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SUBTOURNAMENTS ISOMORPHIC TO W_5 OF AN INDECOMPOSABLE TOURNAMENT

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Abstract. We consider a tournament $T = (V, A)$. For each subset X of V is associated the subtournament $T(X) = (X, A \cap (X \times X))$ of T induced by X. We say that a tournament T' embeds into a tournament T when T' is isomorphic to a subtournament of T. Otherwise, we say that T omits T'. A subset X of V is a clan of T provided that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. For example, $\emptyset$, $\{x\} (x \in V)$ and V are clans of T, called trivial clans. A tournament is indecomposable if all its clans are trivial. In 2003, B. J. Latka characterized the class $\mathcal{T}$ of indecomposable tournaments omitting a certain tournament W_5 on 5 vertices. In the case of an indecomposable tournament T, we will study the set $W_5(T)$ of vertices $x \in V$ for which there exists a subset X of V such that $x \in X$ and $T(X)$ is isomorphic to W_5. We prove the following: for any indecomposable tournament T, if $T \notin \mathcal{T}$, then $|W_5(T)| \geq |V| - 2$ and $|W_5(T)| \geq |V| - 1$ if $|V|$ is even. By giving examples, we also verify that this statement is optimal.

1. Introduction

A *tournament* $T = (V(T), A(T))$ (or (V, A)) consists of a finite set V of *vertices* together with a set A of ordered pairs of distinct vertices, called *arcs*, such that for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The *order* of T, denoted by $|T|$, is the cardinality of $V(T)$. Given a tournament $T = (V, A)$, with each subset X of V is associated the *subtournament* $T(X) = (X, A \cap (X \times X))$ of T induced by X. For $X \subseteq V$ (resp. $x \in V$), the subtournament $T(V \setminus X)$ (resp. $T(V \setminus \{x\})$) is denoted by $T - X$ (resp. $T - x$). Let $T = (V, A)$ and $T' = (V', A')$ be two tournaments. A bijection f from V onto V' is an *isomorphism* from T onto T' provided that for all x, $y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The tournaments T and T' are then said to be *isomorphic*, which is denoted by $T \simeq T'$. An isomorphism from a tournament T onto itself is called an *automorphism* of T. We say that T' *embeds* into T when T' is isomorphic to a subtournament of T. Otherwise, we say that T *omits* T'. With each tournament $T = (V, A)$ is associated its *dual* tournament

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$T^\star = (V, A^\star)$, where $A^\star = \{(x, y) : (y, x) \in A\}$. The tournament T is then said to be *self-dual* when $T \simeq T^\star$. For every $x \neq y \in V$, the notation $x \longrightarrow y$ signifies that $(x, y) \in A$. Moreover, for every $x \in V$ and $Y \subseteq V \setminus \{x\}$, $x \longrightarrow Y$ (resp. $Y \longrightarrow x$) means that $x \longrightarrow y$ (resp. $y \longrightarrow x$) for every $y \in Y$. For every $x \in V$, we set $N_T^+(x) = \{y \in V\colon\ x \longrightarrow y\}$ and $N_T^-(x) = \{y \in V\colon\ y \longrightarrow x\}$. Furthermore, the *score* of a vertex x of T, denoted by $s_T(x)$, is the cardinality of $N_T^+(x)$.

Given a tournament $T = (V, A)$, a subset I of V is a *clan* [6] (or an *interval* [11, 16]) of T provided that for every $x \in V \setminus I$, $x \longrightarrow I$ or $I \longrightarrow x$. For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are clans of T, called *trivial* clans. A tournament is then said to be *indecomposable* [11, 16] (or *primitive* [6]) if all its clans are trivial and it is *decomposable* otherwise. Notice that a tournament T and its dual $T^\star$ have the same clans, in particular, $T^\star$ is indecomposable precisely if the same holds for T.

The main result of this paper, presented in [2] without proof, concerns the subtournaments of an indecomposable tournament T which are isomorphic to a tournament W_5 defined on $\{0, \ldots, 4\}$ by $A(W_5 - 4) = \{(i, j) : 0 \leq i < j \leq 3\}$ and $N^+_{W_5}(4) = \{0, 2\}$. Note that the tournament W_5 is the tournament W_{2n+1}, introduced in Section 3, by taking $n = 2$ (see Figure 3). In 2003, B. J. Latka characterized the indecomposable tournaments omitting W_5 (see Theorem 3.4). Many classes of tournaments defined by means of embedding, involving inevitable configurations or morphological descriptions, have been studied by several authors [1, 10, 12, 13]. The aim of this paper is to examine the set $W_5(T)$ of the vertices x of an indecomposable tournament T for which there exists a subset $X \in \binom{V(T)}{5}$ such that $x \in X$ and $T(X) \simeq W_5$. So, notice that almost all tournaments $T = (V, A)$ verify $W_5(T) = V$. It is an elementary exercise to show that. Note also that if T satisfies a certain *extension axiom*, then it satisfies $W_5(T) = V$. The extension axioms are introduced in [8, 9], as a first order logic sentences, for the study of 0-1 laws. These axioms form an important tool in the study of the random aspects of finite structures, each of these axioms is satisfied by almost all these structures [5, 8, 9]. We recall these axioms in the case of tournaments. A tournament $T = (V, A)$ is *r-existentially closed* (or r-e.c. [3]), where $r \in \mathbb{N}$, when it satisfies the *r-extension axiom*: for all $X \in \binom{V}{r}$ and $Y \subseteq X$, there is a vertex $x \in V \setminus X$ such that $N^+_{T(X \cup \{x\})}(x) = Y$. For $r \in \mathbb{N}$, almost all tournaments are r-e.c. [3, 8, 9]. As a 4-e.c. tournament T satisfies $W_5(T) = V(T)$, then almost all tournaments T satisfy $W_5(T) = V(T)$. As we are interested in indecomposable tournaments, which is the case of almost all tournaments [7], we deduce the following fact.

Fact 1.1. Almost all the indecomposable tournaments T satisfy $W_5(T) = V(T)$.

Note that these facts extend in a natural way when one considers a tournament other than W_5.

In this paper, we focus on the tournament W_5, we establish the following theorem and we verify that it is optimal.

