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GLOBAL REGULARITY OF SOLUTIONS TO QUASILINEAR CONORMAL DERIVATIVE PROBLEM WITH CONTROLLED GROWTH

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Abstract. We prove the global regularity of weak solutions to a conormal derivative boundary value problem for quasilinear elliptic equations in divergence form on Lipschitz domains under the controlled growth conditions on the low order terms. The leading coefficients are in the class of BMO functions with small mean oscillations.

1. Introduction

We consider the conormal derivative boundary value problem

$$\begin{cases} -D_i\left(A_{ij}(x,u)D_ju + a_i(x,u)\right) = b(x,u,\nabla u) \quad \text{in} \quad \Omega, \\ \left(A_{ij}(x,u)D_ju + a_i(x,u)\right)\cdot \nu(x) = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

Here the equation is a quasilinear elliptic equation in divergence form, Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with a small Lipschitz constant, and $\nu(x)$ is the outward normal vector to the surface $\partial\Omega$. We call $u \in W_2^1(\Omega)$ a weak solution to (1) if

$$\int_\Omega (A_{ij}(x,u)D_ju + a_i(x,u))D_i\phi\,dx = \int_\Omega b(x,u,\nabla u)\phi\,dx$$

for any $\phi \in W_2^1(\Omega)$.

In this paper we study the global regularity of weak solutions to (1) under the *controlled growth conditions* (explained below) on a_i and b. The nonlinear terms $A_{ij}(x,u)$, $a_i(x,u)$, $b(x,u,\xi)$ are of Carathéodory type, i.e., they are measurable in $x \in \mathbb{R}^d$ for all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^d$, and continuous in $(u,\xi) \in \mathbb{R}\times\mathbb{R}^d$ for almost all $x \in \mathbb{R}^d$. The leading coefficients A_{ij} are bounded and uniformly elliptic,

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that is, for some constant $\mu \in (0,1]$,

$$|A_{ij}| \le \mu^{-1}, \quad A_{ij}\xi_i\xi_j \ge \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^d. \tag{2}$$

We also assume that $A_{ij}(x,u)$ are uniformly continuous in u and have small mean oscillations with respect to x. Throughout the paper, we set

$$\gamma = \begin{cases} \dfrac{2d}{d-2}, & d > 2, \\ \text{any number bigger than 2}, & d = 2. \end{cases} \tag{3}$$

By the controlled growth conditions, we mean

$$|a_i(x,u)| \le \mu_1(|u|^{\lambda_1} + f), \quad |b(x,u,\nabla u)| \le \mu_2(|\nabla u|^{\lambda_2} + |u|^{\lambda_3} + g)$$

for some constants $\mu_1, \mu_2 > 0$, where $\lambda_1 = \gamma/2$, $\lambda_2 = 2(1 - 1/\gamma)$, $\lambda_3 = \gamma - 1$, and

$$f \in L_2(\Omega), \quad g \in L_{\frac{\gamma}{\gamma-1}}(\Omega).$$

Since $u \in W_2^1(\Omega)$ implies $u \in L_\gamma(\Omega)$, the controlled growth conditions guarantee the convergence of the integrals in the definition of weak solutions above. If $1 \le \lambda_1 < \gamma/2$, $1 \le \lambda_2 < 2(1-1/\gamma)$, $1 \le \lambda_3 < \gamma - 1$, we say that the equation (1) satisfies the *strictly* controlled growth conditions. As mentioned in [6], the controlled growth conditions are optimal (see, for instance, a counterexample in [13]) unless some additional boundedness conditions on weak solutions are imposed.

Under the above assumptions, we prove that weak solutions to (1) are *globally* Hölder continuous with a Hölder exponent depending only on the dimension and the integrability of f and g. Indeed, as noted in [13] and [6], we have an explicit description of the Hölder exponent in terms of σ and τ if $f \in L_\sigma(\Omega)$ and $g \in L_\tau(\Omega)$, $\sigma > d$, $\tau > d/2$, whereas such an explicit Hölder exponent is not shown in the De Giorgi-Moser-Nash theory. To obtain the desired regularity, we prove higher integrability of solutions. Precisely, we show that a weak solution to (1) is a member of $W_p^1(\Omega)$, where $p > d$ is determined only by σ and τ above. Then the globally Hölder continuity of the weak solution follows easily from the Sobolev embedding theorem. In addition to the fact that the low order terms satisfy the controlled growth conditions, note that the leading coefficients satisfy only a small BMO condition as functions of $x \in \mathbb{R}$. Thus they are not necessarily continuous in x. We remark that in general global regularity cannot be expected for systems (see [7, 16]), and even for partial regularities usually one requires the leading coefficients to possess certain regularity in all involved variables (usually uniform continuity).

When the Dirichlet boundary condition is imposed, Dong and the author established in [6] a similar global Hölder continuity of weak solutions to equations as in (1) with the same controlled growth conditions. In [6] we first proved reverse Hölder inequalities for weak solutions to elliptic and parabolic quasilinear equations, which give slightly better integrability of weak solutions. Specifically, for example, we show that weak solutions in $W_2^1(\Omega)$ to elliptic quasilinear equations are in $W_p^1(\Omega)$ for $p > 2$. The exponent p may not be sufficiently large to give a Hölder continuity of weak solutions via the Sobolev embedding theorem. However, the fact that $p > 2$ is enough to give the boundedness and Hölder continuity of weak solutions by making use of well-known results on quasilinear equations with zero boundary conditions (see [10, 11]). Here the Hölder continuity is for a uniform continuity of weak solutions, but is not necessarily strong enough to give the desired optimal Hölder regularity of solutions. Then using L_p-estimates for linear equations, we increase the exponent p until we have sufficient integrability of solutions guaranteeing the global optimal Hölder regularity of solutions.

In this paper, we continue to investigate the same type of quasilinear equations, but the boundary condition is of the Neumann type. As in [6], the key ingredients of the proofs are a reverse Hölder inequality, boundedness, and Hölder continuity of weak solutions to quasilinear equations, as well as L_p-theory of linear equations with *conormal derivative boundary conditions*. Reverse Hölder inequalities were obtained by Arkhipova [1, 2, 3] for quasilinear equations/systems with conormal derivative boundary conditions under the controlled growth conditions, which we shall briefly discuss in Section 8. As to the boundedness and Hölder continuity of weak solutions, in [6] we referred to the relevant results in [11], where the desired properties of solutions are well explained when the boundary condition is of the Dirichlet type. When conormal derivative boundary conditions are considered, boundedness and Hölder continuity results can be found in [10] and [12] with possibly different growth conditions. In particular, a Hölder continuity estimate is proved in [10, Chapter 10] using a boundary flattening argument when the domain is $C^{1,1}$. Recently, Winkert studied in [17] the boundedness of weak solutions to quasilinear elliptic equations satisfying natural growth conditions with a conormal derivative boundary condition. The growth conditions correspond to the case $\lambda_i = 1$ above if weak solutions are in $W_2^1(\Omega)$. Winkert and Zacher treated in [18] the global boundedness of weak solutions to a conormal derivative problem for nonlinear elliptic equations, where their nonstandard growth conditions cover the *strictly* controlled growth conditions. Since these results cannot be applied directly in the proof of the main theorem (Theorem 2.5), we give detailed proofs about the boundedness and Hölder continuity of solutions to quasilinear equations having the *controlled growth conditions* and *zero conormal derivative boundary value* when the domain is Lipschitz. Regarding the boundary regularity, it may be possible to consider more general regularity conditions than Lipschitz.

The lines of the proof for the boundedness are based on De Giorgi's iteration technique and similar to those in [10, 12, 18]. For the Hölder continuity of weak solutions we follow the argument in [10]. However, in both cases one needs to take care of the conormal boundary condition. In particular, contrary to the Dirichlet case, in the conormal derivative case the technique that extends a solution to be zero outside the domain is not available.

We then run, as in [6], an iteration argument to get the desired L_p regularity by applying L_p estimates for linear equations repeatedly. Note that a similar argument was used by Palagachev in [13], where he derived the global Hölder regularity of solutions. The equations considered in [13] are quasilinear elliptic equations with the Dirichlet boundary condition under the strictly controlled growth conditions, and the leading coefficients are in the class of vanishing mean oscillations (VMO). Also see [14] and [15], where the authors discuss the global Hölder regularity of solutions to Dirichlet problems on Reifenberg flat domains when the leading coefficients have small mean oscillations.

As a final remark, we refer the reader to the paper [6] and references therein for more information about various growth conditions and the (partial) regularity of weak solutions to divergence type elliptic and parabolic equations/systems.

This paper is organized as follows. In Section 2 we introduce our assumptions and the main results of this paper. Then we obtain the boundedness and Hölder continuity of solutions in Sections 3 and 4, respectively. In Section 5 we present some L_p-theory for linear equations which is necessary in the proof of Theorem 2.5 in Section 6. Section 7 is an independent section describing a function class, functions in which satisfy Hölder continuity. Section 8 is devoted to a reverse Hölder inequality.

2. Main results

For a given function $u = u(x)$ defined on $\Omega \subset \mathbb{R}^d$, we use $D_i u$ for $\partial u/\partial x^i$. For $\alpha \in (0, 1]$, we define

$$|u|_{\alpha,\Omega} = |u|_{0,\Omega} + [u]_{\alpha,\Omega} := \sup_{x\in\Omega} |u(x)| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

By $C^\alpha(\Omega)$ we denote the set of all bounded measurable functions u on Ω for which $|u|_{\alpha,\Omega}$ is finite. We write $N(d, p, \ldots)$ if N is a constant depending only on the prescribed quantities $d, p, \ldots$. Throughout the paper, the domain Ω satisfies the following Lipschitz condition, where the constant β will be specified later. Unless specified otherwise, Ω is always bounded.

Assumption 2.1 (β)**.** There is a constant $R_0 \in (0, 1]$ such that, for any $x_0 \in \partial\Omega$ and $r \in (0, R_0]$, there exists a Lipschitz function φ: $\mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega \cap B_r(x_0) = \{x = (x_1, x') \in B_r(x_0) : x_1 > \varphi(x')\}$$

and

$$\sup_{x',y'\in B'_r(x'_0), x'\neq y'} \frac{|\varphi(y') - \varphi(x')|}{|y' - x'|} \leq \beta$$

in an appropriate coordinate system, where $B'_r(x'_0) = \{x' \in \mathbb{R}^{d-1} : |x' - x'_0| < r\}$.

