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TETRAVALENT SYMMETRIC GRAPHS OF ORDER $9p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify tetravalent symmetric graphs of order $9p$ for each prime p.

1. Introduction

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. Throughout this paper, we consider undirected finite connected graphs without loops or multiple edges. For a graph X we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, and automorphism group, respectively. For $u\,, v \in V(X)$, denote by $\{u\,, v\}$ the edge incident to u and v in X.

A graph X is said to be *vertex-transitive* if $\mathrm{Aut}(X)$ acts transitively on $V(X)$. An s-arc in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is called an arc for short and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* and (G, s)-*regular* if G is transitive and regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is simply called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* and s-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular and $(\mathrm{Aut}(X), s)$-transitive, respectively.

Arc-transitive or s-transitive graphs have received considerable attention in the literature. For example, s-transitive graphs of order np was classified in [3, 4, 23] depending on n=1, 2 or 3, where p is a prime. Li [13] showed that there exists an s-transitive graph of odd order if and only if $s \leq 3$. For the case of valency 4, Gardiner and Praeger [8, 9] characterized tetravalent

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symmetric graphs, and Li et al. [14] classified vertex-primitive tetravalent s-transitive graphs. The classification of tetravalent s-transitive Cayley graphs on abelian groups was given by Xu and Xu [25]. We may deduce a classification of tetravalent 1-regular Cayley graphs on dihedral groups from [12, 18, 21, 22]. Zhou [31] gave a classification of tetravalent 1-regular graphs of order $2pq$ for p, q primes. Recently, Zhou [29] classified tetravalent s-transitive graphs of order $4p$, and Zhou and Feng [30] classified tetravalent s-transitive graphs of order $2p^2$. In this paper we classify tetravalent s-transitive graphs of order $9p$.

Throughout the paper we denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by F_n the Frobenius group of order n.

2. Preliminary results

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G.

Proposition 2.1 ([11, Chapter I, Theorem 4.5]). *The quotient group*

$$N_G(H)/C_G(H)$$

is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ *of* H.

The following proposition is due to Burnside.

Proposition 2.2 ([19, Theorem 8.5.3]). *Let p and q be primes, and let m and n be non-negative integers. Then every group of order $p^m q^n$ is solvable.*

Let G be a permutation group on a set Ω. The size of Ω is called the degree of G acting on Ω.

Proposition 2.3 ([6, Corollary 3.5B]). *Every transitive permutation group of prime degree p is either 2-transitive or solvable with a regular normal Sylow p-subgroup.*

The following proposition is about the permutation group of degree p^2 for p a prime.

Proposition 2.4 ([28, Proposition 1]). *Any transitive group of degree p^2 has a regular subgroup.*

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G, S)) = G$ and edge set $E(\mathrm{Cay}(G, S)) = \{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, a Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The

Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of G as a regular group of automorphisms of $\mathrm{Cay}(G,S)$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [20]). For two subsets S and T of G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$, then S and T are said to be *equivalent*, denoted by $S \equiv T$. We may easily show that if $S \equiv T$, then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ and $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Cay}(G,T)$ is normal.

Proposition 2.5 ([26, Proposition 1.5])**.** *A Cayley graph* $\mathrm{Cay}(G,S)$ *is normal if and only if* $\mathrm{Aut}(\mathrm{Cay}(G,S))_1 = \mathrm{Aut}(G,S)$*, where* $\mathrm{Aut}(\mathrm{Cay}(G,S))_1$ *is the stabilizer of* 1 *in* $\mathrm{Aut}(\mathrm{Cay}(G,S))$.

From [1, Corollary 1.3], we have the following proposition.

Proposition 2.6. *Let* $X = \mathrm{Cay}(G,S)$ *be a connected tetravalent Cayley graph on a finite abelian group* G *of odd order. Then* X *is normal except for* $G = \mathbb{Z}_5$ *and* $X = K_5$.

For two subgroups M and N of a group G, $M \rtimes N$ stands for the semidirect product of M by N. The next proposition characterizes the vertex stabilizers of connected tetravalent s-transitive graphs (see [14, Lemma 2.5] and [13, Theorem 1.1]).

