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SPATIAL DECAY BOUNDS FOR A TEMPERATURE DEPENDENT STOKES FLOW

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To the memory of my esteemed professor, teacher and friend, Lawrence E. Payne

Abstract. This paper examines a temperature dependent Stokes flow in a semi-infinite cylinder. Under appropriate initial and boundary conditions the author establishes exponential decay of solutions in energy norm with distance from the finite end of the cylinder.

1. Introduction

We consider a problem of temperature dependent Stokes flow in a semi-infinite cylinder of uniform cross section. With prescribed data on the finite end of the cylinder together with appropriate homogeneous initial conditions and boundary conditions on the lateral surface, Saint-Venant type decay results are established. Other decay results for Darcy flow, Stokes and Navier-Stokes flow have been obtained by Payne and Song [12], Ames *et al.* [1], Song [14], Lin and Payne [8], and Horgan and Wheeler [6]. See for instance the survey papers of Horgan and Knowles [5], Horgan [3, 4] and the book of Straughan [15].

In describing the geometry of the semi-infinite cylinder we let R denote its interior and ∂R its boundary. The generators of the cylinder are assumed to be parallel to the x_3 axis whose entry section is assumed to lie in the plane $x_3 = 0$. Denoting the cross section of the pipe by D, the closure of D by $\overline{D}$ and its boundary by ∂D, we introduce the notation:

$$\begin{aligned} R_z &= \{(x_1, x_2, x_3) \,|\, (x_1, x_2) \in D,\ x_3 > z \geq 0\}, \\ D_z &= \{(x_1, x_2, x_3) \,|\, (x_1, x_2) \in D,\ x_3 = z\}. \end{aligned}$$

Clearly $R_0 = R$.

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Let u_i $(i = 1, 2, 3,)$, p, and T, all of functions of (x_1, x_2, x_3, t) denote respectively the velocity field, the pressure, the temperature. The flow is described by

$$u_{i,t} = -p_{,i} + \nu \Delta u_i + g_i(x)T \quad \text{in } R \times \{t > 0\}, \tag{1.1}$$

$$u_{j,j} = 0 \quad \text{in } R \times \{t > 0\}, \tag{1.2}$$

$$T_{,t} + u_i T_{,i} = \kappa \Delta T \quad \text{in } R \times \{t > 0\}, \tag{1.3}$$

$$u_i(x, 0) = 0,\ T(x, 0) = 0 \quad \text{in } R \times \{t = 0\}, \tag{1.4}$$

$$u_i = 0,\ T = 0 \quad \text{on } \partial R \backslash D_0 \times \{t \geq 0\}, \tag{1.5}$$

$$u_i = f_i(x_1, x_2, t),\ T = F(x_1, x_2, t) \quad \text{on } D_0 \times \{t \geq 0\}, \tag{1.6}$$

where Δ is the Laplace operator, ν and κ the constant kinematic viscosity and the constant conductivity respectively, g_i a given vector function, and a comma is used throughout to denote partial differentiation. We also use the summation convention of summing over Latin subscripts ranging from 1 to 3 and over the Greek subscript α from 1 to 2 unless noted otherwise. The prescribed functions f_i and F are assumed to be continuously differentiable in R and to vanish on ∂D for nonnegative x_3 and t, which is to satisfy the compatibility relationships (1.5). For compatibility we further assume that $f_{i,i} = 0$ and that $f_{\alpha,\alpha}$ is differentiable. By re-scaling the space and time variables, we may take both ν and κ to be 1.

We assume that time lies in some finite interval $[0, T]$. We further assume if the data f_i and F are sufficiently small in L_2, a classical solution of the initial-boundary value problem (1.1)–(1.6) will exist.