Let us recall the controlled growth conditions on the lower order terms:

(4) $|a_i(x,u)| \le \mu_1(|u|^{\gamma/2} + f), \quad |b(x,u,\nabla u)| \le \mu_2(|\nabla u|^{2(1-1/\gamma)} + |u|^{\gamma-1} + g),$

where μ_1, μ_2, $\mu_3 > 0$ are some constants, γ is defined as in (3), and

$$f \in L_2(\Omega), \quad g \in L_{\frac{\gamma}{\gamma-1}}(\Omega).$$

Theorem 2.2 (Reverse Hölder inequality). *Let $u \in W_2^1(\Omega)$ be a weak solution to (1). Suppose in addition that $f \in L_\sigma(\Omega)$ and $g \in L_\tau(\Omega)$ for some $\sigma \in (2,\infty)$ and $\tau \in (\gamma/(\gamma-1),\infty)$. Then there exists $p > 2$ depending only on d, μ, μ_1, μ_2, γ, u, and β, such that*

$$\|u\|_{L_{\gamma p/2}(\Omega)} + \|Du\|_{L_p(\Omega)} \le N,$$

where $N = N(d,\mu,\mu_1,\mu_2,\sigma,\tau,\gamma,u,\|f\|_{L_\sigma(\Omega)},\|g\|_{L_\tau(\Omega)},\beta,R_0,\mathrm{diam}\Omega)$.

This is proved in [1] for $d > 2$. Also see [3] for a linear case with $d > 2$. For the reader's convenience, we give the key proposition (Proposition 8.2) in Section 8 which readily implies the theorem including the case $d = 2$. As in [1], Theorem 2.2 is true for elliptic *systems* under the same conditions.

To get the optimal global regularity for the equation (1), we need a few more assumptions. Let

$$A_R^{\#} = \sup_{1\le i,j\le d} \sup_{\substack{x_0\in\mathbb{R}^d \\ z_0\in\mathbb{R}, r\le R}} \fint_{B_r(x_0)} \fint_{B_r(x_0)} |A_{ij}(x,z_0) - A_{ij}(y,z_0)|\,dx\,dy.$$

The following assumption indicates that $A_{ij}(x,\cdot)$ have small mean oscillations as functions of $x \in \mathbb{R}^d$.

Assumption 2.3 (ρ). There is a constant $R_1 \in (0,1]$ such that $A_{R_1}^{\#} \le \rho$.

We also need a continuity assumption on $A_{ij}(\cdot,z)$ as functions of $z \in \mathbb{R}$.

Assumption 2.4. There exists a continuous nonnegative function $\omega(r)$ defined on $[0,\infty)$ such that $\omega(0) = 0$ and

$$|A_{ij}(x_0,z_1) - A_{ij}(x_0,z_2)| \le \omega\left(|z_1 - z_2|\right)$$

for all $x_0 \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}$.

Set

$$q^* = \begin{cases} \dfrac{qd}{d-q} & \text{if} \quad q < d, \\ \text{arbitrary large number} > 1 & \text{if} \quad q \ge d. \end{cases}$$

Note that if $q < d$, then $1/q^* = 1/q - 1/d$. We now state the main result of this paper.

Theorem 2.5 (Optimal global regularity). *Let $u \in W_2^1(\Omega)$ be a weak solution to (1). Suppose in addition that $f \in L_\sigma(\Omega)$ and $g \in L_\tau(\Omega)$ for some $\sigma \in (d,\infty)$ and $\tau \in (d/2,\infty)$. Then there exist positive $\beta = \beta(d,\mu,\sigma,\tau)$ and $\rho =$*

$\rho(d,\mu,\sigma,\tau)$ such that, under Assumption 2.1 (β) and Assumption 2.3 (ρ), we have

$$\|u\|_{W^1_p(\Omega)} \le N, \quad \text{where} \quad p = \min\{\sigma, \tau^*\} > d \tag{5}$$

and $N = N(d,\mu,\mu_1,\mu_2,\sigma,\tau,\gamma,u,\|f\|_{L_\sigma(\Omega)},\|g\|_{L_\tau(\Omega)},R_1,\omega,R_0,\mathrm{diam}\Omega)$. Consequently, we have $u \in C^\alpha(\overline{\Omega})$ for $\alpha = 1 - d/p$.

3. Boundedness of solutions under the controlled growth conditions

In the proof of Theorem 2.5 it is essential to have a Hölder continuity of weak solutions to (1). To achieve this, in this section we prove that weak solutions are globally bounded.

Lemma 3.1. *Under the conditions* (2) *and* (4) *with $f \in L_\sigma(\Omega)$ and $g \in L_\tau(\Omega)$ for some $\sigma \in (d,\infty)$ and $\tau \in (d/2,\infty)$, we have*

$$(A_{ij}\xi_j + a_i)\,\xi_i \ge \frac{\mu}{2}|\xi|^2 - N|u|^\gamma - N|u|^2\psi(x), \tag{6}$$

$$|b(x,u,\xi)u| \le \frac{\mu}{4}|\xi|^2 + N|u|^\gamma + N|u|^2\psi(x) \tag{7}$$

for $\xi \in \mathbb{R}^d$ and $|u| \ge 1$, where $\psi \in L_q(\Omega)$, $q := \min\{\sigma/2, \tau\} > \frac{d}{2}$, and $N = N(\mu,\mu_1,\mu_2)$.

Proof. To prove (6), we first see

$$\begin{aligned} |a_i(x,u)\xi_i| &\le \mu_1|\xi|(|u|^{\gamma/2} + f) \\ &\le \varepsilon\mu_1|\xi|^2 + N(\varepsilon)\mu_1|u|^\gamma + N(\varepsilon)\mu_1|f|^2 \\ &\le \varepsilon\mu_1|\xi|^2 + N(\varepsilon)\mu_1|u|^\gamma + N(\varepsilon)\mu_1|u|^2|f|^2, \end{aligned}$$

provided that $|u| \ge 1$. By taking $\varepsilon = \mu/(2\mu_1)$, we have

$$(A_{ij}(x,u)\xi_j + a_i(x,u))\,\xi_i \ge \frac{\mu}{2}|\xi|^2 - N(\mu,\mu_1)|u|^\gamma - N(\mu,\mu_1)|u|^2|f|^2.$$

Now we take $\psi = |f|^2 + g$. Then the inequality (6) follows.

For the inequality (7), we have

$$\begin{aligned} |b(x,u,\xi)||u| &\le \mu_2\left(|u||\xi|^{2(1-1/\gamma)} + |u|^\gamma + |u|g\right) \\ &\le \mu_2\left(\varepsilon|\xi|^2 + N(\varepsilon)|u|^\gamma + |u|^2 g\right) \\ &= \frac{\mu}{4}|\xi|^2 + N(\mu,\mu_2)|u|^\gamma + N(\mu,\mu_2)|u|^2 g \end{aligned}$$

for $|u| \ge 1$. Upon recalling the definition of ψ we obtain the desired inequality. □

Let σ, τ be numbers satisfying $\sigma \in (d,\infty)$ and $\tau \in (d/2,\infty)$, respectively. Find $q_1 \in (1,\infty)$ satisfying

$$\frac{1}{2} < \frac{1}{q_1} \le \frac{\gamma}{4}\left(1 - \frac{2(\gamma-2)}{\gamma p}\right), \quad \frac{1}{2} < \frac{1}{q_1} \le \frac{\gamma}{4}\left(1 - \frac{1}{\min\{\sigma/2,\tau\}}\right), \tag{8}$$

where $p > 2$ is the exponent from Theorem 2.2. Indeed, this is possible since $p > 2$ and

$$\frac{1}{2} = \frac{\gamma}{4}\left(1 - \frac{1}{d/2}\right) < \frac{\gamma}{4}\left(1 - \frac{1}{\min\{\sigma/2, \tau\}}\right) \quad \text{if} \quad d > 2.$$

When $d = 2$, we take $\gamma > 2$ so that

$$\gamma > \frac{2\min\{\sigma/2, \tau\}}{\min\{\sigma/2, \tau\} - 1}.$$

Note $1 < q_1 < 2$ and $\gamma q_1 > 4$.

Lemma 3.2. *Let $u \in W_2^1(\Omega)$ be a solution to* (1) *and $f \in L_\sigma(\Omega)$, $g \in L_\tau(\Omega)$ for some $\sigma \in (d, \infty)$ and $\tau \in (d/2, \infty)$. Then*

$$\int_{A_k} |\nabla u|^2 \, dx \le N \left(\int_{A_k} |u|^{\gamma q_1/2} \, dx\right)^{\frac{4}{\gamma q_1}} \left(\int_{A_k} |u|^{\frac{\gamma q_1(\gamma-2)}{\gamma q_1 - 4}} \, dx\right)^{\frac{\gamma q_1 - 4}{\gamma q_1}} + \left(\int_{A_k} |u|^{\gamma q_1/2} \, dx\right)^{\frac{4}{\gamma q_1}} \left(\int_{A_k} \psi^{\frac{\gamma q_1}{\gamma q_1 - 4}} \, dx\right)^{\frac{\gamma q_1 - 4}{\gamma q_1}},$$

where q_1 is from (8)*, $N = N(\mu, \mu_1, \mu_2)$, and*

$$A_k = \{x \in \Omega : u(x) > k\}, \quad k \ge 1, \quad \psi = |f|^2 + g.$$

Proof. First note that by Theorem 2.2, the definition of ψ, and (8), i.e.,

$$\frac{\gamma q_1(\gamma - 2)}{\gamma q_1 - 4} \le \gamma p/2, \quad \frac{\gamma q_1}{\gamma q_1 - 4} \le \min\{\sigma/2, \tau\},$$

we have

$$\int_{A_k} |u|^{\frac{\gamma q_1(\gamma-2)}{\gamma q_1 - 4}} \, dx < \infty, \quad \int_{A_k} \psi^{\frac{\gamma q_1}{\gamma q_1 - 4}} \, dx < \infty.$$

By taking $\phi = (u - k)_+ \in W_2^1(\Omega)$ as a test function, we obtain

$$\int_{A_k} (A_{ij} D_j u D_i u + a_i D_i u) \, dx = \int_{A_k} b(u - k) \, dx.$$