Proposition 2.7. *Let* X *be a connected tetravalent* (G,s)*-transitive graph of odd order. Then* $s \leq 3$ *and the stabilizer* G_v *of a vertex* $v \in V(X)$ *in* G *is as follows*:

(1) G_v *is a* 2*-group for* $s = 1$;
(2) $G_v \cong \mathrm{A}_4$ *or* S_4 *for* $s = 2$;
(3) $G_v \cong \mathbb{Z}_3 \times \mathrm{A}_4$, $\mathbb{Z}_3 \rtimes \mathrm{S}_4$, *or* $\mathrm{S}_3 \times \mathrm{S}_4$ *for* $s = 3$.

To introduce tetravalent symmetric graphs of order $3p$ for p a prime, we define some graphs. Let $p > 3$ be a prime and let $\mathbb{Z}_{3p} = \mathbb{Z}_3 \times \mathbb{Z}_p = \langle a\rangle \times \langle b\rangle$ be the cyclic group of order $3p$. Define $\mathcal{C}A_{3p} = \mathrm{Cay}(\mathbb{Z}_{3p}, \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\})$. By the definition of $G(3p,2)$ given in [23, Example 3.4], it is easy to see that $\mathcal{C}A_{3p} \cong G(3p,2)$ and $\mathrm{Aut}(\mathcal{C}A_{3p}) = \mathbb{Z}_{3p} \rtimes \mathbb{Z}_2^2$. The next proposition is about the classification of connected tetravalent symmetric graphs of order $3p$ (see [23, Theorem]).

Proposition 2.8. *Let* $p > 7$ *be a prime and* X *a connected tetravalent symmetric graph of order* $3p$*. Then* $X \cong \mathcal{C}A_{3p}$.

3. Graph constructions and isomorphisms

In this section we introduce connected tetravalent symmetric graphs of order $9p$ for p a prime. The first example is the lexicographic product of C_9 and $2K_1$.

Example 3.1. The lexicographic product $C_9[2K_1]$ is defined as the graph with vertex set $V(C_9) \times V(2K_1)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(C_9[2K_1])$, u is adjacent to v in $C_9[2K_1]$ if and only if $\{x_1, x_2\} \in E(C_9)$. Then $C_9[2K_1]$ is a connected tetravalent 1-transitive Cayley graph on the group $\mathbb{Z}_9 \times \mathbb{Z}_2$ and $\mathrm{Aut}(C_9[2K_1]) = \mathbb{Z}_2^9 \rtimes D_{18}$.

From [25, Example 3.2], we have the following example.

Example 3.2. Let $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$. The Cayley graph $\mathcal{G}_{18} = \mathrm{Cay}(G, \{ca, ca^{-1}, cb, cb^{-1}\})$ is 1-transitive and $\mathrm{Aut}(\mathcal{G}_{18}) = G \rtimes D_8$.

Xu and Xu [25] gave a classification of tetravalent arc-transitive Cayley graphs on finite abelian groups. The following example is extracted from [25, Example 3.2 and Theorem 3.5].

Example 3.3. Let $p \geq 3$ be a prime and $G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_{3p}$. Then the Cayley graph $\mathcal{C}A^1_{(3,3p)} = \mathrm{Cay}(G, \{b, b^{-1}, ab, a^{-1}b^{-1}\})$ is 1-regular and

$$\mathrm{Aut}(\mathcal{C}A^1_{(3,3p)}) = G \rtimes \mathbb{Z}_2^2.$$

Furthermore, if $p \equiv 3 \pmod 4$, then there is only one connected tetravalent symmetric Cayley graph on the group G, that is, $\mathcal{C}A^1_{(3,3p)}$, and if $p \equiv 1 \pmod 4$ there are exactly two connected tetravalent symmetric Cayley graphs on the group G, that is, $\mathcal{C}A^1_{(3,3p)}$ and $\mathcal{C}A^2_{(3,3p)}$, where $\mathcal{C}A^2_{(3,3p)} = \mathrm{Cay}(G, \{b, b^{-1}, ab^w, a^{-1}b^{-w}\})$ and $\mathrm{Aut}(\mathcal{C}A^2_{(3,3p)}) = G \rtimes \mathbb{Z}_4$ with w an element of order 4 in $\mathbb{Z}_p^*$.