We note from (1.2) and (1.5) that

$$\begin{aligned}\int_{D_z} u_3 dA &= \int_{D_0} u_3 dA + \int_0^z \int_{D_\xi} u_{3,3} dA d\xi \\ &= \int_{D_0} u_3 dA - \int_0^z \int_{D_\xi} u_{\alpha,\alpha} dA d\xi \\ &= \int_{D_0} f_3 dA,\end{aligned}$$

where dA denotes the element of area in D. If the mean value of f_3 over D is zero, we expect the solution in some appropriate measure to vanish exponentially (see [1, p. 1399]). However, here we assume that the net entry flow in the pipe

$$\int_D f_3 dA = Q(t)$$

is nonzero and we suppose that $T \to o(x_3^{-1})$ uniformly in x, x_2, t as $x_3 \to \infty$ in view of the assumption (1.18), we expect that for sufficiently small data in (1.1)-(1.6) the velocity field (u_1, u_2, u_3) will tend to a transient Poiseuille flow

$(0,0,V)$ as $x_3 \to \infty$, where $V(x_1,x_2,t)$ satisfies

$$V_{,t} = V_{,\alpha\alpha} - \hat{p}_{,3} \quad \text{in } D \times \{t > 0\}, \tag{1.7}$$

$$V = 0 \quad \text{on } \partial D \times \{t \geq 0\}, \tag{1.8}$$

$$V = 0 \quad \text{in } D \times \{t = 0\}. \tag{1.9}$$

The gradient of the pressure $\hat{p}$ in (1.7) has the form $\hat{p}_{,i} = -P\delta_{3i}$ where P is a positive function of t only. This function $P(t)$ is not prescribed but is determined by the net inflow condition

$$\int_D V(x_1,x_2,t)dA = \int_D f_3 dA = Q(t). \tag{1.10}$$

For given $Q(t)$, the problem (1.7)-(1.10) is viewed as an inverse problem for determining $P(t)$ and $V(x_1,x_2,t)$ (see [14, pp. 506–507] and [8, pp. 459–460] and refer also to the similar argument for the stationary Navier-Stokes entry flow [1, p. 791] and [6, p. 99]).

We now let

$$w_i = u_i - v_i, \quad q_{,i} = p_{,i} - P(t)\delta_{i3}, \tag{1.11}$$

where $(v_1,v_2,v_3) = (0,0,V)$. Then we rewrite (1.1)–(1.6) as

$$w_{i,t} = -q_{,i} + \Delta w_i + g_i(x)T \quad \text{in } R \times \{t > 0\}, \tag{1.12}$$

$$w_{j,j} = 0 \quad \text{in } R \times \{t > 0\}, \tag{1.13}$$

$$T_{,t} + (w_i + v_i)T_{,i} = \Delta T \quad \text{in } R \times \{t > 0\}, \tag{1.14}$$

$$w_i(x,0) = 0, \ T(x,0) = 0 \quad \text{in } R \times \{t = 0\}, \tag{1.15}$$

$$w_i = 0, \ T = 0 \quad \text{on } \partial R \backslash D_0 \times \{t \geq 0\}, \tag{1.16}$$

$$w_i = f_i - V\delta_{i3}, \ T = F(x_1,x_2,t) \quad \text{on } D_0 \times \{t \geq 0\}. \tag{1.17}$$

We assume further that for any finite positive constants k_1 and k_2 the weighted energy expression

$$\int_0^t \int_R x_3 w_{i,j} w_{i,j} \, dx d\eta + k_1 \int_0^t \int_R x_3 w_{i,\eta} w_{i,\eta} \, dx d\eta + k_3 \int_0^t \int_R x_3 T_{,i} T_{,i} \, dx d\eta \tag{1.18}$$

is bounded. Here dx denotes the element of volume and $d\eta$ the element of time and there is no summation over a running time variable η.

In the next section we record some of the inequalities that will be used in our derivations. In Section 3 we derive a differential inequality for a weighted energy which integrates to yield exponential decay for finite energy solutions, and in the final section we establish a bound for the weighted total energy. To attempt to derive the absolute sharpest result would lead to a long and perhaps confusing paper, so in this paper we do not attempt to determine optimal results. Some detail, however, is necessary in order to show that the bounds we obtain are actually valid.

2. Auxiliary inequalities

We list in this section a number of inequalities in addition to the Schwarz inequality and the arithmetic-geometric mean inequality used throughout this paper.

