) \cup \phi_1(C4)\phi_2(C7)$	72	8
C_{31}	$\phi_1(24)\phi_2(41)$	$\phi_1(C4)\phi_2(C5) \cup \phi_1(C5)\phi_2(C4)$	72	8
C_{32}	$\phi_1(37)\phi_2(16)$	$\phi_1(C7)\phi_2(C5) \cup \phi_1(C5)\phi_2(C7)$	72	8
C_{33}	$\phi_1(46)\phi_2(25)$	$\phi_1(C3)\phi_2(C5) \cup \phi_1(C5)\phi_2(C3)$	144	8
C_{34}	$\phi_1(3)\phi_2(24)$	$\phi_1(C3)\phi_2(C4) \cup \phi_1(C4)\phi_2(C3)$	144	8
C_{35}	$\phi_1(1)\phi_2(27)\gamma$	$C_{10}C_6$	288	8
C_{36}	$\phi_1(37)\phi_2(14)$	$\phi_1(C7)\phi_2(C2) \cup \phi_1(C2)\phi_2(C7)$	96	12
C_{37}	$\phi_1(12)\phi_2(35)$	$\phi_1(C7)\phi_2(C6) \cup \phi_1(C6)\phi_2(C7)$	96	12
C_{38}	$\phi_1(48)\phi_2(4)\gamma$	$C_{16}C_6$	384	12
C_{39}	$\phi_1(1)\phi_2(16)\gamma$	$C_{26}C_6$	288	16
C_{40}	$\phi_1(48)\phi_2(16)\gamma$	$C_{25}C_6$	288	16
C_{41}	$\phi_1(47)\phi_2(13)$	$\phi_1(C6)\phi_2(C4) \cup \phi_1(C4)\phi_2(C6)$	96	24
C_{42}	$\phi_1(2)\phi_2(36)$	$\phi_1(C2)\phi_2(C5) \cup \phi_1(C5)\phi_2(C2)$	96	24
C_{43}	$\phi_1(47)\phi_2(36)$	$\phi_1(C6)\phi_2(C5) \cup \phi_1(C5)\phi_2(C6)$	96	24
C_{44}	$\phi_1(2)\phi_2(13)$	$\phi_1(C2)\phi_2(C4) \cup \phi_1(C4)\phi_2(C2)$	96	24

We define i and j of $\phi_1(i)\phi_2(j)$ as j in [Table 1, [4]].

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Hyun Jin Kim
Institute of Mathematical Sciences
Ewha Womans University
Seoul 120-750, Korea
E-mail address: guswls41@ewha.ac.kr

June Bok Lee
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail address: leejb@yonsei.ac.kr