From Lemma 3.1 it follows that

$$\text{LHS} \ge \frac{\mu}{2} \int_{A_k} |\nabla u|^2 \, dx - N \int_{A_k} |u|^\gamma \, dx - N \int_{A_k} |u|^2 \psi \, dx,$$

and

$$\text{RHS} \le \int_{A_k} |b||u| \, dx \le \frac{\mu}{4} \int_{A_k} |\nabla u|^2 \, dx + N \int_{A_k} |u|^\gamma \, dx + \int_{A_k} |u|^2 \psi \, dx,$$

where $N = N(\mu, \mu_1, \mu_2)$. Combining the above two inequalities gives

$$\int_{A_k} |\nabla u|^2 \, dx \le N \int_{A_k} |u|^\gamma \, dx + N \int_{A_k} |u|^2 \psi \, dx.$$

Then we use Hölder's inequality to obtain the desired inequality (recall that $\gamma q_1/4 > 1$). That is,

$$\int_{A_k} |u|^\gamma \, dx = \int_{A_k} |u|^2 |u|^{\gamma-2} \, dx$$
$$\leq \left(\int_{A_k} |u|^{2\frac{\gamma q_1}{4}} \, dx \right)^{\frac{4}{\gamma q_1}} \left(\int_{A_k} |u|^{\frac{\gamma q_1(\gamma-2)}{\gamma q_1 - 4}} \, dx \right)^{\frac{\gamma q_1 - 4}{\gamma q_1}},$$
$$\int_{A_k} |u|^2 \psi \, dx \leq \left(\int_{A_k} |u|^{2\frac{\gamma q_1}{4}} \, dx \right)^{\frac{4}{\gamma q_1}} \left(\int_{A_k} \psi^{\frac{\gamma q_1}{\gamma q_1 - 4}} \, dx \right)^{\frac{\gamma q_1 - 4}{\gamma q_1}}.$$
□

A similar estimate as in the above lemma is needed on the set $\mathcal{B}_k = \{x \in \Omega : u(x) < k\}$.

Lemma 3.3. *Let $u \in W_2^1(\Omega)$ be a solution to (1) and $f \in L_\sigma(\Omega)$, $g \in L_\tau(\Omega)$ for some $\sigma \in (d, \infty)$ and $\tau \in (d/2, \infty)$. Then*

$$\int_{\mathcal{B}_k} |\nabla u|^2 \, dx \leq N \left(\int_{\mathcal{B}_k} |u|^{\gamma q_1/2} \, dx \right)^{\frac{4}{\gamma q_1}} \left(\int_{\mathcal{B}_k} |u|^{\frac{\gamma q_1(\gamma-2)}{\gamma q_1 - 4}} \, dx \right)^{\frac{\gamma q_1 - 4}{\gamma q_1}}$$
$$+ \left(\int_{\mathcal{B}_k} |u|^{\gamma q_1/2} \, dx \right)^{\frac{4}{\gamma q_1}} \left(\int_{\mathcal{B}_k} \psi^{\frac{\gamma q_1}{\gamma q_1 - 4}} \, dx \right)^{\frac{\gamma q_1 - 4}{\gamma q_1}},$$

where q_1 is from (8), $N = N(\mu, \mu_1, \mu_2)$, and

$$\mathcal{B}_k = \{x \in \Omega : u(x) < k\}, \quad k \leq -1, \quad \psi = |f|^2 + g.$$

Proof. The proof follows from that of Lemma 3.2 with $\phi = (k - u)_+$. □

In the proof of the boundedness of weak solutions, we need the following well-known result. It can be found, for example, in [10, 11] if $\delta_1 = \delta_2$.

Lemma 3.4. *Let $\{\Psi_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of positive numbers satisfying*

$$\Psi_{n+1} \leq K b^n \left(\Psi_n^{1+\delta_1} + \Psi_n^{1+\delta_2} \right), \quad n = 0, 1, 2, \ldots,$$

for some $b > 1$, $K > 0$, and $\delta_2 \geq \delta_1$. If

$$\Psi_0 \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}},$$

then

$$\Psi_n \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2} - \frac{n}{\delta_1}}, \quad n \in \mathbb{N}.$$

Thus, in particular, $\Psi_n \to 0$ as $n \to \infty$.

In the following theorem we prove the boundedness of weak solutions to (1) using an iteration argument of De Giorgi type.

Theorem 3.5. *Let $u \in W_2^1(\Omega)$ be a solution to (1) and $f \in L_\sigma(\Omega)$, $g \in L_\tau(\Omega)$ for some $\sigma \in (d, \infty)$ and $\tau \in (d/2, \infty)$. Then for some number M, depending only on d, μ, μ_1, μ_2, γ, p, σ, τ, u, $\|f\|_{L_\sigma(\Omega)}$, $\|g\|_{L_\tau(\Omega)}$, β, R_0, and* diamΩ, *we have*

$$\|u\|_{L_\infty(\Omega)} \le M.$$

Here p is from Theorem 2.2.

Proof. We take an increasing sequence

$$k_n = k\left(2 - \frac{1}{2^n}\right), \quad n = 0, 1, 2, \dots,$$

where $k \ge 1$ will be specified later. Fix q_1 so that it satisfies (8). Then set

$$\gamma_* = \frac{\gamma q_1}{2} > 2, \quad \Psi_n := \int_{A_{k_n}} (u - k_n)^{\gamma_*}\, dx,$$

where $A_{k_n} = \{x \in \Omega : u(x) > k_n\}$. Note that $\frac{\gamma}{\gamma_*} = \frac{2}{q_1} > 1$ since $q_1 \in (1, 2)$.

Using the fact that $A_{k_{n+1}} \subset A_{k_n}$, we have

$$\begin{aligned}\Psi_n = \int_{A_{k_n}} (u - k_n)^{\gamma_*}\, dx &\ge \int_{A_{k_{n+1}}} (u - k_n)^{\gamma_*}\, dx \\ &\ge \int_{A_{k_{n+1}}} u^{\gamma_*} \left(1 - \frac{k_n}{k_{n+1}}\right)^{\gamma_*} dx \\ &\ge \frac{1}{2^{\gamma_*(n+2)}} \int_{A_{k_{n+1}}} u^{\gamma_*}\, dx.\end{aligned}$$

That is,

$$\int_{A_{k_{n+1}}} u^{\gamma_*}\, dx \le 2^{\gamma_*(n+1)} \Psi_n. \tag{9}$$

We also have

$$\begin{aligned}|A_{k_{n+1}}| &\le \int_{A_{k_{n+1}}} \left(\frac{u - k_n}{k_{n+1} - k_n}\right)^{\gamma_*} dx \\ &\le \int_{A_{k_n}} \left(\frac{2^{n+1}}{k}\right)^{\gamma_*} (u - k_n)^{\gamma_*}\, dx = \frac{2^{\gamma_*(n+1)}}{k^{\gamma_*}} \Psi_n.\end{aligned} \tag{10}$$

We now observe that by Hölder's inequality

$$\begin{aligned}\Psi_{n+1} &= \int_{A_{k_{n+1}}} (u - k_{n+1})^{\gamma_*}\, dx \\ &\le \left(\int_{A_{k_{n+1}}} (u - k_{n+1})^{\gamma}\, dx\right)^{\frac{\gamma_*}{\gamma}} |A_{k_{n+1}}|^{\frac{\gamma - \gamma_*}{\gamma}}.\end{aligned} \tag{11}$$

Note that by the Sobolev embedding theorem,

$$
\begin{aligned}
&\left(\int_{A_{k_{n+1}}} (u-k_{n+1})^\gamma \, dx\right)^{1/\gamma} \\
&\le N\left(\int_\Omega |\nabla(u-k_{n+1})_+|^2 \, dx\right)^{1/2} + N\left(\int_\Omega |(u-k_{n+1})_+|^2 \, dx\right)^{1/2} \\
&\le N\left(\int_{A_{k_{n+1}}} |\nabla u|^2 \, dx\right)^{1/2} + N\left(\int_{A_{k_{n+1}}} |u-k_{n+1}|^2 \, dx\right)^{1/2} \\
&:= I_1 + I_2,
\end{aligned} \tag{12}
$$

where $N = N(d, \gamma, \beta, R_0, \mathrm{diam}\Omega)$. To estimate I_1 in (12), we use Lemma 3.2 and (9) (recall that $\gamma_* = \gamma q_1/2$) to get

$$
\int_{A_{k_{n+1}}} |\nabla u|^2 \, dx \le N\left(\int_{A_{k_{n+1}}} u^{\gamma_*} \, dx\right)^{2/\gamma_*} \le N 2^{2(n+1)} \Psi_n^{\frac{2}{\gamma_*}}, \tag{13}
$$

where

$$
N = N\left(\mu, \mu_1, \mu_2, \gamma, p, \sigma, \tau, \int_\Omega |u|^{\gamma p/2} \, dx, \int_\Omega \psi^{\min\{\sigma/2, \tau\}} \, dx\right).
$$

Using the facts that $\gamma_* > 2$ and $\Psi_{n+1} \le \Psi_n$, the term I_2 in (12) is estimated as

(14)

$$
\begin{aligned}
\left(\int_{A_{k_{n+1}}} |u-k_{n+1}|^2 \, dx\right)^{1/2} &\le |A_{k_{n+1}}|^{1/2-1/\gamma_*} \left(\int_{A_{k_{n+1}}} (u-k_{n+1})^{\gamma_*} \, dx\right)^{1/\gamma_*} \\
&= |A_{k_{n+1}}|^{1/2-1/\gamma_*} \Psi_{n+1}^{1/\gamma_*} \\
&\le |A_{k_{n+1}}|^{1/2-1/\gamma_*} \Psi_n^{1/\gamma_*}.
\end{aligned}
$$