Let v be a Dirichlet integrable function defined on a bounded plane domain D and vanishing on ∂D, then

$$\lambda \int_D v^2 dA \leq \int_D v_{,\alpha} v_{,\alpha} dA, \tag{2.1}$$

where λ is the smallest eigenvalue of

$$w_{,\alpha\alpha} + \hat{\lambda} w = 0 \quad \text{in } D, \quad w = 0 \quad \text{on } \partial D. \tag{2.2}$$

Lower bounds for λ are well known (see [9]).

We also make use of the following lemma.

Lemma. *Let R be a bounded simply connected region in $\mathbb{R}^3$ with Lipschitz boundary ∂R. Then, given any a Dirichlet integrable function v satisfying $\int_R v dx = 0$, there exists a vector field with components χ_i $(i = 1, 2, 3)$ which is Dirichlet integrable and vanishes on ∂R and a dimensionless constant C depending only on the geometry of R such that*

$$\chi_{j,j} = v \quad \text{in } R \tag{2.3}$$

and

$$\int_R \chi_{i,j}\chi_{i,j} dx \leq C \int_R [\chi_{j,j}]^2 dx. \tag{2.4}$$

This lemma is established by Ladyzhenskaya and Solonnikov [7] and in two dimensions this inequality by Babusuka and Aziz [2] (see also Horgan and Wheeler [6]). Recently, a bound for the optimal constant C is obtained by Payne [10].

3. Decay bounds

We now consider the energy expression for any finite positive constants k_1 and k_2,

$$\begin{aligned} F(z,t) &= \int_0^t \int_{R_z} (\xi - z) w_{i,j} w_{i,j} \, dx d\eta + k_1 \int_0^t \int_{R_z} (\xi - z) w_{i,\eta} w_{i,\eta} \, dx d\eta \\ &\quad + k_2 \int_0^t \int_{R_z} (\xi - z) T_{,i} T_{,i} \, dx d\eta \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.1}$$

where positive parameters k_1 and k_2 are to be determined later. By assumption (1.18), $F(z,t)$ is clearly bounded. Our goal is to show that for specific choices of k_1 and k_2, $F(z,t)$ actually decays exponentially in z. Upon integration by parts and using (1.12)-(1.16), we obtain

(3.2)
$$I_1 = -\int_0^t\int_{R_z} w_i w_{i,3}\,dxd\eta + \int_0^t\int_{R_z} w_3 q\,dxd\eta + \int_0^t\int_{R_z}(\xi - z)w_i g_i T\,dxd\eta - \frac{1}{2}\int_{R_z}(\xi - z)w_i w_i dx\big|_{\eta=t},$$

(3.3)
$$I_2 = k_1\int_0^t\int_{R_z} w_{3,\eta} q\,dxd\eta - k_1\int_0^t\int_{R_z} w_{i,\eta}w_{i,3}\,dxd\eta + k_1\int_0^t\int_{R_z}(\xi - z)w_{i,\eta}g_i T\,dxd\eta - \frac{k_1}{2}\int_{R_z}(\xi - z)w_{i,j}w_{i,j}dx\big|_{\eta=t},$$

(3.4)
$$I_3 = -k_2\int_0^t\int_{R_z} TT_{,3}\,dxd\eta + \frac{k_2}{2}\int_0^t\int_{R_z}(w_3 + v_3)T^2\,dxd\eta - \frac{k_2}{2}\int_{R_z} T^2 dx.$$