Theorem 1.2. *Let T be an indecomposable tournament. If W_5 embeds into T, then $|W_5(T)| \geq |T| - 2$. If, moreover, $|T|$ is even, then $|W_5(T)| \geq |T| - 1$.*

2. The indecomposable tournaments

Definition. Given a tournament $T = (V, A)$, with each subset X of V, such that $|X| \geq 3$ and $T(X)$ is indecomposable, are associated the following subsets of $V \setminus X$.

- $[X] = \{x \in V \setminus X : x \longrightarrow X \text{ or } X \longrightarrow x\}$.
- For every $u \in X$, $X(u) = \{x \in V \setminus X : \{u, x\}$ is a clan of $T(X \cup \{x\})\}$.
- $Ext(X) = \{x \in V \setminus X : T(X \cup \{x\})$ is indecomposable$\}$.

Lemma 2.1 ([6]). *Let $T = (V, A)$ be a tournament and let X be a subset of V such that $|X| \geq 3$ and $T(X)$ is indecomposable. The nonempty elements of the family $\{X(u) : u \in X\} \cup \{Ext(X), [X]\}$ constitute a partition of $V \setminus X$ and the following assertions are satisfied.*

- *Let $u \in X$, $x \in X(u)$ and $y \in V \setminus (X \cup X(u))$. If $T(X \cup \{x, y\})$ is decomposable, then $\{u, x\}$ is a clan of $T(X \cup \{x, y\})$.*
- *Let $x \in [X]$ and $y \in V \setminus (X \cup [X])$. If $T(X \cup \{x, y\})$ is decomposable, then $X \cup \{y\}$ is a clan of $T(X \cup \{x, y\})$.*
- *Let $x \neq y \in Ext(X)$. If $T(X \cup \{x, y\})$ is decomposable, then $\{x, y\}$ is a clan of $T(X \cup \{x, y\})$.*

From this lemma follows the next result.

Corollary 2.2 ([6]). *Let $T = (V, A)$ be an indecomposable tournament. If X is a subset of V such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $T(X)$ is indecomposable, then there are distinct x, $y \in V \setminus X$ such that $T(X \cup \{x, y\})$ is indecomposable.*

We also recall the following result concerning the indecomposable tournaments.

Lemma 2.3 ([15]). *Let $T = (V, A)$ be an indecomposable tournament. If X is a subset of V such that $|X| \geq 3$, $|V \setminus X| \geq 4$ and $T(X)$ is indecomposable, then there are distinct x, $y \in V \setminus X$ such that $T - \{x, y\}$ is indecomposable.*

3. The critical tournaments and Latka's theorem

Given an indecomposable tournament T with $V(T) \neq \emptyset$, T is said to be *critical* if for all vertex x of T, the tournament $T - x$ is decomposable. The critical tournaments are one of the tools of the proof of Theorem 1.2. Moreover, an important part of these tournaments form the class of indecomposable tournaments of order > 7 and omitting W_5 due to B. J. Latka [12]. In order to present the characterization of the critical tournaments due to J. H.

Schmerl and W. T. Trotter [16], we introduce the following notations and tournaments. A *transitive* tournament is a tournament omitting the tournament $C_3 = (\{0,1,2\},\{(0,1),(1,2),(2,0)\})$. For $n \in \mathbb{N}$, we set $\mathbb{N}_n = \{0,\ldots,n\}$, $2\mathbb{N}_n = \{2i : i \in \mathbb{N}_n\}$ and for every finite set X of $\mathbb{N}$, we denote by $\underline{X}$ the transitive tournament defined on X by $A(\underline{X}) = \{(i,j) \in X \times X : i < j\}$. Now, we introduce the tournaments T_{2n+1}, U_{2n+1} and W_{2n+1} defined on $\mathbb{N}_{2n}$, where $n \geq 2$, as follows.

(1) $A(T_{2n+1}) = \{(i,j) : j - i \in \{1,\ldots,n\} \mod 2n+1\}$ (see Figure 1).
(2) $U_{2n+1}(\mathbb{N}_n) = \underline{\mathbb{N}_n}$, $U^{\star}_{2n+1}(\mathbb{N}_{2n} \setminus \mathbb{N}_n) = \underline{\mathbb{N}_{2n} \setminus \mathbb{N}_n}$ and for $i \in \mathbb{N}_{n-1}$, $\{i+1,\ldots,n\} \longrightarrow i+n+1 \longrightarrow \mathbb{N}_i$ (see Figure 2).
(3) $W_{2n+1}(\mathbb{N}_{2n-1}) = \underline{\mathbb{N}_{2n-1}}$ and $N^{+}_{W_{2n+1}}(2n) = 2\mathbb{N}_{n-1}$ (see Figure 3).

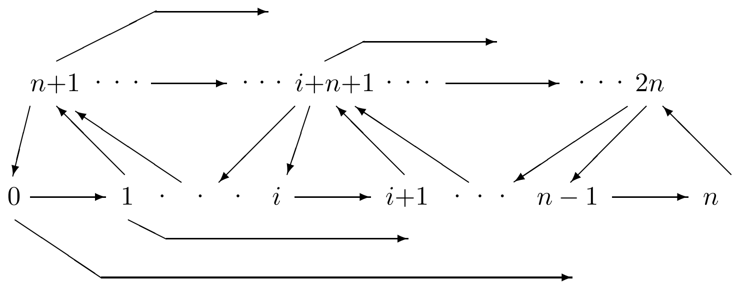

FIGURE 1. The tournament T_{2n+1}.

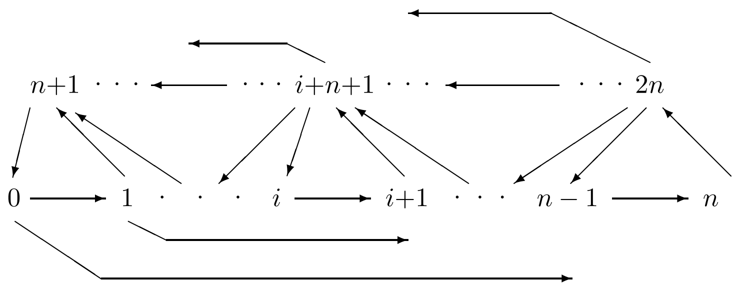

FIGURE 2. The tournament U_{2n+1}.