By [27, Theorems 1 and 3], there is only one connected tetravalent symmetric Cayley graph on the cyclic group of order $9p$ for each prime $p \geq 5$.

Example 3.4. Let $p \geq 5$ be a prime and $G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_p$. The unique connected tetravalent symmetric Cayley graph on G is $\mathcal{C}A_{9p} = \mathrm{Cay}(G, \{ab, a^{-1}b^{-1}, a^{-1}b, ab^{-1}\})$, which is 1-regular and its automorphism group $\mathrm{Aut}(\mathcal{C}A_{9p}) = G \rtimes \mathbb{Z}_2^2$.

Let $X = \mathrm{Cay}(H, T)$ be a connected tetravalent symmetric Cayley graph on a non-abelian group H of order 27. Then $\langle T\rangle = H$, $T^{-1} = T$ and $|T| = 4$. By [7, Corollary 3.2], X is normal, and hence $\mathrm{Aut}(X)_1 = \mathrm{Aut}(H, T)$ by Proposition 2.5. Since $|H| = 27$, we may assume that $T = \{x, x^{-1}, y, y^{-1}\}$. Thus, $\mathrm{Aut}(H, T)$ is a 2-group and faithful on T, forcing that $\mathrm{Aut}(H, T) \leq D_8$. Since X is symmetric, $4 \mid |\mathrm{Aut}(H, T)|$. By the elementary group theory, there are two non-abelian groups of order 27:

$G_1(27) = \langle a, b \mid a^9 = b^3 = 1, b^{-1}ab = a^4\rangle$;

$G_2(27) = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle$.

If $H = G_1(27)$, then $4 \nmid |\mathrm{Aut}(H)|$ because each automorphism $\alpha \in \mathrm{Aut}(H)$ has the following form:

$$\alpha : \begin{cases} a \mapsto a^i b^j, & (i, 9) = 1, 0 \leq j \leq 2; \\ b \mapsto a^{3k} b, & 0 \leq k \leq 2. \end{cases}$$

This is impossible because $4 \mid |\mathrm{Aut}(H,T)|$. Thus, $H = G_2(27)$ and $o(x) = o(y) = 3$, where $o(x)$ denotes the order of x in $G_2(27)$. Since $\langle x, y\rangle = H$ and $[x,y] \in Z(H) = \langle c\rangle$, a, b and c have the same relations as do x, y and $[x,y]$, which implies that the map $a \mapsto x$, $b \mapsto y$, $c \mapsto [x,y]$ induces an automorphism of $G_2(27)$. It follows that $X \cong \mathrm{Cay}(G_2(27), S)$, where $S = \{a, a^{-1}, b, b^{-1}\}$.

Clearly, the maps $a \mapsto b$, $b \mapsto a$, $c \mapsto c$ and $a \mapsto b$, $b \mapsto a^{-1}$, $c \mapsto c$ induce automorphisms of $G_2(27)$, say α_1 and α_2, respectively. Then $\alpha_1, \alpha_2 \in \mathrm{Aut}(G_2(27), S)$ and $\langle \alpha_1, \alpha_2\rangle \cong D_8$, forcing that X is symmetric. On the other hand, since $\mathrm{Aut}(G_2(27), S) \leq D_8$, one has that $\mathrm{Aut}(G_2(27), S) = D_8$ and $\mathrm{Aut}(X) = G_2(27) \rtimes D_8$. Thus, we have the following example.

Example 3.5. Let $G = G_2(27) = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a,b] = c, [a,c] = [b,c] = 1\rangle$ and $S = \{a, a^{-1}, b, b^{-1}\}$. Define

$$\mathcal{G}_{27} = \mathrm{Cay}(G, S).$$

Then $\mathrm{Aut}(\mathcal{G}_{27}) = G \rtimes D_8$ and $\mathcal{G}_{27}$ is the only connected tetravalent symmetric Cayley graph on non-abelian group of order 27.