In bounding I_3, we assume that the velocity is uniformly bounded in R. This allows us to conclude that T satisfies a maximum principle in R. Integrating by parts, applying Schwarz's inequality, the arithmetic-geometric mean inequality, and (2.1) in (3.2)-(3.4), and dropping negative terms, we have

$$\begin{aligned}
I_1 \le{}& \left(\frac{1}{\lambda}\int_0^t\int_{R_z} w_{i,\alpha}w_{i,\alpha}\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z} w_{i,3}w_{i,3}\,dxd\eta\right)^{1/2} \\
&+ \int_0^t\int_{R_z} w_3 q\,dxd\eta \\
&+ \frac{g}{\lambda}\left(\int_0^t\int_{R_z}(\xi - z)w_{i,j}w_{i,j}\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}(\xi - z)T_{,i}T_{,i}\,dxd\eta\right)^{1/2}, \\
I_2 \le{}& k_1\int_0^t\int_{R_z} w_{3,\eta} q\,dxd\eta - k_1\int_0^t\int_{R_z} w_{i,\eta}w_{i,3}\,dxd\eta \\
&+ k_1\left(\int_0^t\int_{R_z}(\xi - z)w_{i,\eta}w_{i,\eta}\,dxd\eta\right)^{1/2}\left(\frac{g^2}{\lambda}\int_0^t\int_{R_z}(\xi - z)T_{,i}T_{,i}\,dxd\eta\right)^{1/2}, \\
I_3 \le{}& k_2\left(\frac{1}{\lambda}\int_0^t\int_{R_z} T_{,\alpha}T_{,\alpha}\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z} T_{,3}T_{,3}\,dxd\eta\right)^{1/2} \\
&+ \frac{k_2 T_M}{2}\left(\int_0^t\int_{R_z} w_3^2\,dxd\eta\right)^{1/2}\left(\frac{1}{\lambda}\int_0^t\int_{R_z} T_{,\alpha}T_{,\alpha}\,dxd\eta\right)^{1/2} \\
&+ \frac{k_2|V|_M}{2\lambda}\int_0^t\int_{R_z} T_{,\alpha}T_{,\alpha}\,dxd\eta,
\end{aligned}$$

where

$$g=\max_{\overline{D}}(g_ig_i)^{1/2},\quad T_M=\max_{D\times\{t>0\}}F(x_1,x_2,t),\quad |V|_M=|V|_{\max},$$

with $|V|_M$ given in [8]. Substituting the bounds for I_1, I_2, and I_3 into (3.1), we have

$$\begin{aligned}
&\left(1-\frac{\epsilon_1}{2\lambda}\right)\int_0^t\int_{R_z}(\xi-z)w_{i,j}w_{i,j}\,dxd\eta\\
&+k_1\left(1-\frac{\epsilon_2}{2\lambda}\right)\int_0^t\int_{R_z}(\xi-z)w_{i,\eta}w_{i,\eta}\,dxd\eta\\
&+\left(k_2-\frac{g^2}{2\epsilon_1\lambda}-\frac{k_1g^2}{2\epsilon_2\lambda}\right)\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}\,dxd\eta\\
&\le\frac{1}{\sqrt{\lambda}}\int_0^t\int_{R_z}w_{i,j}w_{i,j}\,dxd\eta+\int_0^t\int_{R_z}w_3q\,dxd\eta+k_1\int_0^t\int_{R_z}w_{3,\eta}q\,dxd\eta\\
&-k_1\int_0^t\int_{R_z}w_{i,\eta}w_{i,3}\,dxd\eta+\frac{k_2}{2\sqrt{\lambda}}\int_0^t\int_{R_z}T_{,i}T_{,i}\,dxd\eta\\
&+\frac{k_2T_M}{2}\left(\int_0^t\int_{R_z}w_3^2\,dxd\eta\right)^{1/2}\left(\frac{1}{\lambda}\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}\,dxd\eta\right)^{1/2}\\
&+\frac{k_2|V|_M}{2\lambda}\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}\,dxd\eta
\end{aligned}\tag{3.5}$$

for positive constants k_1 and k_2 to be specified. Choosing

$$k_1=\frac{1}{\lambda},\ k_2=\frac{1}{2}+\frac{g^2}{\lambda^2},\ \epsilon_1=\lambda,\ \epsilon_2=1,\tag{3.6}$$