Proposition 3.1 ([16]). *Up to isomorphisms, the critical tournaments are the tournaments T_{2n+1}, U_{2n+1} and W_{2n+1}, where $n \geq 2$.*

Notice that the critical tournaments are self-dual and recall the remarks below, which follow easily from the definitions of these tournaments.

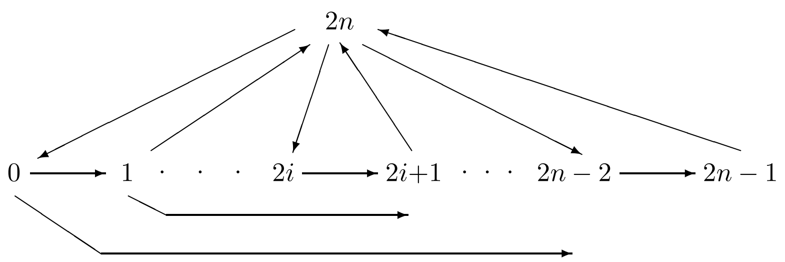

FIGURE 3. The tournament W_{2n+1}.

Remark 3.2. Up to isomorphisms, the indecomposable subtournaments of T_{2n+1} (resp. U_{2n+1}, W_{2n+1}) on at least 5 vertices, where $n \geq 2$, are the tournaments T_{2m+1} (resp. U_{2m+1}, W_{2m+1}), where $2 \leq m \leq n$. In particular, for all integers p, q, $l \geq 2$, the tournaments T_{2p+1}, U_{2q+1} and W_{2l+1} are incomparable with respect to the embedding.

Remark 3.3. Let $T = (\mathbb{N}_6, A)$ be an indecomposable tournament such that $T(\mathbb{N}_4) = U_5$. The tournament T is isomorphic to U_7 if and only if, by interchanging the vertices 5 and 6, one of the six following configurations occurs.

- $N_T^+(5) = \{0, 1, 2\}$ and $N_T^+(6) = \{5\}$.
- $N_T^+(5) = \{1, 2, 6\}$ and $N_T^+(6) = \{0\}$.
- $N_T^+(5) = \{1, 2, 3\}$ and $N_T^+(6) = \{0, 3, 5\}$.
- $N_T^+(5) = \{2, 3, 6\}$ and $N_T^+(6) = \{1, 3, 0\}$.
- $N_T^+(5) = \{2, 3, 4\}$ and $N_T^+(6) = \{0, 1, 3, 4, 5\}$.
- $N_T^+(5) = \{3, 4, 6\}$ and $N_T^+(6) = \{0, 1, 2, 3, 4\}$.

In order to present the characterization of the indecomposable tournaments omitting W_5, due to B. J. Latka, we also introduce the *Paley* tournament P_7 defined on $\mathbb{N}_6$ by $A(P_7) = \{(i, j) : j - i \in \{1, 2, 4\} \mod 7\}$. Notice that for every $x \neq y \in \mathbb{N}_6$, $P_7 - x \simeq P_7 - y$ and set $B_6 = P_7 - 6$. Moreover, for all $x \neq y \in \mathbb{N}_5$, $B_6 - x \simeq B_6 - y \simeq U_5$. Notice also that the tournaments B_6 and P_7 are self-dual.

Theorem 3.4 ([12]). *Up to isomorphisms, the indecomposable tournaments on at least* 5 *vertices and omitting* W_5 *are the tournaments* B_6, P_7, T_{2n+1} *and* U_{2n+1}, *where* $n \geq 2$.

With this characterization, we obtain the following statement of Theorem 1.2: *Let* T *be an indecomposable tournament of order* ≥ 5 *such that* $T \not\simeq B_6$, P_7, T_{2n+1} *or* U_{2n+1} *for* $n \geq 2$. *Then* $|W_5(T)| \geq |T| - 2$. *If, moreover,* $|T|$ *is even, then* $|W_5(T)| \geq |T| - 1$.

Given five distinct vertices x_i ($i \in \mathbb{N}_4$) of a tournament T. For convenience, we write $T(x_0, x_1, x_2, x_3, x_4) \simeq W_5$ to signify that the bijection $\tau : i \mapsto x_i$ is an isomorphism from W_5 onto $T(\{x_0, x_1, x_2, x_3, x_4\})$. Similarly, for another choice of five distinct vertices y_i ($i \in \mathbb{N}_4$) of T, we write $T(x_0, x_1, x_2, x_3, x_4) \simeq T(y_0, y_1, y_2, y_3, y_4)$ to signify that the bijection $\sigma : x_i \mapsto y_i$ is an isomorphism from $T(\{x_0, x_1, x_2, x_3, x_4\})$ onto $T(\{y_0, y_1, y_2, y_3, y_4\})$.

4. The minimal tournaments

The minimal tournaments are involved in the proof of Theorem 1.2. These tournaments have been introduced in 1998 by A. Cournier and P. Ille [4] as follows. Given an indecomposable tournament $T = (V, A)$ and two distinct vertices $x \neq y \in V$, T is said to be *minimal* for x and for y (or $\{x, y\}$-minimal) whenever for every proper subset X of V ($X \neq V$), if $\{x, y\} \subset X$ ($| X | \geq 3$), then $T(X)$ is decomposable. We say that T is minimal when there exist $x \neq y \in V(T)$ such that T is $\{x, y\}$-minimal. A. Cournier and P. Ille characterized the minimal tournaments. In order to recall this characterization, we introduce the tournaments F_n and G_n in the following manner.

(1) For $n \geq 4$, F_n is defined on $\mathbb{N}_{n-1}$ as follows: for i, $j \in \mathbb{N}_{n-1}$, $(i, j) \in A(F_n)$ if and only if $j = i + 1$ or $i \geq j + 2$ (see Figure 4).
(2) For $n \geq 6$, G_n is defined on $\mathbb{N}_{n-1}$ as follows: $G_n(\mathbb{N}_{n-3}) = F_{n-2}$, $N^+_{G_n}(n-2) = \{n-3\}$ and $N^+_{G_n}(n-1) = \{n-2\}$ (see Figure 5).