Let X be a symmetric graph, and A an arc-transitive subgroup of $\mathrm{Aut}(X)$. Let $\{u, v\}$ be an edge of X. Assume that $H = A_u$ is the stabilizer of $u \in V(X)$ and that $g \in A$ interchanges u and v. It is easy to see that the core H_A of H in A (the largest normal subgroup of A contained in H) is trivial, and that HgH consists of all elements of A which maps u to one of its neighbors in X. By [16, 20], the graph X is isomorphic to the coset graph $\mathrm{Cos}(A, H, HgH)$, which is defined as the graph with vertex set $\{Ha \mid a \in A\}$, the set of right cosets of H in A, and edge set $\{\{Ha, Hda\} \mid a \in A, d \in HgH\}$. The valency of $\mathrm{Cos}(A, H, HgH)$ is $|HgH|/|H| = |H : H \cap H^g|$, and $\mathrm{Cos}(A, H, HgH)$ is connected if and only if HgH generates A. By right multiplication, every element in A induces an automorphism of $\mathrm{Cos}(A, H, HgH)$. Since $H_A = 1$, the induced action of A on $V(\mathrm{Cos}(A, H, HgH))$ is faithful, and hence we may view A as a group of automorphisms of $\mathrm{Cos}(A, H, HgH)$.

From [14], one can see that, up to isomorphism, there is only one primitive tetravalent symmetric graph of order n if $n = 45$ or 153.

Example 3.6. Let $G = \mathrm{Aut}(\mathrm{A}_6) \cong \mathrm{S}_6 \rtimes \mathbb{Z}_2$ and let P be a Sylow 2-subgroup of G. By [5], P is a maximal subgroup of G and hence $N_G(P) = P$. Let H be an elementary abelian 2-subgroup of P of order 8. Then $N_G(H) \cong \mathrm{S}_4 \times \mathbb{Z}_2$. Let d be an involution in $N_G(H) \setminus P$. Define

$$\mathcal{G}_{45} = \mathrm{Cos}(G, P, PdP).$$

Then $\mathcal{G}_{45}$ is a connected tetravalent 1-transitive graph and $\mathrm{Aut}(\mathcal{G}_{45}) \cong \mathrm{Aut}(\mathrm{A}_6)$.

Example 3.7. Let $G = \mathrm{PSL}(2, 17)$ and let $P = \langle a, b \mid a^8 = b^2 = 1, bab = a^{-1}\rangle \cong D_{16}$ be a Sylow 2-subgroup of G. By [5], P is a maximal subgroup of G and hence $N_G(P) = P$. Let $H = \langle a^4, b\rangle$. Then $N_G(H) \cong \mathrm{S}_4$. Let d be an involution in $N_G(H) \setminus P$. Define

$$\mathcal{G}_{153} = \mathrm{Cos}(G, P, PdP).$$

Then $\mathcal{G}_{153}$ is a connected tetravalent 1-transitive graph and $\mathrm{Aut}(\mathcal{G}_{153}) \cong \mathrm{PSL}(2, 17)$.

Since the automorphism groups of the graphs defined in Examples 3.1-3.7 are pairwise non-isomorphic, we have the following lemma.

Lemma 3.8. *$C_9[2K_1], \mathcal{G}_{18}, \mathcal{CA}^1_{(3,3p)}, \mathcal{CA}^2_{(3,3p)}, \mathcal{CA}_{9p}, \mathcal{G}_{27}, \mathcal{G}_{45}$ and $\mathcal{G}_{153}$ are connected pairwise non-isomorphic tetravalent symmetric graphs.*

4. Classification

This section is devoted to classifying tetravalent symmetric graphs of order $9p$ for p a prime. First we have the following lemma.

Lemma 4.1. *Let p be a prime greater than 3 and G a non-abelian group of order $9p$. Then any connected tetravalent normal Cayley graph on G cannot be symmetric.*

Proof. Let $X = \mathrm{Cay}(G, S)$ be a connected tetravalent normal Cayley graph. Then $\langle S\rangle = G$, $S^{-1} = S$ and $|S| = 4$. Since $|G| = 9p$, we may assume $S = \{x, x^{-1}, y, y^{-1}\}$, and since X is normal, $\mathrm{Aut}(G, S) = \mathrm{Aut}(X)_1$ by Proposition 2.5.