we have

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{R_z}(\xi-z)w_{i,j}w_{i,j}\,dxd\eta+\frac{1}{2\lambda}\int_0^t\int_{R_z}(\xi-z)w_{i,\eta}w_{i,\eta}\,dxd\eta\\
&+\frac{1}{2}\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}\,dxd\eta\\
\le&\frac{1}{\sqrt{\lambda}}\int_0^t\int_{R_z}w_{i,j}w_{i,j}\,dxd\eta+\int_0^t\int_{R_z}u_3q\,dxd\eta+k_1\int_0^t\int_{R_z}w_{3,\eta}q\,dxd\eta\\
&-k_1\int_0^t\int_{R_z}w_{i,\eta}w_{i,3}\,dxd\eta+\frac{k_2}{2\sqrt{\lambda}}\int_0^t\int_{R_z}T_{,i}T_{,i}\,dxd\eta\\
&+\frac{k_2T_M}{2}\left(\int_0^t\int_{R_z}w_3^2\,dxd\eta\right)^{1/2}\left(\frac{1}{\lambda}\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}\,dxd\eta\right)^{1/2}\\
&+\frac{k_2|V|_M}{2\lambda}\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}\,dxd\eta.
\end{aligned}\tag{3.7}$$

We now set

$$E(z,t) = \frac{1}{2}\int_0^t\int_{R_z}(\xi - z)w_{i,j}w_{i,j}\,dxd\eta + \frac{1}{2\lambda}\int_0^t\int_{R_z}(\xi - z)w_{i,\eta}w_{i,\eta}\,dxd\eta + \frac{1}{2}\int_0^t\int_{R_z}(\xi - z)T_{,i}T_{,i}\,dxd\eta \tag{3.8}$$

from which we seek to derive a first-order differential inequality. We, therefore, bound the terms on the right in (3.7) in terms of $-\frac{\partial E}{\partial z}$. Upon Schwarz's inequality and the arithmetic-geometric mean inequality, we can easily estimate most of terms except for the two terms involving the pressure. To seek a bound for $\int_0^t\int_{R_z} w_3 q\,dxd\eta$, we note that, for any $z \geq 0$,

$$\int_{R_z} w_3 dx = 0.$$

Accordingly by Lemma, there exists a vector function χ_i such that

$$\chi_{i,i} = w_3 \quad \text{in } R_z, \quad \chi_i = 0 \quad \text{on } \partial R_z \tag{3.9}$$

and for χ_i inequality (2.4) holds. Using this χ_i and (2.4), we have

$$\begin{aligned}\int_0^t\int_{R_z} w_3 q\,dxd\eta &= \int_0^t\int_{R_z}\chi_i(w_{i,\eta} - \Delta w_i - g_i T)\,dxd\eta \\ &\leq \left(\int_0^t\int_{R_z}\chi_i\chi_i\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}w_{i,\eta}w_{i,\eta}\,dxd\eta\right)^{1/2} \\ &\quad + \left(\int_0^t\int_{R_z}\chi_{i,j}\chi_{i,j}\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}w_{i,j}w_{i,j}\,dxd\eta\right)^{1/2} \\ &\quad + g\left(\int_0^t\int_{R_z}\chi_i\chi_i\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}T^2\,dxd\eta\right)^{1/2}.\end{aligned}$$

By (2.1), we proceed to bound

$$\begin{aligned}\int_0^t\int_{R_z} w_3 q\,dxd\eta &\leq \left(\frac{C}{\lambda}\int_0^t\int_{R_z}w_3^2\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}w_{i,\eta}w_{i,\eta}\,dxd\eta\right)^{1/2} \\ &\quad + \left(C\int_0^t\int_{R_z}w_3^2\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}w_{i,j}w_{i,j}\,dxd\eta\right)^{1/2} \\ &\quad + \frac{g}{\lambda}\left(C\int_0^t\int_{R_z}w_3^2\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z}T_{,i}T_{,i}\,dxd\eta\right)^{1/2} \\ &\leq A\left(-\frac{\partial E}{\partial z}\right),\end{aligned} \tag{3.10}$$

where $A = 2\sqrt{\frac{C}{\lambda}}\left(2 + \frac{g}{\lambda}\right)$.