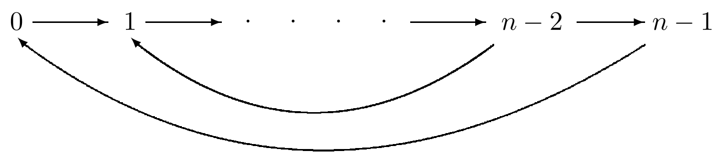

FIGURE 4. The tournament F_n.

Proposition 4.1 ([4]). *Up to isomorphisms, the minimal tournaments of order* ≥ 3 *are the tournaments* C_3, U_5, W_5, F_n, G_n *and* $G^\star_n$, *where* $n \geq 6$.

Corollary 4.2. *Given a minimal tournament* T *of order* $n \geq 6$, *we have* $| W_5(T) | \geq n - 1$.

Proof. As W_5 is self-dual, it suffices to prove the result for the tournaments F_n and G_n for $n \geq 6$. For $n \geq 5$, we have $| W_5(F_n) | = n$ because for all $i \in \mathbb{N}_{n-5}$, $F_n(i+3, i+4, i, i+1, i+2) \simeq W_5$. For $n = 6$, $| W_5(G_6) | \geq 5$ because $G_6(1, 2, 5, 4, 3) \simeq W_5$. For $n \geq 7$, $| W_5(G_n) | = n$ because $G_n(n-5, n-4, n-1, n-2, n-3) \simeq W_5$ and $| W_5(G_n(\mathbb{N}_{n-3})) | = | W_5(F_{n-2}) | = n-2$. □

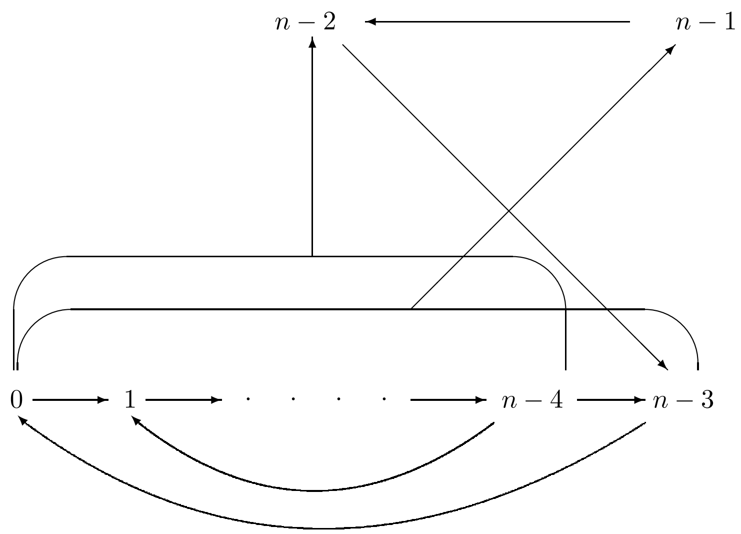

FIGURE 5. The tournament G_n.

5. Proof of Theorem 1.2 for $|T| \leq 8$

Theorem 1.2 is trivial for $|T| \leq 7$. In this section, we establish this theorem for $|T| = 8$. Up to isomorphisms, there are 6880 tournaments of 8 vertices, 3785 tournaments of them are indecomposable [14], but our verification, made by hand, is not exhaustive. We begin by the case where B_6 embeds into T. So, notice the following additional remarks concerning B_6 and P_7. The Paley tournament P_7 is regular: for every $x \in \mathbb{N}_6$, $s_{P_7}(x) = 3$. The tournament B_6 is quasi-regular: for every $x \in \mathbb{N}_5$, $s_{B_6}(x) = 2$ if $x \in \{2, 4, 5\}$, and $s_{B_6}(x) = 3$ if $x \in \{0, 1, 3\}$. Moreover, the automorphism group of B_6 is generated by the permutation $\pi = (013)(254)$.

Lemma 5.1. *If B_6 embeds into an indecomposable tournament T on 7 vertices and if $T \not\simeq P_7$, then $|W_5(T)| = 7$.*

Proof. We set $V(T) = \mathbb{N}_6$ and $T(\mathbb{N}_5) = B_6$. By interchanging T and $T^\star$, we can assume that $s_T(6) \leq 3$. The automorphisms of B_6 restrict the proof to the following cases. When $s_T(6) = 1$, we can assume that $N_T^+(6) = \{0\}$ or $\{2\}$. If $N_T^+(6) = \{0\}$, then $T(1, 5, 2, 6, 0) \simeq T(2, 3, 4, 6, 0) \simeq W_5$. If $N_T^+(6) = \{2\}$, then $T(0, 1, 6, 2, 3) \simeq T(5, 0, 6, 2, 4) \simeq W_5$. When $s_T(6) = 2$, we can assume that $N_T^+(6) = \{4, 5\}$, $\{0, 1\}$, $\{0, 5\}$, $\{3, 5\}$ or $\{1, 5\}$. If $N_T^+(6) = \{1, 5\}$, then T is decomposable because $6 \in \mathbb{N}_5(4)$. If $N_T^+(6) = \{4, 5\}$, then $T(3, 0, 6, 4, 1) \simeq T(0, 1, 2, 6, 5) \simeq W_5$. If $N_T^+(6) = \{0, 1\}$, then $T(3, 5, 6, 0, 2) \simeq T(3, 4, 5, 6, 1) \simeq W_5$. If $N_T^+(6) = \{0, 5\}$, then $T(2, 3, 4, 6, 0) \simeq T(4, 1, 6, 5, 2) \simeq W_5$. If $N_T^+(6) =$