Suppose to the contrary that X is symmetric. Then $\mathrm{Aut}(G, S)$ is transitive on S, forcing that $o(x) = o(y)$. Note that $p > 3$. By Sylow Theorem, G has a normal Sylow p-subgroup, which means that $o(x) \neq p$ because $\langle S\rangle = G$. Denote by $Z(G)$ the center of G. From the elementary group theory, up to isomorphism, there are three non-abelian groups of order $9p$ for a prime $p > 3$:

$G_1=\langle a, b \mid a^p = b^9 = 1, b^{-1}ab = a^r\rangle$, where $r \in \mathbb{Z}_p^*$ and $o(r) = 3$;

$G_2=\langle a, b \mid a^p = b^9 = 1, b^{-1}ab = a^s\rangle$, where $s \in \mathbb{Z}_p^*$ and $o(s) = 9$;

$G_3=\langle a, b, c \mid a^p = b^3 = c^3 = [b, c] = [a, b] = 1, c^{-1}ac = a^t\rangle$, where $t \in \mathbb{Z}_p^*$ and $o(t) = 3$.

Case 1: $G = G_1$.

In this case, $Z(G) = \langle b^3\rangle$ and $Z(G)$ is the unique subgroup of order 3 in G. Since $\langle S\rangle = G$, one has $o(x) \neq 3$ and hence $o(x) = o(y) = 3p$ or 9. Similarly, if $o(x) = 3p$, then $G = \langle S\rangle \subseteq Z(G) \times \langle a\rangle$, a contradiction. Thus, $o(x) = 9$ and x, y have the form a^ib^{3j+1} or a^ib^{3j-1}. Each automorphism α in $\mathrm{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i, & 1 \le i \le p-1; \\ b \mapsto a^jb^{3k+1}, & 0 \le j \le p-1,\ 0 \le k \le 2. \end{cases}$$

Clearly, $\mathrm{Aut}(G)$ is transitive on the set $\{\{g, g^{-1}\} \mid g \in G, o(g) = 9\}$. We may assume that $x = b$ and $y = a^ib^{3k+1}$. Since $a \mapsto a^i$, $b \mapsto b$ induces an automorphism of G, $S \equiv \{b, b^{-1}, ab^{3k+1}, (ab^{3k+1})^{-1}\}$. Note that every automorphism of G cannot map b to a^ib^{3k-1}. It follows that $\mathrm{Aut}(G,S) \lesssim \mathbb{Z}_2$. Thus, $\mathrm{Aut}(G,S)$ cannot be transitive on S, a contradiction.

Case 2: $G = G_2$.

Since $o(x) \neq p$, each element in S has order 3 or 9, and since $\langle a, b^3\rangle$ is a metacyclic normal subgroup of order $3p$ containing all elements of order 3, one has $o(x) \neq 3$. Thus, $o(x) = o(y) = 9$ and x, y have the form a^ib^{3j+1} or a^ib^{3j-1}. Each automorphism α in $\mathrm{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i, & 1 \leq i \leq p-1; \\ b \mapsto a^jb, & 0 \leq j \leq p-1. \end{cases}$$

Note that $a \mapsto a^i$, $b \mapsto b$ and $a \mapsto a$, $b^j \mapsto a^kb^j$ induce automorphisms of G. Then $S \equiv \{b^{3k_1+1}, (b^{3k_1+1})^{-1}, ab^{3k_2+1}, (ab^{3k_2+1})^{-1}\}$. Since every automorphism of G cannot map b^i to a^jb^{-i}, one has $\mathrm{Aut}(G,S) \lesssim \mathbb{Z}_2$. Thus, $\mathrm{Aut}(G,S)$ cannot be transitive on S, a contradiction.

Case 3: $G = G_3$.