For $\int_0^t\int_{R_z} w_{3,\eta}q\,dxd\eta$, with a derivation similar to (3.10), we obtain

(3.11)
$$\begin{aligned}\int_0^t\int_{R_z} w_{3,\eta}q\,dxd\eta \le{}& \left(\frac{C}{\lambda}\int_0^t\int_{R_z} w_{3,\eta}^2\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z} w_{i,\eta}w_{i,\eta}\,dxd\eta\right)^{1/2}\\ &+\left(C\int_0^t\int_{R_z} w_{3,\eta}^2\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z} w_{i,j}w_{i,j}\,dxd\eta\right)^{1/2}\\ &+\frac{g}{\lambda}\left(C\int_0^t\int_{R_z} w_{3,\eta}^2\,dxd\eta\right)^{1/2}\left(\int_0^t\int_{R_z} T_{,i}T_{,i}\,dxd\eta\right)^{1/2}\\ \le{}& B\left(-\frac{\partial E}{\partial z}\right),\end{aligned}$$

where $B = 2\sqrt{C}\big(2\sqrt{\lambda}+\frac{g}{\lambda^{3/2}}\big)$. On applying Schwarz's inequality, the arithmetic-geometric mean inequality, (2.1), and (2.4) on the other remaining terms on the right in (3.7), it then follows that

$$E(z,t) \le K\left(-\frac{\partial E}{\partial z}\right), \tag{3.12}$$

where K is a computable constant. Upon integration we have

$$E(z,t) \le E(0,t)e^{-z/K}. \tag{3.13}$$

In order to make the exponential decay inequality (3.13) explicit, we require a bound for $E(0,t)$ in terms of the boundary data. In the next section we will indicate a procedure for obtaining bounds for the total weighed energy $E(0,t)$.

4. Bounds for $E(0,t)$

In this section we sketch how one can derive the total energy needed to complete our decay results.

We first note from (3.12) that

$$E(0,t) \le -K\frac{\partial E}{\partial z}(0,t). \tag{4.1}$$

Thus, we must bound $-\frac{\partial E}{\partial z}(0,t)$, which implies that we need to bound

$$\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta,\ \int_0^t\int_R w_{i,\eta}w_{i,\eta}\,dxd\eta,\ \int_0^t\int_R w_{i,j}w_{i,j}\,dxd\eta.$$

We first derive a bound for $\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta$ in terms of $\int_0^t\int_R w_{i,j}w_{i,j}\,dxd\eta$, then seek a bound for $\int_0^t\int_R w_{i,j}w_{i,j}\,dxd\eta$ in terms of data, and finally bound $\int_0^t\int_R w_{i,\eta}w_{i,\eta}\,dxd\eta$ in terms of data. We will use arguments similar to those employed in [11, 13, 12] to bound $\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta$. Thus

$$\begin{aligned}
&\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta \\
&= -\int_0^t\int_D FT_{,3}\,dAd\eta - \int_0^t\int_R T(T_{,\eta}+u_iT_{,i})\,dxd\eta \qquad (4.2)\\
&= -\int_0^t\int_D FT_{,3}\,dAd\eta + \frac{1}{2}\int_0^t\int_D f_3F^2\,dAd\eta - \frac{1}{2}\int_R T^2dx\Big|_{\eta=t}.
\end{aligned}$$

To bound the first term in (4.2) we set

$$S = F(x_1,x_2,t)e^{-\gamma z} \qquad (4.3)$$

for some positive γ. Then

$$\begin{aligned}
&-\int_0^t\int_D FT_{,3}\,dAd\eta \\
&= \int_0^t\oint_{\partial R} ST_{,i}n_i ds d\eta \\
&= \int_0^t\int_R S_{,i}T_{,i}\,dxd\eta + \int_0^t\int_R S(T_{,\eta}+u_iT_{,i})\,dxd\eta \qquad (4.4)\\
&= \int_0^t\int_R S_{,i}T_{,i}\,dxd\eta + \int_R STdx\Big|_{\eta=t} - \int_0^t\int_R TS_{,\eta}\,dxd\eta \\
&\quad -\int_0^t\int_R S_{,i}(w_i+v_i)T\,dxd\eta,
\end{aligned}$$

where ds is the element of surface area on ∂R. Inserting (4.4) into (4.2) and using the arithmetic-geometric mean inequality and the inequality (2.1) we have for some positive $\hat\epsilon_1, \hat\epsilon_2$, and $\hat\epsilon_3$