Combining (11), (12), (13), (14), and (10), we obtain

$$
\begin{aligned}
\Psi_{n+1} &\le N |A_{k_{n+1}}|^{\frac{\gamma-\gamma_*}{\gamma}} \left[2^{\gamma_*(n+1)} \Psi_n + |A_{k_{n+1}}|^{\frac{\gamma_*-2}{2}} \Psi_n\right] \\
&\le N\left(\frac{2^{\gamma_*(n+1)}}{k^{\gamma_*}} \Psi_n\right)^{\frac{\gamma-\gamma_*}{\gamma}} 2^{\gamma_*(n+1)} \Psi_n \\
&\quad + N\left(\frac{2^{\gamma_*(n+1)}}{k^{\gamma_*}} \Psi_n\right)^{\frac{\gamma_*(\gamma-2)}{2\gamma}} \Psi_n \\
&=: J_1 + J_2,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= N k^{-\frac{\gamma_*(\gamma-\gamma_*)}{\gamma}} 2^{\gamma_*\left(\frac{\gamma-\gamma_*}{\gamma}+1\right)} 2^{\gamma_*\left(\frac{\gamma-\gamma_*}{\gamma}+1\right)n} \Psi_n^{1+\frac{\gamma-\gamma_*}{\gamma}}, \\
J_2 &= N k^{-\frac{\gamma_*^2(\gamma-2)}{2\gamma}} 2^{\frac{\gamma_*^2(\gamma-2)}{2\gamma}} 2^{\frac{\gamma_*^2(\gamma-2)}{2\gamma} n} \Psi_n^{1+\frac{\gamma_*(\gamma-2)}{2\gamma}}.
\end{aligned}
$$

Set

$$\delta_1 = \frac{\gamma - \gamma_*}{\gamma}, \quad \delta_2 = \frac{\gamma_*(\gamma - 2)}{2\gamma},$$
$$b = \max\left\{2^{\gamma_*\left(\frac{\gamma_*(\gamma-2)}{2\gamma}+1\right)}, 2^{\frac{\gamma_*^2(\gamma-2)}{2\gamma}}\right\}, \quad K = Nk^{-\frac{\gamma_*(\gamma-\gamma_*)}{\gamma}}b.$$

Then $\delta_2 \geq \delta_1 > 0$, $b > 1$, and

$$J_1 \leq Kb^n\Psi_n^{1+\delta_1}, \quad J_2 \leq Kb^n\Psi_n^{1+\delta_2}.$$

Hence

$$\Psi_{n+1} \leq Kb^n\left(\Psi_n^{1+\delta_1} + \Psi_n^{1+\delta_2}\right).$$

Observe that

$$\begin{aligned}\Psi_0 &= \int_{u>k}(u-k)^{\gamma_*}\,dx\\ &\leq \int_\Omega u_+^{\gamma_*}\,dx\\ &= \left((2K)^{\frac{1}{\delta_1}}b^{\frac{1}{\delta_1^2}}\int_\Omega u_+^{\gamma_*}\,dx\right)(2K)^{-\frac{1}{\delta_1}}b^{-\frac{1}{\delta_1^2}}\\ &\leq (2K)^{-\frac{1}{\delta_1}}b^{-\frac{1}{\delta_1^2}}\end{aligned}$$

provided that

$$(2K)^{\frac{1}{\delta_1}}b^{\frac{1}{\delta_1^2}}\int_\Omega u_+^{\gamma_*}\,dx \leq 1,$$

that is, if we take $k \geq 1$ so that

$$k = \max\left\{1, 2^{\frac{1}{\gamma_*\delta_1}}N^{\frac{1}{\gamma_*\delta_1}}b^{\frac{1+\delta_1}{\gamma_*\delta_1^2}}\left(\int_\Omega u_+^{\gamma_*}\,dx\right)^{1/\gamma_*}\right\}.$$

Then by Lemma 3.4 it follows that $u \leq 2k$ on Ω. To prove that u is bounded below, we repeat the above argument using Lemma 3.3. The theorem is proved. □

4. Hölder continuity

The inequality (6) holds true for $|u| \geq 1$. However, from the proof of Lemma 3.1 it is possible to have

$$(A_{ij}D_ju + a_i)\,D_iu \geq \frac{\mu}{2}|\nabla u|^2 - N|u|^\gamma - N|f|^2 \tag{15}$$

for all values of u, where $N = N(\mu, \mu_1)$. Observe that from the condition (4) on b, we obtain

$$\begin{aligned}|b(x,u,\nabla u)(u-k)_+| &\leq \mu_2(u-k)_+\left(|\nabla u|^{2(1-1/\gamma)} + |u|^{\gamma-1} + g\right)\\ &\leq \frac{\mu}{4}|\nabla u|^2 + N(\mu,\mu_2)(u-k)_+^\gamma + \mu_2(u-k)_+\left(|u|^{\gamma-1} + g\right)\end{aligned}$$

$$\le \frac{\mu}{4}|\nabla u|^2 + N\left(1 + |u|^{\gamma-1} + g\right)$$

for $k \ge u - 1$, where $N = N(\mu, \mu_2)$. By the same reasoning

$$|b(x, u, \nabla u)(k-u)_+| \le \frac{\mu}{4}|\nabla u|^2 + N\left(1 + |u|^{\gamma-1} + g\right)$$

for $k \le u + 1$. Set

$$\varphi_0 = |u|^\gamma + |f|^2, \quad \varphi_1 = |u|^{\gamma/2} + f, \quad \varphi_2 = |u|^{\gamma-1} + g + 1. \tag{16}$$

Then by (15) and the condition on a_i we have

$$(A_{ij}D_j u + a_i)D_i u \ge \frac{\mu}{2}|\nabla u|^2 - N\varphi_0, \quad |a_i| \le N\varphi_1 \tag{17}$$

for all values of u, where $N = N(\mu, \mu_1)$. We also have

$$\begin{aligned} &|b(x, u, \nabla u)(u-k)_+| \le \frac{\mu}{4}|\nabla u|^2 + N\varphi_2 \quad \text{for} \quad k \ge u - 1, \\ &|b(x, u, \nabla u)(k-u)_+| \le \frac{\mu}{4}|\nabla u|^2 + N\varphi_2 \quad \text{for} \quad k \le u + 1, \end{aligned} \tag{18}$$

where $N = N(\mu, \mu_2)$. As shown in Theorem 3.5, $|u| \le M$ on Ω for some constant M. Thus, if $f \in L_\sigma(\Omega)$ and $g \in L_\tau(\Omega)$ for some $\sigma \in (d, \infty)$ and $\tau \in (d/2, \infty)$ as in Theorem 2.5, we have

$$\varphi_0, \varphi_2 \in L_q(\Omega) \quad \varphi_1 \in L_{2q}(\Omega), \quad q = \min\{\sigma/2, \tau\} > d/2.$$

Lemma 4.1. *Let $\zeta \in W_2^1(\mathbb{R}^d)$ have a compact support, and k be a real number such that $k \ge u - 1$ on the support of ζ. Let $u \in W_2^1(\Omega)$ be a solution to* (1) *and $f \in L_\sigma(\Omega)$, $g \in L_\tau(\Omega)$ for some $\sigma \in (d, \infty)$ and $\tau \in (d/2, \infty)$. Then we have*

$$\begin{aligned} &\int_{\{u>k\}\cap\Omega} |\zeta \nabla u|^2 \, dx \\ &\le N \int_{\{u>k\}\cap\Omega} |\zeta_x|^2 (u-k)^2 \, dx \\ &\quad + N \left(\int_{\{u>k\}\cap\Omega} \zeta^{2\bar{q}} \, dx \right)^{1/\bar{q}} \left(\int_{\{u>k\}\cap\Omega} \left(\varphi_0 + \varphi_1^2 + \varphi_2\right)^q \, dx \right)^{1/q}, \end{aligned} \tag{19}$$

where φ_i, $i = 0, 1, 2$, are those in (16), *$q = \min\{\sigma/2, \tau\}$, $1/\bar{q} + 1/q = 1$, and $N = N(\mu, \mu_1, \mu_2)$.*

Now if $k \le u + 1$ on the support of ζ, then

$$\begin{aligned} &\int_{\{u<k\}\cap\Omega} |\zeta \nabla u|^2 \, dx \\ &\le N \int_{\{u<k\}\cap\Omega} |\zeta_x|^2 (k-u)^2 \, dx \\ &\quad + N \left(\int_{\{u<k\}\cap\Omega} \zeta^{2\bar{q}} \, dx \right)^{1/\bar{q}} \left(\int_{\{u<k\}\cap\Omega} \left(\varphi_0 + \varphi_1^2 + \varphi_2\right)^q \, dx \right)^{1/q}. \end{aligned} \tag{20}$$

Proof. Using $\phi = \zeta^2(u-k)_+ \in W_2^1(\Omega)$ as a test function, we obtain

$$\int_\Omega (A_{ij}(x,u)D_j u + a_i(x,u))\, \phi_{x_j}\, dx = \int_\Omega b(x,u,\nabla u)\phi\, dx, \tag{21}$$

which is equal to

$$\begin{aligned}&\int_{\{u>k\}\cap\Omega} (A_{ij}D_j u + a_i)\, \zeta^2 D_j u\, dx \\ &= \int_{\{u>k\}\cap\Omega} b\zeta^2(u-k)\, dx - \int_{\{u>k\}\cap\Omega} (A_{ij}D_j u + a_i)\, 2\zeta\zeta_{x_j}(u-k)\, dx.\end{aligned}$$

Note that by (17)

$$\begin{aligned}&\int_{\{u>k\}\cap\Omega} \zeta^2 \left(A_{ij}D_j u + a_i\right) D_j u\, dx \\ &\geq \frac{\mu}{2}\int_{\{u>k\}\cap\Omega} |\zeta\nabla u|^2\, dx - N\int_{\{u>k\}\cap\Omega} \zeta^2\varphi_0\, dx\end{aligned}$$

and

$$\begin{aligned}&-\int_{\{u>k\}\cap\Omega} (A_{ij}D_j u + a_i)\, 2\zeta\zeta_{x_j}(u-k)\, dx \\ &\leq \frac{\mu}{8}\int_{\{u>k\}\cap\Omega} |\zeta\nabla u|^2\, dx + N\int_{\{u>k\}\cap\Omega} \left(|\zeta_x|^2(u-k)^2 + \zeta^2\varphi_1^2\right)\, dx.\end{aligned}$$

Since $k \geq u-1$ on the support of ζ, by (18)

$$\int_{\{u>k\}\cap\Omega} b\zeta^2(u-k)\, dx \leq \frac{\mu}{4}\int_{\{u>k\}\cap\Omega} |\zeta\nabla u|^2\, dx + N\int_{\{u>k\}\cap\Omega} \zeta^2\varphi_2\, dx.$$

Hence (21) is written as

$$\begin{aligned}&\int_{\{u>k\}\cap\Omega} |\zeta\nabla u|^2\, dx \\ &\leq N\int_{\{u>k\}\cap\Omega} |\zeta_x|^2(u-k)^2\, dx + N\int_{\{u>k\}\cap\Omega} \zeta^2\left(\varphi_0 + \varphi_1^2 + \varphi_2\right)\, dx,\end{aligned}$$

where $N = N(\mu, \mu_1, \mu_2)$. Finally, by applying Hölder's inequality we obtain the desired inequality in the lemma. The second assertion follows by the same reasoning as above with $\phi = \zeta^2(k-u)_+$. □

Proposition 4.2. *Let $u \in W_2^1(\Omega)$ be a solution to (1) and $f \in L_\sigma(\Omega)$, $g \in L_\tau(\Omega)$ for some $\sigma \in (d,\infty)$ and $\tau \in (d/2,\infty)$. Then $u \in C^{\alpha_0}(\overline{\Omega})$ and*

$$|u|_{\alpha_0,\Omega} \leq N,$$

where $\alpha_0 \in (0,1)$ and N depend only on the parameters for the bound of u in Theorem 3.5.