Since $o(x) \neq p$, each element in S has order $3p$ or 3. Since $\langle a, b\rangle$ contains all elements of order $3p$ in G, one has $o(x) = 3$ because $\langle S\rangle = G$. Note that $Z(G) = \langle b\rangle$. Thus, $b, b^2 \notin S$, and x, y have the form a^ib^jc or $a^ib^jc^{-1}$ with $1 \leq i \leq p$ and $1 \leq j \leq 3$. Each automorphism α in $\mathrm{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i & 1 \leq i \leq p-1; \\ b \mapsto b^j & 1 \leq j \leq 2; \\ c \mapsto a^kb^lc & 0 \leq k \leq p-1,\ 0 \leq l \leq 2. \end{cases}$$

Thus, we may assume that $x = c$, and since the map $a \mapsto a^i$, $b \mapsto b^j$, $c \mapsto c$ induces an automorphism of G, $S \equiv \{c, c^{-1}, abc, (abc)^{-1}\}$. Since every automorphism of G cannot map a^ib^jc to $(a^ib^jc)^{-1}$, one has $\mathrm{Aut}(G,S) \lesssim \mathbb{Z}_2$. Thus, $\mathrm{Aut}(G,S)$ cannot be transitive on S, a contradiction. □

To state the main theorem, we introduce the so called quotient graph. Let X be a graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup on X. Assume that G is imprimitive on $V(X)$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ is a complete block system of G. The *block graph* or *quotient graph* $X_{\mathcal{B}}$ of X relative to $\mathcal{B}$ is defined as the graph with vertex set the complete block system $\mathcal{B}$, and with the two blocks adjacent if and only if there is an edge in X between those two blocks. Clearly, if X is G-symmetric, then $X_{\mathcal{B}}$ is G/K-symmetric, where K is the kernel of K on $\mathcal{B}$. For a normal subgroup N of G, the set of the orbits of N forms a complete block system of G. In this case we denote by X_N the quotient graph of X relative to the set of the orbits of N. The following is the main result of this paper.

Theorem 4.2. *Let p be a prime. Then any connected tetravalent symmetric graph of order $9p$ is isomorphic to one of the graphs in Table 1. Furthermore, all graphs in Table 1 are pairwise non-isomorphic.*

Table 1. Tetravalent s-transitive graphs of order $9p$

X	s-transitive	$\mathrm{Aut}(X)$	Comments
$C_9[2K_1]$	1-transitive	$\mathbb{Z}_2^9 \rtimes D_{18}$	Example 3.1, $p = 2$
$\mathcal{G}_{18}$	1-transitive	$(\mathbb{Z}_3^2 \times \mathbb{Z}_2) \rtimes D_8$	Example 3.2, $p = 2$
$\mathcal{G}_{27}$	1-transitive	$(\mathbb{Z}_3^2 \rtimes \mathbb{Z}_3) \rtimes D_8$	Example 3.5, $p = 3$
$\mathcal{G}_{45}$	1-transitive	$\mathrm{Aut}(\mathrm{A}_6)$	Example 3.6, $p = 5$
$\mathcal{G}_{153}$	1-transitive	$\mathrm{PSL}(2, 17)$	Example 3.7, $p = 17$
$\mathcal{C}A_{9p}$	1-regular	$\mathbb{Z}_{9p} \rtimes \mathbb{Z}_2^2$	Example 3.4, $p \geq 5$
$\mathcal{C}A^1_{(3,3p)}$	1-regular	$(\mathbb{Z}_3 \times \mathbb{Z}_{3p}) \rtimes \mathbb{Z}_2^2$	Example 3.3, $p \geq 3$
$\mathcal{C}A^2_{(3,3p)}$	1-regular	$(\mathbb{Z}_3 \times \mathbb{Z}_{3p}) \rtimes \mathbb{Z}_4$	Example 3.3, $p \equiv 1 (\mathrm{mod}\ 4)$

Proof. By Lemma 3.8, all graphs in Table 1 are connected pairwise non-isomorphic tetravalent symmetric graphs. Let X be a connected tetravalent symmetric graph of order $9p$. To finish the proof, it suffices to show that X is isomorphic to one of the graphs listed in Table 1.

If $p \leq 7$, then by [17, 24], there are ten connected tetravalent symmetric graphs of order $9p$: two graphs for $p = 2$, two graphs for $p = 3$, four graphs for $p = 5$ and two graphs for $p = 7$. Thus, X is isomorphic to $C_9[2K_2]$, $\mathcal{G}_{18}$, $\mathcal{G}_{27}$, $\mathcal{C}A^1_{(3,9)}$, $\mathcal{G}_{45}$, $\mathcal{C}A_{45}$, $\mathcal{C}A^1_{(3,15)}$, $\mathcal{C}A^2_{(3,15)}$, $\mathcal{C}A_{63}$ or $\mathcal{C}A^1_{(3,21)}$. Let $p > 7$ and assume that X is a normal Cayley graph. Then by Examples 3.3, 3.4 and Lemma 4.1, X is isomorphic to $\mathcal{C}A_{9p}$, $\mathcal{C}A^1_{(3,3p)}$ or $\mathcal{C}A^2_{(3,3p)}$.