$\{3,5\}$, then $T(0,1,2,6,5) \simeq T(1,2,6,3,4) \simeq W_5$. When $s_T(6) = 3$, we can assume that $N_T^+(6) = \{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, $\{0,1,5\}$, $\{0,4,5\}$, $\{1,4,5\}$, $\{2,4,5\}$ or $\{3,4,5\}$. If $N_T^+(6) = \{0,1,3\}$, then $T \simeq P_7$. If $N_T^+(6) = \{1,4,5\}$ (resp. $N_T^+(6) = \{0,4,5\}$), then T is decomposable because $6 \in \mathbb{N}_5(4)$ (resp. $6 \in \mathbb{N}_5(3)$). If $N_T^+(6) = \{2,4,5\}$, then $T(1,3,6,5,0) \simeq T(0,6,2,4,5) \simeq W_5$. If $N_T^+(6) = \{0,1,2\}$, then $T(6,0,1,2,4) \simeq T(3,4,5,6,1) \simeq W_5$. If $N_T^+(6) = \{0,1,5\}$, then $T(3,6,5,0,1) \simeq T(2,3,4,6,0) \simeq W_5$. If $N_T^+(6) = \{0,1,4\}$, then $T(3,5,6,0,2) \simeq T(6,0,4,1,2) \simeq W_5$. If $N_T^+(6) = \{3,4,5\}$, then $T(6,3,4,5,0) \simeq T(0,1,2,6,5) \simeq W_5$. □

Lemma 5.2. *If P_7 embeds into an indecomposable tournament T on 8 vertices, then $|W_5(T)| = 8$.*

Proof. We set $V(T) = \mathbb{N}_7$ and $T(\mathbb{N}_6) = P_7$. By interchanging T and $T^\star$, we can assume that $s_T(7) \leq 3$. First, assume that $s_T(7) = 1$. We take $N_T^+(7) = \{x\}$ and $y \in \mathbb{N}_6 \setminus \{x\}$. We have $T - \{7, y\} \simeq B_6$, $T - y \not\simeq P_7$ and $T - y$ is indecomposable by Lemma 2.1. It follows from Lemma 5.1 that $|W_5(T-y)| = 7$. By changing y by $z \in \mathbb{N}_6 \setminus \{x, y\}$, we obtain $|W_5(T-z)| = 7$. Therefore, $|W_5(T)| = 8$. Second, assume that $s_T(7) = 2$ and set $N_T^+(7) = \{x, y\}$. For $\alpha \in \{x, y\}$, we have $T - \{7, \alpha\} \simeq B_6$, $T - \alpha \not\simeq P_7$ and $T - \alpha$ is indecomposable by Lemma 2.1. It follows from Lemma 5.1 that $|W_5(T-\alpha)| = 7$ and hence $|W_5(T)| = 8$. Finally, assume that $s_T(7) = 3$, set $N_T^+(7) = \{x, y, z\}$ and let $\alpha \in \{x, y, z\}$. We have $T(X) \simeq B_6$, where $X = \mathbb{N}_6 \setminus \{\alpha\}$ and $T - \alpha \not\simeq P_7$. Moreover, $T - \alpha$ is indecomposable. Otherwise, as $7 \notin [X]$, then by Lemma 2.1, there is $u \in X$ such that $7 \in X(u)$. Since $s_{T-\alpha}(7) = 2$, then $s_{T(X)}(u) = 2$. As moreover, $\{u, 7\} \longrightarrow \alpha$, then $\{u, 7\}$ is a nontrivial clan of T, which contradicts the indecomposability of T. It follows from Lemma 5.1 that $|W_5(T-\alpha)| = 7$ and thus $|W_5(T)| = 8$. □

Lemma 5.3. *If B_6 embeds into an indecomposable tournament T on 8 vertices, then $|W_5(T)| \geq 7$.*

Proof. We set $V(T) = \mathbb{N}_7$ and $T(\mathbb{N}_5) = B_6$. By Lemma 5.1 and Lemma 5.2 we obtain the following remark. If $Ext(\mathbb{N}_5) \neq \emptyset$, then $|W_5(T)| \geq 7$. So, assume that $Ext(\mathbb{N}_5) = \emptyset$. By Lemma 2.1, assume first that $6 \in \mathbb{N}_5(u)$, where $u \in \mathbb{N}_5$, and that $7 \in [\mathbb{N}_5]$. By interchanging T and $T^\star$, we can assume that $7 \longrightarrow \mathbb{N}_5$ and hence $6 \longrightarrow 7$ by Lemma 2.1. We have $T(X) \simeq B_6$ where $X = \mathbb{N}_6 \setminus \{u\}$. Since $s_{T-u}(7) = 5 \notin \{2,3,4,0,6\}$, then $7 \in Ext(X)$ by Lemma 2.1. It follows, from the remark above, that $|W_5(T)| \geq 7$. Now, suppose that $6 \in \mathbb{N}_5(u)$ and $7 \in \mathbb{N}_5(v)$, where $u \neq v \in \mathbb{N}_5$. By interchanging T and $T^\star$ and by considering the automorphisms of B_6, we may assume that $u \in \{0,1,3,4\}$ and $v = 5$. We have $T(Y) \simeq U_5$ and $7 \in Y(5)$, where $Y = \mathbb{N}_5 \setminus \{u\}$. Moreover, $6 \notin Y(5)$. Indeed, if $u = 1$ or 4, then $5 \longrightarrow 0 \longrightarrow 6$ and if $u = 0$ or 3, then $6 \longrightarrow 4 \longrightarrow 5$. As furthermore, $\{5, 7\}$ is not a clan of $T - u$, it ensues, from Lemma 2.1, that $T - u$ is indecomposable. Therefore, $|W_5(T)| \geq 7$ by the remark above. □

Lemma 5.4. *Let $T = (\mathbb{N}_6, A)$ be an indecomposable tournament such that $T(\mathbb{N}_4) = U_5$. If $T \not\simeq U_7$ and $Ext(\mathbb{N}_4) = \emptyset$, then $W_5(T) \cap \{3, 4\} \neq \emptyset$.*