Proof. Let $r_0 > 0$ and $B_{r_0} \subset \Omega$. For $r \le r_0$ and $\delta \in (0,1)$, let B_r and $B_{r(1-\delta)}$ be balls concentric with B_{r_0}. Let ζ be an infinitely differentiable function such that $0 \le \zeta \le 1$, $\zeta = 1$ on $B_{r(1-\delta)}$ and $\zeta = 0$ outside B_r. We may assume that $|D\zeta| \le 1/(\delta r)$. Then from (19) we obtain

$$\begin{aligned} &\int_{\{u>k\}\cap B_{r(1-\delta)}} |\nabla u|^2 \, dx \\ &\le C \left(\frac{1}{\delta^2 r^{2-d/q}} \max_{\{u>k\}\cap B_r} (u-k)^2 + 1 \right) |\{u>k\} \cap B_r|^{1-1/q} \end{aligned} \tag{22}$$

for $B_r \subset \Omega$ and $k \ge \max_{B_r} u - 1$, where $\delta \in (0,1)$, $q = \min\{\sigma/2, \tau\} > d/2$, and the constant C depends only on the parameters for the bound of u in Theorem 3.5. Using (20) we also obtain (39) for u. Hence $u \in H(\Omega, M, C, 1, q)$ in Definition 7.1 when $\Omega = B_{r_0}$. Therefore, we have the oscillation estimate in Theorem 7.5, which indeed implies

$$\operatorname{osc}_{B_r} u \le N \left(\frac{r}{r_0} \right)^{\alpha} \operatorname{osc}_{B_{r_0}} u + N r_0^{\alpha_1} r^{\alpha} \tag{23}$$

for all $r \le r_0$, where $\alpha > 0$, $\alpha_1 > 0$, and $N > 0$ depend only on d, C, and q.

Let $x_0 \in \partial\Omega$ and $r_0 < R_0$, where R_0 is from Assumption 2.1. Without loss of generality we assume that $x_0 = 0$ and $\varphi(0) = 0$, where φ is a Lipschitz function such that $\Omega_{R_0} = \Omega \cap B_{R_0} = \{x \in B_{R_0} : x_1 > \varphi(x')\}$. Under this assumption, since $|\varphi(x')| \le \beta |x'|$, we observe that

$$\begin{aligned} \Phi(B_r) \subset \Omega_{r_\beta} \quad &\text{for} \quad r < \frac{R_0}{\sqrt{2(1+\beta^2)}}, \\ \Phi^{-1}(\Omega_r) \subset B_{r_\beta} \quad &\text{for} \quad r < \frac{R_0}{2(1+\beta^2)}, \end{aligned} \tag{24}$$

where $\Phi(y) = (y_1 + \varphi(y'), y')$ and $r_\beta = r\sqrt{2(1+\beta^2)}$. Set $v(y) := u(\Phi(y))$ and $r_1 := \frac{r_0}{\sqrt{2(1+\beta^2)}}$. From (24) we have $v \in W_2^1(B_{r_1}^+)$. For $r \in (0, r_1]$, let ψ be an infinitely differentiable function such that $0 \le \psi \le 1$, $\psi = 0$ on $B_{r(1-\delta)}$, and $\psi = 0$ outside B_r. We may assume that $|D\psi| \le 1/(\delta r)$. By using

$$\zeta(x) = \psi(\Phi^{-1}(x)),$$

we obtain from (19)

$$\int_{\{u>k\}\cap\Omega} |\zeta \nabla u|^2 \, dx \le N \int_{\{u>k\}\cap\Omega} |\zeta_x|^2 (u-k)^2 \, dx + C \left(\int_{\{u>k\}\cap\Omega} \zeta^{2\bar{q}} \, dx \right)^{1/\bar{q}}$$

for $k \ge u - 1$ on the support of ζ, where C is a constant as in (22). By the change of variables, this turns into

$$\int_{\{v>k\}\cap\Omega_{r(1-\delta)}} |\nabla v|^2 \, dx$$

$$\leq C\left(\frac{1}{\delta^2 r^{2-d/q}} \max_{\{v>k\}\cap\Omega_r}(v-k)^2+1\right)|\{v>k\}\cap\Omega_r|^{1-1/q},$$

where $\Omega_r = B_r^+ = \{y \in \mathbb{R}^d : |y| < r, y_1 > 0\}$. Similarly, the inequality (39) is proved for v. Hence $v \in H(\Omega, M, C, 1, q)$ in Definition 7.1 when $\Omega = B_{r_1}^+$. Therefore, by Theorem 7.5 we have

$$\operatorname{osc}_{B_r^+} v \leq N\left(\frac{r}{r_1}\right)^{\alpha}$$

for all $r \leq r_1$, where $\alpha = \alpha(d, C, q)$ and $N = N(d, C, q, r_1, M)$. This together with the definition of v and (24) shows that, for any $r < \frac{r_0}{2(1+\beta^2)} =: r_2$,

$$\operatorname{osc}_{\Omega_r} u \leq \operatorname{osc}_{B_{r_\beta}^+} v \leq N\left(\frac{r}{r_2}\right)^{\alpha}.$$

Finally, we use this inequality and (23) to finish the proof (for details, see Theorem 8.29 in [9]). □

5. L_p-estimates for linear equations

In the proof of Theorem 2.5 where we prove the global Hölder regularity result, it is essential to use some results from L_p-theory for linear elliptic equations. In this section, we consider the linear equation

$$\begin{cases} -D_i(a_{ij}D_jv)+\lambda v = D_ih_i+h & \text{in } \Omega, \\ (a_{ij}D_jv+h_i)\,\nu_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{25}$$

where ν is the outward normal vector to the surface $\partial\Omega$ and $\lambda > 0$, and present some L_p-solvability as well as L_p-estimates necessary for the proof of Theorem 2.5.

We assume that the leading coefficients a_{ij} have small mean oscillations with respect to $x \in \mathbb{R}^d$. To describe this assumption, we set

$$a_R^{\#} = \sup_{1\leq i,j\leq d} \sup_{\substack{x_0\in\mathbb{R}^d \\ r\leq R}} \fint_{B_r(x_0)} \fint_{B_r(x_0)} |a_{ij}(x)-a_{ij}(y)|\,dx\,dy.$$

Assume that $|a_{ij}(x)| \leq \mu^{-1}$ and $a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Also we assume

Assumption 5.1 (ρ_1)**.** There is a constant $R_1 \in (0,1]$ such that $a_{R_1}^{\#} \leq \rho_1$.

We use the following result from [4]. By a half space, we mean, for example, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_1 > 0\}$.

Proposition 5.2. *Let Ω be the whole space $\mathbb{R}^d$, a half space, or a bounded Lipschitz domain. Let $\hat{p} \in (2,\infty)$, $p \in [\hat{p}/(\hat{p}-1), \hat{p}]$, $h_i \in L_p(\Omega)$, and $h \in L_p(\Omega)$.*

(1) *If Ω is the whole space $\mathbb{R}^d$ or a half space $\mathbb{R}^d_+$, there exists a positive $\rho_1 = \rho_1(d, \mu, \hat{p})$ such that, under Assumption 5.1 (ρ_1), there is a unique $v \in W^1_p(\Omega)$ satisfying* (25) *and*

$$\sqrt{\lambda}\|v_x\|_{L_p(\Omega)} + \lambda\|v\|_{L_p(\Omega)} \le N\sqrt{\lambda}\|h_i\|_{L_p(\Omega)} + N\|h\|_{L_p(\Omega)} \tag{26}$$

provided that $\lambda \ge \lambda_0$, where $N > 0$ and $\lambda_0 > 0$ are constants depending only on d, μ, $\hat{p}$, and R_1.

(2) *If Ω is a bounded Lipschitz domain, there exist positive $\beta = \beta(d, \mu, \hat{p})$ and $\rho_1 = \rho_1(d, \mu, \hat{p})$ such that, under Assumption 2.1 (β) and Assumption 5.1 (ρ_1), there is a unique $v \in W^1_p(\Omega)$ satisfying* (25) *and* (26) *provided that $\lambda \ge \lambda_0$, where $N > 0$ and $\lambda_0 \ge 0$ are constants depending only on d, μ, $\hat{p}$, R_0, R_1, and* $\operatorname{diam}\Omega$.

The proposition above was proved in [4] so that the choices of β and ρ_1 may be different depending on p. Also see [5]. To find uniform ρ_1 and β for all $p \in [\hat{p}/(\hat{p}-1), \hat{p}]$, we use the cited result and an interpolation argument as in [6]. Indeed, if we have the $W^1_{\hat{p}}$ solvability of (25) for some a_{ij} and Ω, by the duality, the $W^1_{\hat{p}/(\hat{p}-1)}$ solvability follows. Then we apply Marcinkiewicz's theorem to get the W^1_p solvability for any $p \in [\hat{p}/(\hat{p}-1), \hat{p}]$.

By using Proposition 5.2, we derive the following theorem, where h may have less integrability than those in Proposition 5.2. Again, the constants β and ρ_1 are found independent of σ and q as long as σ and q^* are in an a prior fixed interval. Recall the definition of q^* given above Theorem 2.5.