Thus, in what follows one may assume that $p > 7$ and X is not a normal Cayley graph, that is, A has no normal regular subgroup on $V(X)$. Then, to finish the proof it suffices to show that $X \cong \mathcal{G}_{153}$.

Set $A = \mathrm{Aut}(X)$ and let A_v be the stabilizer of $v \in V(X)$ in A. Since X is symmetric, either A_v is a 2-group or $A_v \cong \mathrm{A}_4$, S_4, $\mathbb{Z}_3 \times \mathrm{A}_4$, $\mathbb{Z}_3 \rtimes \mathrm{S}_4$ or $\mathrm{S}_3 \times \mathrm{S}_4$ by Proposition 2.7. It follows that $|A| \mid 2^4 \cdot 3^4 \cdot p$ or $2^t \cdot 3^2 \cdot p$ for some integer t. Since $p > 7$, every Sylow 2-subgroup of A is also a Sylow 2-subgroup of a stabilizer of some vertex in A, implying that A has no non-trivial normal 2-subgroups.

Suppose that A has an intransitive minimal normal subgroup, say N. Since $|V(X)| = 9p$ and $|A| \mid 2^4 \cdot 3^4 \cdot p$ or $2^t \cdot 3^2 \cdot p$, N is either a non-abelian simple group, or an elementary abelian 3- or p-group. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be the set of orbits of N and K the kernel of A acting on $\mathcal{B}$. Then $N \leq K$. Let $m = |B_1|$. Then $mn = 9p$ with $1 < m, n < 9p$. The quotient graph X_N has vertex set $\mathcal{B}$ and $A/K \leq \mathrm{Aut}(X_N)$. Moreover, assume that B_1 is adjacent to B_2 in X_N with $v \in B_1$ and $u \in B_2$ being adjacent in X. Clearly, X_N has valency 2 or 4.

Case 1: X_N has valency 2.

In this case, X_N is a cycle and $A/K \cong D_{2n}$. Since X is symmetric, the induced subgraph $\langle B_1 \cup B_2 \rangle$ of $B_1 \cup B_2$ in X is a union of several cycles of the same length greater than 4, implying that K_v is a 2-group and K acts faithfully on B_1. Since $A/K \cong D_{2n}$, one has $|A| = 2^s mn = 2^s 9p$ for some integer s. This implies that if A has a Hall $\{3, p\}$-subgroup, then it is regular on $V(X)$. Note that $mn = 9p$ with $1 < m, n < 9p$. Thus, $\langle B_1 \cup B_2 \rangle \cong C_{2m}$, $3C_6$, $3C_{2p}$ or pC_6.

Let $\langle B_1 \cup B_2 \rangle \cong C_{2m}$. Since $\mathrm{Aut}(C_{2m}) \cong D_{4m}$, one has $\mathbb{Z}_m \lesssim K \lesssim D_{2m}$, and since $A/K \cong D_{2n}$, A has a normal subgroup of order $9p$, which is regular on $V(X)$ because A_v is a 2-group. Thus, A has a normal regular subgroup, a contradiction.

Let $\langle B_1 \cup B_2 \rangle \cong 3C_6$. Then N has blocks of length 3 on B_1 and since K acts faithfully on B_1, N must be an elementary abelian 3-group and hence K is a $\{2, 3\}$-group. By Proposition 2.2, K is solvable, and since $A/K \cong D_{2p}$, A is solvable. Thus, A has a Hall $\{3, p\}$-subgroup, say G, which is regular on $V(X)$. Since $N \trianglelefteq G$, G cannot be isomorphic to G_1, G_2 or G_3 as listed in Lemma 4.1. It follows that G is abelian, and by Proposition 2.6, X is a normal Cayley graph on G, a contradiction.