Proof. Notice first that $Ext(\mathbb{N}_4) = \emptyset$ if and only if $\{N^+_{T-6}(5), N^+_{T-5}(6)\} \subset \mathcal{C} = \{\emptyset, \mathbb{N}_4, \{0\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{0, 3\}, \{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{0, 1, 3\}, \{0, 1, 3, 4\}\}$. As $T - 6$ is decomposable, then $T \not\simeq P_7$. Similarly, $T \not\simeq T_7$ by Remark 3.2. It follows from Theorem 3.4 that W_5 embeds into T. If $T - \{3, 4\} \not\simeq W_5$, then there exists $X \in \binom{\mathbb{N}_6}{5}$ such that $T(X) \simeq W_5$ and $X \cap \{3, 4\} \neq \emptyset$. So, assume that $T - \{3, 4\} \simeq W_5$. Since $T(\mathbb{N}_2) = \underline{\mathbb{N}_2}$, then, by considering the subtournaments of W_5 which are isomorphic to $\underline{\mathbb{N}_2}$ and by taking $5 \longrightarrow 6$, we obtain that $W_5 \simeq T(5, 0, 6, 1, 2)$, $T(5, 0, 2, 6, 1)$, $T(6, 0, 1, 2, 5)$, $T(1, 5, 2, 6, 0)$, $T(0, 1, 2, 5, 6)$, $T(0, 1, 6, 2, 5)$ or $T(0, 5, 1, 2, 6)$. If $W_5 \simeq T(6, 0, 1, 2, 5)$ or $T(1, 5, 2, 6, 0)$ (resp. $W_5 \simeq T(5, 0, 6, 1, 2)$, $T(5, 0, 2, 6, 1)$ or $T(0, 1, 2, 5, 6)$), then $N^+_{T-6}(5) \notin \mathcal{C}$ (resp. $N^+_T(6) \notin \mathcal{C}$). If $W_5 \simeq T(0, 1, 6, 2, 5)$, then $N^+_T(6) = \{2, 3\}$ or $\{2, 3, 4\}$ and $N^+_{T-6}(5) = \{0\}$ or $\{0, 3\}$. If $N^+_T(6) = \{2, 3\}$, then $T(4, 1, 5, 6, 2) \simeq W_5$. If $N^+_T(6) = \{2, 3, 4\}$, then $T(4, 1, 3, 5, 6) \simeq W_5$ when $3 \longrightarrow 5$ and $T(1, 5, 6, 3, 0) \simeq W_5$ when $5 \longrightarrow 3$. Finally, if $W_5 \simeq T(0, 5, 1, 2, 6)$, then $N^+_T(6) = \{0, 1, 3\}$ or $\{0, 1, 3, 4\}$ and $N^+_{T-6}(5) = \{1, 2\}$ or $\{1, 2, 3\}$. If $N^+_{T-6}(5) = \{1, 2\}$ (resp. $N^+_{T-6}(5) = \{1, 2, 3\}$), then $T(2, 4, 6, 3, 5) \simeq W_5$ (resp. $T(4, 5, 6, 1, 2) \simeq W_5$) when $4 \longrightarrow 6$ and $T(6, 4, 1, 3, 5) \simeq W_5$ (resp. $T(6, 4, 3, 0, 5) \simeq W_5$) when $6 \longrightarrow 4$. □

Proposition 5.5. *Given an indecomposable tournament T of order 8, we have $|W_5(T)| \geq 7$.*

Proof. Suppose, by contradiction, that there are $x \neq y \in V(T)$ such that $\{x, y\} \cap W_5(T) = \emptyset$. Let X be a minimal subset of $V(T)$ such that $\{x, y\} \subset X$ ($|X| \geq 3$) and $T(X)$ is indecomposable. $T(X)$ is $\{x, y\}$-minimal. By Proposition 4.1 and Corollary 4.2, $T(X) \simeq C_3$ or U_5. If $T(X) \simeq C_3$, then, by Lemma 2.3 and Theorem 3.4, B_6 embeds into T. By Lemma 5.3, $|W_5(T)| \geq 7$, a contradiction. Therefore, $T(X) \simeq U_5$. We take $V(T) = \mathbb{N}_7$ and $T(X) = U_5$. By observing the subtournaments of U_5 which are isomorphic to C_3, we obtain that $\{x, y\} = \{3, 4\}$. We have $Ext(\mathbb{N}_4) = \emptyset$. Otherwise, by Theorem 3.4, there is $\alpha \in \{5, 6, 7\}$ such that $T(\mathbb{N}_4 \cup \{\alpha\}) \simeq B_6$, contradiction by Lemma 5.3. By Corollary 2.2, we may assume that $T - 7$ is indecomposable. If $T - 7 \not\simeq U_7$, then, by Lemma 5.4, we have $W_5(T - 7) \cap \{3, 4\} \neq \emptyset$, a contradiction. To finish, it remains to examine the case where $T - 7 \simeq U_7$. By interchanging T and $T^\star$ and by using Remark 3.3, it suffices to consider the following three cases: $(N^+_{T-7}(5), N^+_{T-7}(6)) = (\{0, 1, 2\}, \{5\})$, $(\{1, 2, 6\}, \{0\})$ or $(\{1, 2, 3\}, \{0, 3, 5\})$. If $(N^+_{T-7}(5), N^+_{T-7}(6)) = (\{0, 1, 2\}, \{5\})$ (resp. $(\{1, 2, 6\}, \{0\})$, $(\{1, 2, 3\}, \{0, 3, 5\})$), then $5 \in \mathbb{N}_4(0)$ and $6 \in [\mathbb{N}_4]$ (resp. $5 \in \mathbb{N}_4(0)$ and $6 \in \mathbb{N}_4(3)$, $5 \in \mathbb{N}_4(1)$ and $6 \in \mathbb{N}_4(3)$). It follows that $7 \in \mathbb{N}_4(0)$ or $[\mathbb{N}_4]$ (resp. $7 \in \mathbb{N}_4(u)$ for $u \in \{0, 3\}$, $7 \in \mathbb{N}_4(u)$ for $u \in \{1, 3\}$). Otherwise, since $\{v, 7\}$, where $v \in \{1, 2, 3, 4\}$ (resp. $v \in \{1, 2, 4\}$, $v \in \{0, 2, 4\}$), and $[\mathbb{N}_6]$ are not clans of T, then, by Lemma 2.1, there is $\alpha \in \{5, 6\}$ such that $T - \alpha$ is indecomposable. By Remark 3.3, $T-\alpha \not\simeq U_7$, which contradicts Lemma 5.4. Thus, we distinguish the following cases.