Theorem 5.3. *Let Ω be a bounded Lipschitz domain, $\sigma, q \in (1, \infty)$, $\hat{p} \in (2, \infty)$, $h_i \in L_\sigma(\Omega)$, and $h \in L_q(\Omega)$. Assume that $\sigma, q^* \in [\hat{p}/(\hat{p}-1), \hat{p}]$. Then there exist positive $\beta = \beta(d, \mu, \hat{p})$ and $\rho_1 = \rho_1(d, \mu, \hat{p})$ such that, under Assumption 2.1 (β) and Assumption 5.1 (ρ_1), there is a unique $v \in W^1_p(\Omega)$ satisfying* (25) *and*

$$\|v\|_{W^1_p(\Omega)} \le N\|h_i\|_{L_\sigma(\Omega)} + N\|h\|_{L_q(\Omega)},$$

provided that $\lambda \ge \lambda_0$, where $p := \min\{\sigma, q^*\}$ and

$$\begin{aligned} &\lambda_0 = \lambda_0(d, \mu, \hat{p}, R_0, R_1, \operatorname{diam}\Omega) \ge 0, \\ &N = N(d, \mu, \hat{p}, \sigma, q, q^*, R_0, R_1, \lambda, \operatorname{diam}\Omega) > 0. \end{aligned} \tag{27}$$

Proof. We split the equation (25) into two linear equations with $h \equiv 0$ and $h_i \equiv 0$, $i = 1, \dots, d$, respectively. Since $h_i \in L_p(\Omega) \subset L_\sigma(\Omega)$ and $p \in [\hat{p}/(\hat{p}-1), \hat{p}]$, by Proposition 5.2, we have constants β and ρ_1, depending only on d, μ, and $\hat{p}$, such that, under Assumption 2.1 (β) and Assumption 5.1 (ρ_1), there exists a unique solution $v \in W^1_p(\Omega)$ to the equation (25) with $h \equiv 0$ satisfying

$$\|v\|_{W^1_p(\Omega)} \le N\|h_i\|_{L_\sigma(\Omega)},$$

provided that $\lambda \ge \lambda_0$, where λ_0 and N depend only on the parameters in (27).

Now let $h_i \equiv 0$, $i = 1, \dots, d$. Thanks to the localization argument using a partition of unity, it is enough to show the existence and uniqueness of a solution in $W^1_{q^*}(\Omega)$ of the equation (25) along with the following estimate when $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^d_+$:

$$\|v\|_{W^1_{q^*}(\Omega)} \le N\|h\|_{L_q(\Omega)}. \tag{28}$$

In case $\Omega = \mathbb{R}^d$, since $h \in L_q(\Omega)$, we find a unique solution $w \in W^2_q(\mathbb{R}^d)$ to the equation

$$-\Delta w + \lambda w = h \quad \text{in} \quad \Omega$$

satisfying

$$\|w\|_{W^2_q(\Omega)} \le N\|h\|_{L_q(\Omega)},$$

where $\lambda > 0$ and $N = N(d, q, \lambda)$. From the above inequality and the Sobolev imbedding theorem, we know that $w \in W^1_{q^*}(\Omega)$ and

$$\|w\|_{W^1_{q^*}(\Omega)} \le N\|h\|_{L_q(\Omega)}. \tag{29}$$

Since $q^* \in [\hat{p}/(\hat{p}-1), \hat{p}]$, by Proposition 5.2 we have $\rho_1 = \rho_1(d, \mu, \hat{p}) > 0$ such that, under Assumption 5.1 (ρ_1), there is a unique solution $\hat{w} \in W^1_{q^*}(\Omega)$ to the equation

$$-D_i(a_{ij}D_j\hat{w}) + \lambda\hat{w} = D_i((a_{ij} - \delta_{ij})D_j w) \quad \text{in} \quad \Omega$$

satisfying

$$\|\hat{w}\|_{W^1_{q^*}(\Omega)} \le N\|Dw\|_{L_{q^*}(\Omega)}, \tag{30}$$

provided that $\lambda \ge \lambda_0$, where $\lambda_0 = \lambda_0(d, \mu, \hat{p}, R_1)$ and $N = N(d, \mu, \hat{p}, R_1, \lambda)$. Clearly $v := w + \hat{w} \in W^1_{q^*}(\Omega)$ is a unique solution to (25) when $h_i \equiv 0$, $i = 1, \ldots, d$. By (30) and (29) the solution v satisfies (28).

In the case that $\Omega = \mathbb{R}^d_+$, let w be the unique $W^2_q(\mathbb{R}^d)$ solution to $-\Delta w + \lambda w = \bar{h}$ in $\mathbb{R}^d$, where $\bar{h}$ is the even extension of h with respect to x_1. Clearly $w_{x_1} = 0$ on $\partial\mathbb{R}^d$, and as before we know that $w \in W^1_{q^*}(\Omega)$ and satisfies (29). Now we argue as in the previous case. In particular, note that $v := w + \hat{w}$ satisfies the boundary condition in (25). □

Remark 5.4. A Dirichlet problem version of the above theorem is proved in [13], where an $F \in L_{q^*}(\Omega)$ satisfying $\operatorname{div} F = h$ is found directly, thanks to the Dirichlet boundary condition, by using a Newtonian potential. Here, since we have the conormal derivative boundary condition, the argument in [13] is not applicable. Instead, we have gone through the interior estimates (when $\Omega = \mathbb{R}^d$), the boundary estimates (when $\Omega = \mathbb{R}^d_+$), and the well-known partition of unity argument. In the above theorem as well as Proposition 5.2 for a bounded Lipschitz domain Ω, the λ_0 can be made equal to zero (If $\lambda = 0$ in (25) we need $\int_\Omega h\,dx = 0$). See Section 7 in [4]. However, we do not pursue this direction here.

6. Proof of Theorem 2.5

Under the assumptions in Theorem 2.5, Proposition 4.2 says that u is globally Hölder continuous on Ω. Then one can have an extension $\bar{u}(x)$ on $\mathbb{R}^d$ of $u(x)$ such that $\bar{u}(x)$ is Hölder continuous on $\mathbb{R}^d$ with the same Hölder exponent. Now we define

$$a_{ij}(x) := A_{ij}(x, \bar{u}(x)).$$

Also define

$$h_i(x) := a_i(x, u(x)), \quad h(x) := b(x, u(x), \nabla u(x)).$$

Then the equation (1) turns into

$$\begin{cases} -D_i(a_{ij}D_j u) = h_i(t,x)) + h(t,x) \quad \text{in} \quad \Omega, \\ (a_{ij}D_j v + h_i)\nu_i = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{31}$$

where ν is the outward normal vector to the surface $\partial\Omega$. Note that

$$|h_i(x)| \le \mu_1(|u|^{\gamma/2} + f) \le \mu_1(M + f) \in L_\sigma(\Omega), \tag{32}$$

where M is from Theorem 3.5, and

$$|h(x)| \le \mu_2(|\nabla u|^{2(1-1/\gamma)} + |u|^{\gamma-1} + g). \tag{33}$$

The coefficients a_{ij} in (31) satisfy, for any $x_0 \in \mathbb{R}^d$,

$$\begin{aligned} & ⨍_{x,y \in B_r(x_0)} |a_{ij}(x) - a_{ij}(y)| \, dx \, dy \\ = & ⨍_{x,y \in B_r(x_0)} |A_{ij}(x, \bar{u}(x)) - A_{ij}(y, \bar{u}(y))| \, dx \, dy \\ \le & ⨍_{x \in B_r(x_0)} |A_{ij}(x, \bar{u}(x)) - A_{ij}(x, \bar{u}(x_0))| \, dx \\ & + ⨍_{x,y \in B_r(x_0)} |A_{ij}(x, \bar{u}(x_0)) - A_{ij}(y, \bar{u}(x_0))| \, dx \, dy \\ & + ⨍_{y \in B_r(x_0)} |A_{ij}(y, \bar{u}(x_0)) - A_{ij}(y, \bar{u}(y))| \, dy \\ \le & 2\omega(N r^{\alpha_0}) + A_r^\#, \end{aligned}$$

where the last inequality is due to Assumption 2.3 and Proposition 4.2. That is, by using the notation in Section 5, we have

$$a_R^\# \le 2\omega(N R^{\alpha_0}) + A_R^\#.$$

Then by Assumptions 2.3 and 2.4 there exists $R_2 \in (0, R_1]$ such that

$$a_{R_2}^\# \le 2\rho, \tag{34}$$

where R_2 depends on the function ω.

Proof of Theorem 2.5. We set $\hat{p}$ to be $\max\{\sigma, \tau^*\}$, and fix

$$\beta = \beta(d, \hat{p}, \mu), \quad \rho = \frac{1}{2}\rho_1(d, \hat{p}, \mu), \tag{35}$$

where $\beta(d, \hat{p}, \mu)$ and $\rho_1(d, \hat{p}, \mu)$ are those in Theorem 5.3. Also fix $\lambda \geq \lambda_0$, where $\lambda_0 = \lambda_0(d, \mu, \hat{p}, R_0, R_1, \mathrm{diam}\Omega)$ is taken from Theorem 5.3.

By Theorem 2.2 there exists $p_0 > 2$ such that $u \in W^1_{p_0}(\Omega)$. If $p_0 \geq \min\{\sigma, \tau^*\}$, we immediately obtain (5). Otherwise, we see that u satisfies (31). By (32) and (33), $h_i \in L_\sigma(\Omega)$ and $h \in L_{q_1}(\Omega)$, where

$$q_1 = \min\left\{\frac{\gamma}{2(\gamma-1)}p_0, \tau\right\}.$$

By taking $\left(\frac{\gamma}{2(\gamma-1)}p_0\right)^*$ to be τ in the case that

$$\frac{\gamma}{2(\gamma-1)}p_0 \geq d,$$

we see that

$$2 < p_0 < q_1^* \leq \tau^*.$$

Indeed, it is easily verified because

$$q_1^* = \left(\frac{\gamma}{2(\gamma-1)}p_0\right)^* = \frac{\gamma d p_0}{2\gamma d - 2d - \gamma p_0} > p_0 \quad \text{when} \quad \frac{\gamma}{2(\gamma-1)}p_0 < d. \tag{36}$$

Moreover, $2 < \sigma \leq \hat{p}$. Hence we have

$$\sigma, q_1^* \in [\hat{p}/(\hat{p}-1), \hat{p}]. \tag{37}$$

Set $p_1 = \min\{\sigma, q_1^*\}$. Then

$$p_1 = \begin{cases} \min\{\sigma, \tau^*\} & \text{if} \quad \dfrac{\gamma}{2(\gamma-1)}p_0 \geq d, \\ \min\{\sigma, \left(\dfrac{\gamma}{2(\gamma-1)}p_0\right)^*, \tau^*\} & \text{if} \quad \dfrac{\gamma}{2(\gamma-1)}p_0 < d. \end{cases}$$