Now let $\langle B_1 \cup B_2 \rangle \cong 3C_{2p}$ or pC_6. Then $|B_1| = 3p$ and since N is transitive on B_1, N must be a non-abelian simple group, say T. By [5, pp. 12–14], T is one of the following groups in Table 2.

TABLE 2. Non-abelian simple $\{2, 3, p\}$-groups extracted from [5]

Group	Order	Out
A_5	$2^2 \cdot 3 \cdot 5$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	2
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	3
$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$	2
$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$	2
$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$	3
$\mathrm{PSU}(4,2)$	$2^6 \cdot 3^4 \cdot 5$	2

If $\langle B_1 \cup B_2 \rangle \cong 3C_{2p}$, then N has a transitive action of degree 3, which is impossible because N is a non-abelian simple group. Thus, $\langle B_1 \cup B_2 \rangle \cong pC_6$. Since $|A| = 2^s mn = 2^s 9p$ and N is intransitive, $9p \nmid |N|$. Then by Table 2, one has $N \cong \mathrm{PSL}(2, 7)$. This is impossible because $p > 7$.

Case 2: X_N has valency 4.

In this case, K_v fixes the neighborhood of v in X pointwise. Thus, $K = N$ is semiregular on $V(X)$ and $A/N \lesssim \mathrm{Aut}(X_N)$. Since $|V(X)| = 9p$, one has $N = \mathbb{Z}_p$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_3$.

Let $N \cong \mathbb{Z}_p$. Then the quotient graph X_N has order 9. By Proposition 2.4, A/N contains a regular subgroup, say B/N, on $V(X_N)$, that is, X_N is a Cayley graph on B/N. It follows that $|B/N| = 9$ and hence B/N is abelian. By

Proposition 2.6, $B/N \trianglelefteq A/N$ and hence $B \trianglelefteq A$. Thus, B is a normal regular subgroup of A on $V(X)$, a contradiction.

Let $N \cong \mathbb{Z}_3^2$. Then X_N is a tetravalent A/N-symmetric graph of order p. Since $p > 7$, X_N is not a complete graph, and hence A/N has a normal regular Sylow p-subgroup by Proposition 2.3. This implies that A has a normal regular subgroup, a contradiction.

Let $N \cong \mathbb{Z}_3$. Then X_N is a connected tetravalent symmetric graph of order $3p$. Since $p > 7$, by Proposition 2.8 one has $X_N \cong \mathcal{C}A_{3p}$. It follows that A/N has a normal regular subgroup on $V(X_N)$ because $\mathrm{Aut}(\mathcal{C}A_{3p}) \cong \mathbb{Z}_{3p} \rtimes \mathbb{Z}_2^2$, which implies that A has a normal regular subgroup on $V(X)$, a contradiction.

Now we may assume that A has no intransitive minimal normal subgroup. Thus, every non-trivial normal subgroup of A is transitive on $V(X)$. Again let N be a minimal normal subgroup of A. Then N is transitive on $V(X)$ and since $|V(X)| = 9p$, N is a non-abelian simple group as listed in Table 2. Recall that $p > 7$ and either $|N_v| = 2^t$ or $|N_v| = 3 \cdot 2^2$, $3 \cdot 2^3$, $3^2 \cdot 2^2$, $3^2 \cdot 2^3$ or $3^2 \cdot 2^4$. It follows that $N \cong \mathrm{PSL}(2, 17)$. Set $C = C_A(N)$, the centralizer of N in A. Then $C \cap N = 1$ and C is a $\{2, 3\}$-group. If $C \neq 1$, then C is an intransitive normal subgroup of A because $|V(X)| = 9p$, which is contrary to our assumption. Thus, $C = 1$ and $A = A/C \lesssim \mathrm{Aut}(N)$ by Proposition 2.1. Since $N \cong \mathrm{PSL}(2, 17)$, one has that $A = \mathrm{PSL}(2, 17)$ or $\mathrm{PGL}(2, 17)$, and the stabilizer A_v is a Sylow 2-subgroup of A, which is maximal in A by [5]. It follows that A is primitive on $V(X)$, and by [14, Theorem 1.5] and Example 3.7, $X \cong \mathcal{G}_{153}$ and $A \cong \mathrm{PSL}(2, 17)$. □

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