(4.5)

$$\begin{aligned}
&\left[1-\left(\frac{\hat\epsilon_1}{2}+\frac{\hat\epsilon_2}{2}+\frac{\hat\epsilon_3}{2}\right)\right]\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta \\
&\le \left(\frac{1}{2\hat\epsilon_1}+\frac{|V|_M^2}{2\lambda\hat\epsilon_3}\right)\int_0^t\int_R S_{,i}S_{,i}\,dxd\eta + \frac{1}{2\lambda\hat\epsilon_2}\int_0^t\int_R S_{,\eta}^2\,dxd\eta \\
&\quad + \frac{1}{2}\int_R S^2dx\Big|_{\eta=t} + T_M\left(\int_0^t\int_R S_{,i}S_{,i}\,dxd\eta\int_0^t\int_R w_iw_i\,dxd\eta\right)^{1/2} + \text{data}.
\end{aligned}$$

Choosing $\hat\epsilon_1 = 1/2, \hat\epsilon_2 = 1/4, \hat\epsilon_3 = 1/4$, by (2.1) and the arithmetic-geometric mean inequality we can write for some positive ϵ

$$\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta \le \epsilon\int_0^t\int_R w_{i,j}w_{i,j}\,dxd\eta + \text{data}, \qquad (4.6)$$

where the data involve parameters λ, T_M, $|V|_M$ and $\int_0^t\int_R S_{,i}S_{,i}\,dxd\eta$, $\int_0^t\int_R S_{,\eta}^2\,dxd\eta$, $\int_R S^2dx\Big|_{\eta=t}$ and $\int_0^t\int_D f_3F^2\,dAd\eta$ which are clearly data.

To obtain a bound for $\int_0^t \int_R w_{i,j} w_{i,j}\, dx d\eta$, by the triangle inequality we have

$$\begin{aligned} \int_0^t \int_R w_{i,j} w_{i,j}\, dx d\eta \leq\ & 2 \int_0^t \int_R (w_i - \tilde{w}_i)_{,j} (w_i - \tilde{w}_i)_{,j}\, dx d\eta \\ & + 2 \int_0^t \int_R \tilde{w}_{i,j} \tilde{w}_{i,j}\, dx d\eta, \end{aligned} \tag{4.7}$$

where

$$\tilde{w}_i = [f_i(x_1, x_2, t) - V\delta_{i3}] e^{-\sigma z} \tag{4.8}$$

for some positive σ. Using integration by parts and the lateral surface boundary condition (1.5), we obtain

$$\begin{aligned} & \int_0^t \int_R (w_i - \tilde{w}_i)_{,j} (w_i - \tilde{w}_i)_{,j}\, dx d\eta \\ = & -\int_0^t \int_R (w_i - \tilde{w}_i) \Delta (w_i - \tilde{w}_i)\, dx d\eta \\ = & -\int_0^t \int_R (w_i - \tilde{w}_i) [(w_{i,\eta} - \tilde{w}_{i,\eta}) + q_{,i} - g_i T + (\tilde{w}_{i,\eta} - \Delta \tilde{w}_i)]\, dx d\eta. \end{aligned} \tag{4.9}$$

Upon integration by parts and application of Schwarz's inequality, the arithmetic-geometric mean inequality and (2.1), we find for some positive γ_1 and γ_2,

$$\begin{aligned} & \left(1 - \frac{\gamma_1 + \gamma_2}{2\lambda}\right) \int_0^t \int_R (w_i - \tilde{w}_i)_{,j} (w_i - \tilde{w}_i)_{,j}\, dx d\eta \\ \leq\ & \frac{g}{2\gamma_1 \lambda} \int_0^t \int_R T_{,i} T_{,i}\, dx d\eta + \frac{1}{2\gamma_2} \int_0^t \int_R (\tilde{w}_{i,\eta} - \Delta \tilde{w}_i)(\tilde{w}_{i,\eta} - \Delta \tilde{w}_i)\, dx d\eta + \text{data}. \end{aligned} \tag{4.10}$$