Observe that u satisfies

$$\begin{cases} -D_i(a_{ij}D_j u) + \lambda u = D_i h_i + h + \lambda u \quad \text{in} \quad \Omega, \\ \quad (a_{ij}D_j v + h_i)\,\nu_i = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where a_{ij}, h_i, and h are those in (31). Also observe that $u \in L_{q_1}(\Omega)$ because $\frac{\gamma}{2(\gamma-1)} < 1$. Thus by Theorem 5.3 along with (34) and (35) applied to (31) we have $u \in W^1_{p_1}(\Omega)$ and

$$\|u\|_{W^1_{p_1}(\Omega)} \leq N\left(\|h_i\|_{L_\sigma(\Omega)} + \|h\|_{L_{q_1}(\Omega)} + \|u\|_{L_{q_1}(\Omega)}\right),$$

where $N = N(d, \mu, \hat{p}, \sigma, q_1, q_1^*, R_0, R_2, \lambda, \mathrm{diam}\Omega)$. Bearing in mind the definitions of h_i and h as well as using Theorem 2.2, we obtain (5) unless

$$\frac{\gamma}{2(\gamma-1)}p_0 < d \quad \text{and} \quad \left(\frac{\gamma}{2(\gamma-1)}p_0\right)^* < \min\{\sigma, \tau^*\}. \tag{38}$$

In this case, $p_1 = q_1^* = \left(\frac{\gamma}{2(\gamma-1)}p_0\right)^*$ and, as seen in (36), $p_1 > p_0$. Now, since $u \in W^1_{p_1}(\Omega)$, by (33) it follows that

$$h \in L_{q_2}(\Omega), \quad q_2 = \min\left\{\frac{\gamma}{2(\gamma-1)}p_1, \tau\right\}.$$

Note that $q_2 > q_1$. We define $p_2 = \min\{\sigma, q_2^*\} > p_1$. Then we see that (37) is satisfied with q_2^* in place of q_1^* and $u \in L_{q_2}(\Omega)$. By repeating the above argument, we obtain (5) unless (38) holds with p_1 in place of p_0. We continue, if necessary, repeating the above argument to obtain $p_3, p_4, \dots$ with the recursion formula

$$p_{k+1} = \left(\frac{\gamma}{2(\gamma-1)}p_k\right)^* = \frac{\gamma d p_k}{2\gamma d - 2d - \gamma p_k}, \quad k = 0, 1, 2, \dots.$$

Since

$$p_{k+1} - p_k \geq \frac{p_0(\gamma p_0 + 2d - \gamma d)}{2\gamma d - 2d},$$

there has to be an integer k_0 such that $p = p_{k_0} = \min\{\sigma, \tau^*\}$. Note that (37) holds true with q_k^* in place of q_1^* for all $k = 1, \dots, k_0$. This allows us to use Theorem 5.3 in the above iteration process with the same β and ρ in (35) for all $k = 1, \dots, k_0$. □

7. Functions in the class *H*

Throughout the section, the domain Ω is either B_{r_0} or $B^+_{r_0} = \{x \in B_{r_0}, x_1 > 0\}$, and $\Omega_r = \Omega \cap B_r$, where B_r is concentric with B_{r_0}. The results in this section are those in [10, Chater 2, section 6], where the interior Hölder regularity is proved. We slightly modified the statements in [10] so that they also work for the boundary Hölder regularity. We also give precise parameters on which the constants in the statements depend. We omit here the proofs since they can be done in the same way as in [10].

Definition 7.1. Let $r_0 > 0$, $M > 0$, $C > 0$, $\kappa > 0$, $q > d/2$ be real numbers, and $\Omega = B_{\rho_0}$ or $\Omega = B^+_{\rho_0}$. We say $v \in H(\Omega, M, C, \kappa, q)$ if $v \in W^1_2(\Omega)$ satisfies $|v| \leq M$ as well as the following two inequalities for any $r \in (0, r_0]$ and $\delta \in (0, 1)$:

$$\int_{\{v>k\}\cap\Omega_{r(1-\delta)}} |\nabla v|^2 \, dx$$
$$\leq C\left(\frac{1}{\delta^2 r^{2-d/q}} \max_{\{v>k\}\cap\Omega_r} (v-k)^2 + 1\right) |\{v > k\} \cap \Omega_r|^{1-1/q}$$

for $k \geq \max_{\Omega_r} v - \kappa$, and

$$\begin{aligned}&\int_{\{v<k\}\cap\Omega_{r(1-\delta)}} |\nabla v|^2\, dx \\ &\leq C\left(\frac{1}{\delta^2 r^{2-d/q}} \max_{\{v<k\}\cap\Omega_r}(k-v)^2 + 1\right) |\{v<k\}\cap\Omega_r|^{1-1/q}\end{aligned} \tag{39}$$

for $k \leq \min_{\Omega_r} v + \kappa$.

The following lemma is Lemma 2.3.5 in [10] with $\Omega = B_r$. As noted there, it also works for any convex domains.

Lemma 7.2. *Let $\Omega = B_r$ or $\Omega = B_r^+$. Then for an arbitrary function v in $W_1^1(\Omega)$ and for arbitrary k and l such that $k \leq l$,*

$$(l-k)|\{v>l\}\cap\Omega|^{1-1/d} \leq N\frac{r^d}{|\{v\leq k\}\cap\Omega|}\int_{\{k<v\leq l\}\cap\Omega}|\nabla v|\, dx,$$

where $N = N(d)$.

Lemma 7.3. *Let $v \in H(\Omega, M, C, \kappa, q)$. Then there exists a $\theta_1 = \theta_1(d, C, q) > 0$ such that, for any $\Omega_r \subset \Omega$ and for any number $k \geq \max_{\Omega_r} u(x) - \kappa$, the inequality*

$$|\Omega_r \cap \{v > k\}| \leq \theta_1 r^d$$

implies

$$|\Omega_{r/2} \cap \{v > k + K/2\}| = 0,$$

provided that

$$K = \max_{\Omega_r} u - k \geq r^{1-\frac{d}{2q}}.$$

Lemma 7.4. *Let $v \in H(\Omega, M, C, \kappa, q)$. Then there exists a positive integer $s = s(d, C, q)$ such that, for any $\Omega_r \subset \Omega_{4r} \subset \Omega$, at least one of the following two inequalities holds*:

$$\operatorname{osc}_{\Omega_r} v \leq 2^s r^{1-\frac{d}{2q}}, \quad \operatorname{osc}_{\Omega_r} v \leq \left(1 - \frac{1}{2^{s-1}}\right)\operatorname{osc}_{\Omega_{4r}} v.$$

Theorem 7.5. *Let $v \in H(\Omega, M, C, \kappa, q)$. Then for all $r \leq r_0$, we have*

$$\operatorname{osc}_{\Omega_r} v \leq N\left(\frac{r}{r_0}\right)^\alpha,$$

where

$$\alpha = \min\left\{-\log_4\left(1 - \frac{1}{2^{s-1}}\right), 1 - \frac{d}{2q}\right\},$$

$$N = 4^\alpha \max\left\{\operatorname{osc}_{\Omega_{r_0}} u, 2^s r_0^{1-\frac{d}{2q}}\right\},$$

and the number s is taken from Lemma 7.4.

8. Reverse Hölder inequality

Recall the definition of γ in (3). Also recall that, throughout the section, Ω is a bounded domain satisfying Assumption 2.1. Let

$$c = (u)_{B_R(x_0)} = \fint_{B_R} u\,dx \quad \text{and} \quad q = \frac{2d}{d+2}.$$

Then by the Poincaré inequality, we have

$$\int_{B_R} |u-c|^2\,dx \le N(d) R^{d+2-2d/q} \left(\int_{B_R} |\nabla u|^q\,dx \right)^{2/q},$$

$$\int_{B_R} |u-c|^\gamma\,dx \le N(d,\gamma) R^{d+\gamma-\gamma d/2} \left(\int_{B_R} |\nabla u|^2\,dx \right)^{\gamma/2}.$$

These inequalities also hold true if B_R is replaced by $B_R^+ = \{|x| < R : x_1 > 0\}$. As before, we write $\Omega_r(x) = \Omega \cap B_r(x)$.

Lemma 8.1. *Let Ω be a Lipschitz domain satisfying Assumption 2.1, $u \in W_2^1(\Omega)$, and $x_0 \in \partial\Omega$. Then for $R \le R_0$, we have*

$$\int_{\Omega_r(x_0)} |u-c|^2\,dx \le N_1 R^{d+2-2d/q} \left(\int_{\Omega_R(x_0)} |\nabla u|^q\,dx \right)^{2/q}, \tag{40}$$

$$\int_{\Omega_r(x_0)} |u-c|^\gamma\,dx \le N_2 R^{d+\gamma-\gamma d/2} \left(\int_{\Omega_R(x_0)} |\nabla u|^2\,dx \right)^{\gamma/2},$$

where $r = \frac{R}{2(1+\beta^2)}$, $q = 2d/(d+2)$, $N_1 = N_1(d,\beta)$, $N_2 = N_2(d,\gamma,\beta)$, and

$$c = (u)_{\Omega_r(x_0)} = \fint_{\Omega_r(x_0)} u\,dx.$$

Proof. Without loss of generality we assume that $x_0 = 0$ and $\varphi(0) = 0$, where φ is a Lipschitz function such that $\Omega_{R_0} = \Omega \cap B_{R_0} = \{x \in B_{R_0} : x_1 > \varphi(x')\}$. Let $\Phi(y) = (y_1 + \varphi(y'), y')$ and $\Phi^{-1}(x) = \Psi(x) = (x_1 - \varphi(x'), x')$. Also let $v(y) = u(\Phi(y))$. Then by (24) we can say $v \in W_2^1(B_{R_1})$, where $R_1 = \frac{R}{\sqrt{2(1+\beta^2)}}$. From the Poincaré inequality above for a half ball it follows that

$$\int_{B_{R_1}^+} |v - (v)_{B_{R_1}^+}|^2\,dy \le N R^{d+2-2d/q} \left(\int_{B_{R_1}^+} |\nabla v|^q\,dy \right)^{2/q}.$$

Here

$$(v)_{B_{R_1}^+} = \fint_{B_{R_1}^+} v(y)\,dy.$$

From this and the set inclusions in (24) we see that

$$\int_{\Omega_r} |u - (v)_{B_{R_1}^+}|^2\,dx \le \int_{\Phi(B_{R_1}) \cap \{x_1 > \varphi(x')\}} |u - (v)_{B_{R_1}^+}|^2\,dx$$

$$
\begin{aligned}
&= \int_{B^+_{R_1}} |v - (v)_{B^+_{R_1}}|^2 \, dy \\
&\le N R^{d+2-2d/q} \left(\int_{B^+_{R_1}} |\nabla v|^q \, dy \right)^{2/q} \\
&\le N R^{d+2-2d/q} \left(\int_{\Omega_R} |\nabla u|^q \, dy \right)^{2/q},
\end{aligned}
$$

where $N = N(d, \beta)$. Now the inequality (40) follows because

$$
\int_{\Omega_r} |u - (u)_{\Omega_r}|^2 \, dx \le \int_{\Omega_r} |u - C|^2 \, dx
$$

for any constant C. The other inequality follows similarly. The lemma is proved. □

Theorem 2.2 is proved by the following proposition combined with Proposition V.1.1 in [8]. Also see the proof of Theorem 3.6 in [6].