- $N^+_{T-7}(5) = \{0,1,2\}$, $N^+_{T-7}(6) = \{5\}$ and $7 \in \mathbb{N}_4(0)$ or $[\mathbb{N}_4]$. First, suppose that $7 \in \mathbb{N}_4(0)$. If $6 \longrightarrow 7$, then $0 \longrightarrow 7$ because $\{5,7\}$ is not a clan of T. Thus, $T(3,0,6,7,1) \simeq W_5$, a contradiction. If $7 \longrightarrow 6$, as $\{0,7\}$ is not a clan of T, then $7 \longrightarrow 5$. Thus, $T(3,7,6,5,2) \simeq W_5$, a contradiction. Now, assume that $7 \in [\mathbb{N}_4]$. If $7 \longrightarrow \mathbb{N}_4$, then $7 \longrightarrow 5$, otherwise $T(5,7,0,1,3) \simeq W_5$. Since $\mathbb{N}_6$ is not a clan of T, then $6 \longrightarrow 7$ and hence $T(7,4,5,1,6) \simeq W_5$, a contradiction. If $\mathbb{N}_4 \longrightarrow 7$, as $\{6,7\}$ and $\mathbb{N}_6$ are not clans of T, then $5 \longrightarrow 7 \longrightarrow 6$ and thus $T(1,3,7,6,5) \simeq W_5$, a contradiction.
- $N^+_{T-7}(5) = \{1,2,6\}$, $N^+_{T-7}(6) = \{0\}$ and $7 \in \mathbb{N}_4(u)$ for $u \in \{0,3\}$. If $7 \in \mathbb{N}_4(0)$ with $7 \longrightarrow 6$ (resp. $7 \in \mathbb{N}_4(3)$ with $5 \longrightarrow 7$), as $\{5,7\}$ (resp. $\{6,7\}$) is not a clan of T, then $7 \longrightarrow 0$ (resp. $7 \longrightarrow 3$). It follows from Lemma 2.1 and Remark 3.3 that $T-5$ (resp. $T-6$) is indecomposable and not isomorphic to U_7, which contradicts Lemma 5.4. Now, if $7 \in \mathbb{N}_4(0)$ with $6 \longrightarrow 7$ (resp. $7 \in \mathbb{N}_4(3)$ with $7 \longrightarrow 5$), since $\{0,7\}$ (resp. $\{3,7\}$) is not a clan of T, then $5 \longrightarrow 7$ (resp. $6 \longrightarrow 7$). So, $T(3,5,6,7,1) \simeq W_5$ (resp. $T(2,4,6,7,5) \simeq W_5$), a contradiction.
- $N^+_T(5) = \{1,2,3\}$, $N^+_T(6) = \{0,3,5\}$ and $7 \in \mathbb{N}_4(u)$ for $u \in \{1,3\}$. If $7 \in \mathbb{N}_4(1)$ and $6 \longrightarrow 7$ (resp. $7 \in \mathbb{N}_4(3)$ and $7 \longrightarrow 5$), as $\{5,7\}$ (resp. $\{6,7\}$) is not a clan of T, then $1 \longrightarrow 7$ (resp. $3 \longrightarrow 7$). It ensues from Lemma 2.1 and Remark 3.3 that $T-5$ (resp. $T-6$) is indecomposable and not isomorphic to U_7. This contradicts Lemma 5.4. Now, if $7 \in \mathbb{N}_4(1)$ and $7 \longrightarrow 6$ (resp. $7 \in \mathbb{N}_4(3)$ and $5 \longrightarrow 7$), then $7 \longrightarrow 5$ (resp. $7 \longrightarrow 6$) because $\{1,7\}$ (resp. $\{3,7\}$) is not a clan of T. Therefore, $T(4,7,6,5,2) \simeq W_5$ (resp. $T(2,4,7,6,5) \simeq W_5$), a contradiction. □

6. Theorem 1.2: Proof and optimality

Theorem 1.2 *Let T be an indecomposable tournament. If W_5 embeds into T, then $|W_5(T)| \geq |T| - 2$. If, moreover, $|T|$ is even, then $|W_5(T)| \geq |T| - 1$.*

Proof. The result is trivial for $|T| \leq 7$. By Proposition 5.5, we can assume that $|T| = n \geq 9$. First, assume that n is even. Suppose, by contradiction, that $|W_5(T)| \leq n-2$ and consider $x \neq y \in V(T)$ such that $\{x,y\} \cap W_5(T) = \emptyset$. Let X be a minimal subset of $V(T)$ such that $\{x,y\} \subset X$ ($|X| \geq 3$) and $T(X)$ is indecomposable, so that $T(X)$ is $\{x,y\}$-minimal. By Proposition 4.1 and Corollary 4.2, $T(X) \simeq C_3$ or U_5. By applying several times Lemma 2.3, there exists a subset $Y \in \binom{V(T)}{8}$ such that $X \subset Y$ and $T(Y)$ is indecomposable. This contradicts Proposition 5.5. Now, assume that n is odd. If T is critical, then, by Remark 3.2, $T \simeq W_n$ and hence $|W_5(T)| = n$. If T is not critical, then there is $x \in V(T)$ such that $T-x$ is indecomposable. We have $|T-x|$ is even and W_5 embeds into $T-x$ by Theorem 3.4. By the first case, $|W_5(T-x)| \geq n-2$, so that $|W_5(T)| \geq n-2$. □

By constructing examples, we verify that Theorem 1.2 is optimal. By Fact 1.1, we only construct for each integer $m \geq 6$, an indecomposable tournament T of order m with $\mid W_5(T) \mid = m-1$ and, when m is odd, another indecomposable tournament T' of order m with $\mid W_5(T') \mid = m-2$. We then introduce the tournaments Q_{2n+3}, R_{2n+3} defined on $\mathbb{N}_{2n+2}$, where $n \geq 2$, in the following manner.

- $Q_{2n+3}(\mathbb{N}_{2n}) = T_{2n+1}$, $N^+_{Q_{2n+3}}(2n+1) = \{1, \ldots, n\} \cup \{2n+2\}$ and $N^+_{Q_{2n+3}}(2n+2) = \mathbb{N}_{2n}$.
- $R_{2n+3} - \{2n+1\} = Q_{2n+3} - \{2n+1\}$ and $N^+_{R_{2n+3}}(2n+1) = \{0, 2n+2\}$.