Substituting (4.7) into (4.6) and inserting the result back into (4.10), we conclude that

$$\left(1 - \frac{\gamma_1 + \gamma_2}{2\lambda} - \frac{g\epsilon}{\gamma_1 \lambda}\right) \int_0^t \int_R (w_i - \tilde{w}_i)_{,j} (w_i - \tilde{w}_i)_{,j}\, dx d\eta \leq \text{data}. \tag{4.11}$$

Choosing $\gamma_1 = \lambda/4, \gamma_2 = \lambda/4, \epsilon = \gamma_1 \lambda/(4g)$ and combining (4.7), we have

$$\int_0^t \int_R w_{i,j} w_{i,j}\, dx d\eta \leq \text{data}. \tag{4.12}$$

It then follows from (4.6) that we find

$$\int_0^t \int_R T_{,i} T_{,i}\, dx d\eta \leq \text{data}. \tag{4.13}$$

Turning now to the bound for $\int_0^t\int_R w_{i,\eta}w_{i,\eta}\,dxd\eta$, by the triangle inequality we have

$$\left(\int_0^t\int_R w_{i,\eta}w_{i,\eta}\,dxd\eta\right)^{1/2} \le \left(\int_0^t\int_R (w_i-\tilde{w}_i)_{,\eta}(w_i-\tilde{w}_i)_{,\eta}\,dxd\eta\right)^{1/2} + \left(\int_0^t\int_R \tilde{w}_{i,\eta}\tilde{w}_{i,\eta}\,dxd\eta\right)^{1/2}. \tag{4.14}$$

We first note that

$$\begin{aligned}&\int_0^t\int_R (w_i-\tilde{w}_i)_{,\eta}(w_i-\tilde{w}_i)_{,\eta}\,dxd\eta\\ &= \int_0^t\int_R (w_i-\tilde{w}_i)_{,\eta}(-q_{,i}+\Delta(w_i-\tilde{w}_i+\tilde{w}_i)+g_iT-\tilde{w}_{i,\eta})\,dxd\eta.\end{aligned} \tag{4.15}$$

An application of Schwarz's inequality and the arithmetic-geometric mean inequality gives for some positive δ_1, δ_2 and δ_3

$$\begin{aligned}&\left(1-\frac{\delta_1+\delta_2+\delta_3}{2}\right)\int_0^t\int_R (w_i-\tilde{w}_i)_{,\eta}(w_i-\tilde{w}_i)_{,\eta}\,dxd\eta\\ &\le \frac{1}{2\delta_1}\int_0^t\int_R \Delta\tilde{w}_i\Delta\tilde{w}_i\,dxd\eta + \frac{g^2}{2\delta_2}\int_0^t\int_R T^2\,dxd\eta\\ &\quad+\frac{1}{2\delta_3}\int_0^t\int_R \tilde{w}_{i,\eta}\tilde{w}_{i,\eta}\,dxd\eta,\end{aligned} \tag{4.16}$$

where we have dropped a negative spatial integral term. For instance, taking $\delta_1 = 1/4, \delta_2 = 1/2, \delta_3 = 1/4$ and using (2.1), we find

$$\int_0^t\int_R (w_i-\tilde{w}_i)_{,\eta}(w_i-\tilde{w}_i)_{,\eta}\,dxd\eta \le \frac{2g^2}{\lambda}\int_0^t\int_R T_{,i}T_{,i}\,dxd\eta + \text{data}. \tag{4.17}$$

Combining (4.6), (4.14), and (4.17), we have

$$\int_0^t\int_R w_{i,\eta}w_{i,\eta}\,dxd\eta \le \text{data}. \tag{4.18}$$

When these results (4.12), (4.13), and (4.18) are inserted into (4.1), the bound for $E(0,t)$ in terms of data is obtained.

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