Proposition 8.2. *Let $R \le R_0$, $u \in W^1_2(\Omega)$ be a weak solution to* (1)*, $f \in L_2(\Omega)$, and $g \in L_{\frac{\gamma}{\gamma-1}}(\Omega)$. Then, for any $\Omega_R(x_0)$, where either $B_R(x_0) \subset \Omega$ or $x_0 \in \partial\Omega$, we have*

$$
\begin{aligned}
&\fint_{\Omega_{\varrho R}(x_0)} (|\nabla u|^2 + |u|^\gamma) \\
&\le N \left(\fint_{\Omega_R(x_0)} |\nabla u|^q + |u|^{\gamma q/2} \right)^{\frac{2}{q}} + N \fint_{\Omega_R(x_0)} \left(|f|^2 + |F|^2 \right) \\
&\quad + N R^{d\gamma\left(\frac{1}{\gamma} - \frac{1}{2}\right) + \gamma} \left(\int_{\Omega_R(x_0)} |\nabla u|^2 \right)^{\frac{\gamma}{2} - 1} \left(\fint_{\Omega_R(x_0)} |\nabla u|^2 \right),
\end{aligned}
$$

where $\varrho = \frac{1}{4(1+\beta^2)} \in (0,1)$, $q = \frac{2d}{d+2}$, $F = |g|^{\frac{1}{2}\frac{\gamma}{\gamma-1}}$, and $N = N(d, \mu, \mu_1, \mu_2, \gamma, \beta)$.

Note that $R^{d\gamma\left(\frac{1}{\gamma} - \frac{1}{2}\right) + \gamma} = 1$ if $d > 2$ and $R^{d\gamma\left(\frac{1}{\gamma} - \frac{1}{2}\right) + \gamma} = R^2$ if $d = 2$.

Proof. We only show the case $x_0 \in \partial\Omega$. The other case follows the same lines. Let $\eta_0 \in C_0^\infty(\mathbb{R}^d)$ be a function satisfying $0 \le \eta_0 \le 1$ and

$$
\eta_0 = \begin{cases} 1 & \text{for} \quad |x| \le \dfrac{1}{4(1+\beta^2)}, \\[2ex] 0 & \text{for} \quad |x| \ge \dfrac{1}{2(1+\beta^2)}. \end{cases}
$$

Set $r = \frac{R}{2(1+\beta^2)}$, $c = (u)_{\Omega_r(x_0)} = \fint_{\Omega_r(x_0)} u \, dx$, and $\eta = \eta_0(R^{-1}(\cdot - x_0))$.

Using a test function $(u - c)\eta^2$, we have

$$
\int_{\Omega_R(x_0)} A_{ij}(x, u) D_i \left[(u - c)\eta^2 \right] D_j u + \int_{\Omega_R(x_0)} a_i(x, u) D_i \left[(u - c)\eta^2 \right]
$$

$$= \int_{\Omega_R(x_0)} b(x,u,\nabla)(u-c)\eta^2.$$

That is,

$$\begin{aligned}
&\int_{\Omega_R(x_0)} A_{ij}\eta(D_i u)\eta(D_j u) \\
&= -\int_{\Omega_R(x_0)} 2A_{ij}(u-c)\eta(D_i\eta)(D_j u) \\
&\quad -\int_{\Omega_R(x_0)} a_i D_i\left[(u-c)\eta^2\right] + \int_{\Omega_R(x_0)} b(u-c)\eta^2 \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{41}$$

We estimate J_1, J_2 and J_3 by using Young's inequality and the conditions on A_{ij}, a_i, and b.

Estimate of J_1:

$$\begin{aligned}
J_1 &\le 2\mu^{-1}\int_{\Omega_R(x_0)} |\nabla u||u-c|\eta|\nabla\eta| \\
&\le \frac{\mu}{16}\int_{\Omega_R(x_0)} \eta^2|\nabla u|^2 + N\int_{\Omega_R(x_0)} |u-c|^2|\nabla\eta|^2.
\end{aligned}$$

Estimate of J_2:

$$\begin{aligned}
J_2 &\le \mu_1\int_{\Omega_R(x_0)} |u|^{\gamma/2}|\nabla u|\eta^2 + \mu_1\int_{\Omega_R(x_0)} |f||\nabla u|\eta^2 \\
&\quad + 2\mu_1\int_{\Omega_R(x_0)} |u|^{\gamma/2}|u-c||\nabla\eta|\eta + 2\mu_1\int_{\Omega_R(x_0)} |f||u-c||\nabla\eta|\eta \\
&\le \frac{\mu}{16}\int_{\Omega_R(x_0)} |\nabla u|^2\eta^2 + N\int_{\Omega_R(x_0)} |u|^\gamma\eta^2 \\
&\quad + N\int_{\Omega_R(x_0)} |f|^2\eta^2 + N\int_{\Omega_R(x_0)} |u-c|^2|\nabla\eta|^2.
\end{aligned}$$

Estimate of J_3:

$$\begin{aligned}
J_3 &\le \mu_2\int_{\Omega_R(x_0)} |\nabla u|^{2(1-1/\gamma)}|u-c|\eta^2 + \mu_2\int_{\Omega_R(x_0)} |u|^{\gamma-1}|u-c|\eta^2 \\
&\quad + \mu_2\int_{\Omega_R(x_0)} |g||u-c|\eta^2 \\
&\le \frac{\mu}{16}\int_{\Omega_R(x_0)} |\nabla u|^2\eta^2 + N\int_{\Omega_R(x_0)} |u-c|^\gamma\eta^2 \\
&\quad + N\int_{\Omega_R(x_0)} |u|^\gamma\eta^2 + N\int_{\Omega_R(x_0)} |g|^{\frac{\gamma}{\gamma-1}}\eta^2.
\end{aligned}$$

From these estimates of J_i, $i = 1, 2, 3$, and the inequality (41) along with the ellipticity condition in (2) we have

$$\begin{aligned}\int_{\Omega_R(x_0)} |\nabla u|^2 \eta^2 \le N \int_{\Omega_R(x_0)} |u-c|^2 |\nabla \eta|^2 + N \int_{\Omega_R(x_0)} |u-c|^\gamma \eta^2 \\ + N \int_{\Omega_R(x_0)} |u|^\gamma \eta^2 + N \int_{\Omega_R(x_0)} \left(|f|^2 + |g|^{\frac{\gamma}{\gamma-1}} \right) \eta^2 \\ := N(I_1 + I_2 + I_3 + I_4),\end{aligned}$$

where $N = N(\mu, \mu_1, \mu_2)$. Now we get estimates for I_1, I_2, and I_3 as follows.

Estimate of I_1: Recall the definition of η, $r = \frac{R}{2(1+\beta^2)}$, and $q = \frac{2d}{d+2}$. Then by Lemma 8.1,

$$I_1 \le N R^{-2} \int_{\Omega_r(x_0)} |u-c|^2 \le N R^d \left(\fint_{\Omega_R(x_0)} |\nabla u|^q \right)^{2/q}.$$

Estimate of I_2: Again by Lemma 8.1,

$$I_2 \le \int_{\Omega_r(x_0)} |u-c|^\gamma \le N R^{d+\gamma-\gamma d/2} \left(\int_{\Omega_R(x_0)} |\nabla u|^2 \right)^{\gamma/2}.$$

Estimate of I_3: First note that

$$\int_{\Omega_r(x_0)} |c|^\gamma \, dx \le N R^d \left(\fint_{\Omega_r(x_0)} |u| \, dx \right)^\gamma \le N R^d \left(\fint_{\Omega_R(x_0)} |u|^{\gamma q/2} \, dx \right)^{2/q},$$

where, in the last inequality, we have used the fact that $|\Omega_r(x_0)| \ge |B^+_{r_1}|$, $r_1 = R\left(2(1+\beta^2)\right)^{-3/2}$. Hence

$$\begin{aligned}I_3 &\le N \int_{\Omega_r(x_0)} |u-c|^\gamma + N \int_{\Omega_r(x_0)} |c|^\gamma \\ &\le N R^{d+\gamma-\gamma d/2} \left(\int_{\Omega_R(x_0)} |\nabla u|^2 \right)^{\gamma/2} + R^d \left(\fint_{\Omega_R(x_0)} |u|^{\gamma q/2} \, dx \right)^{2/q}.\end{aligned} \tag{42}$$

Therefore,

$$\begin{aligned}\int_{\Omega_R(x_0)} |\nabla u|^2 \eta^2 \le N R^d \left(\fint_{\Omega_R(x_0)} |\nabla u|^q \right)^{\frac{2}{q}} \\ + N R^d \left(\fint_{\Omega_R(x_0)} |u|^{\gamma q/2} \right)^{\frac{2}{q}} + N \int_{\Omega_R(x_0)} \left(|f|^2 + |F|^2 \right) \\ + N R^{d\gamma\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\gamma} \left(\int_{\Omega_R(x_0)} |\nabla u|^2 \right)^{\frac{\gamma}{2}-1} \left(\int_{\Omega_R(x_0)} |\nabla u|^2 \right),\end{aligned}$$

where $N = N(d, \mu, \mu_1, \mu_2, \gamma, \beta)$ and

$$F = |g|^{\frac{1}{2}\frac{\gamma}{\gamma-1}}.$$

Finally, we obtain the desired inequality in the proposition by adding the I_3 term to the above inequality, using (42), and diving all terms by R^d. $\square$

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