The tournaments Q_{2n+3}, R_{2n+3} and $R_{2n+3} - \{2n+2\}$ form the required constructions:

Proposition 6.1. *For $n \geq 2$, the tournaments Q_{2n+3}, R_{2n+3} and $R_{2n+3} - \{2n+2\}$ are indecomposable and satisfy: $W_5(Q_{2n+3}) = \mathbb{N}_{2n+2} \setminus \{0, n+1\}$, $W_5(R_{2n+3}) = \mathbb{N}_{2n+2} \setminus \{n\}$ and $W_5(R_{2n+3} - \{2n+2\}) = \mathbb{N}_{2n+1} \setminus \{n\}$.*

Proof. We begin by verifying the indecomposability of these tournaments by using Lemma 2.1. The tournament Q_{2n+3} is indecomposable because $Q_{2n+3}(\mathbb{N}_{2n})$ is indecomposable, $2n+2 \longrightarrow \mathbb{N}_{2n}$ and, in this tournament, we have $2n+1 \in \mathbb{N}_{2n}(0)$ with $2n+1 \longrightarrow 2n+2$. The tournaments R_{2n+3} and $R_{2n+3} - \{2n+2\}$ are indecomposable by remarking that $R_{2n+3}(\mathbb{N}_{2n})$ is indecomposable, and the vertex $2n+1 \notin [\mathbb{N}_{2n}]$, $2n+1 \notin \mathbb{N}_{2n}(u)$, for a certain $u \in \mathbb{N}_{2n}$ and $2n+2 \longrightarrow \mathbb{N}_{2n}$ with $2n+1 \longrightarrow 2n+2$. At present, we verify that $W_5(Q_{2n+3}) = \mathbb{N}_{2n+2} \setminus \{0, n+1\}$. Since $Q_{2n+3}(2n+1, 2n+2, 1, 2, n+2) \simeq W_5$, then, by Theorem 1.2, it suffices to prove that $\{0, n+1\} \cap W_5(Q_{2n+3}) = \emptyset$. So, let $x \in \{0, n+1\}$ and suppose, by contradiction, that $x \in W_5(Q_{2n+3})$. By Remark 3.2, there exist $i \neq j \in \mathbb{N}_{2n} \setminus \{x\}$ with $i \longrightarrow j$ and $Q_{2n+3}(\{x, 2n+1, 2n+2, i, j\}) \simeq W_5$. As $Q_{2n+3}(\{x, 2n+1, 2n+2\}) \simeq C_3$ and $2n+2 \longrightarrow \mathbb{N}_{2n}$, then, by observing the subtournaments of W_5 which are isomorphic to C_3, we obtain that $W_5 \simeq Q_{2n+3}(2n+1, 2n+2, i, j, x)$, $Q_{2n+3}(2n+2, x, i, j, 2n+1)$ or $Q_{2n+3}(2n+2, i, j, x, 2n+1)$. It follows that $x \neq 0$. Otherwise, $\{x, 2n+1\} = \{0, 2n+1\}$ is not a clan of $Q_{2n+3}(\{0, 2n+1, j\})$, a contradiction because $2n+1 \in \mathbb{N}_{2n}(0)$. If $W_5 \simeq Q_{2n+3}(2n+1, 2n+2, i, j, n+1)$ or $Q_{2n+3}(2n+2, n+1, i, j, 2n+1)$ (resp. $W_5 \simeq Q_{2n+3}(2n+2, i, j, n+1, 2n+1)$), then $i \in N^+_{Q_{2n+3}}(2n+1) \cap N^+_{Q_{2n+3}}(n+1)$ (resp. $i \in N^-_{Q_{2n+3}}(2n+1) \cap N^-_{Q_{2n+3}}(n+1)$). A contradiction because $N^+_{Q_{2n+3}}(2n+1) \cap N^+_{Q_{2n+3}}(n+1) = N^-_{Q_{2n+3}}(2n+1) \cap N^-_{Q_{2n+3}}(n+1) = \emptyset$. Now, we verify that $W_5(R_{2n+3}) = \mathbb{N}_{2n+2} \setminus \{n\}$. For all $\alpha \in \{n+2, \ldots, 2n\}$ and for all $\beta \in \{1, \ldots, n-1\}$, we have $R_{2n+3}(2n+2, \alpha, 0, 1, 2n+1) \simeq R_{2n+3}(n+1, 2n, 2n+1, 0, \beta) \simeq W_5$. Thus, $\mathbb{N}_{2n+2} \setminus \{n\} \subseteq W_5(R_{2n+3})$. So, suppose, by contradiction, that $n \in W_5(R_{2n+3})$. Since $R_{2n+3} - 0$ and $R_{2n+3} - \{2n+1\}$ omits W_5, then there exist $k \neq l \in \mathbb{N}_{2n+2} \setminus \{0, n, 2n+1\}$ with $k \longrightarrow l$ and $R_{2n+3}(\{n, 0, 2n+1, k, l\}) \simeq W_5$. Since $R_{2n+3}(\{0, n, 2n+1\}) \simeq C_3$ and $s_{R_{2n+3}}(2n+1) = 2$, then, by observing again the subtournaments of W_5 which are isomorphic to C_3, we obtain that $W_5 \simeq R_{2n+3}(0, n, k, l, 2n+1)$, $R_{2n+3}(0, k, l, n, 2n+1)$, $R_{2n+3}(n, k, l, 2n+1, 0)$,

$R_{2n+3}(k, l, 0, n, 2n+1)$, $R_{2n+3}(k, l, 2n+1, 0, n)$, or $R_{2n+3}(k, l, , n, 2n+1, 0)$. Thus, $R_{2n+3}(\{0, n, l\})$ is a transitive tournament. This contradicts the fact that for all $l \in \mathbb{N}_{2n} \setminus \{0, n\}$, $R_{2n+3}(\{0, n, l\}) \simeq C_3$. Finally we can deduce that $W_5(R_{2n+3} - \{2n+2\}) = \mathbb{N}_{2n+1} \setminus \{n\}$ by Theorem 1.2 and by the fact that $W_5(R_{2n+3}) = \mathbb{N}_{2n+2} \setminus \{n\}$. □

We end by posing the following problems, motivated by Theorem 1.2, Fact 1.1 and Proposition 6.1.

Problem 6.2. Characterize the indecomposable tournaments T such that $|W_5(T)| = |T| - 2$.

Problem 6.3. Characterize the indecomposable tournaments T such that $|W_5(T)| = |T| - 1$.

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