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CONVERGENCE PROPERTIES OF THE PARTIAL SUMS FOR SEQUENCES OF END RANDOM VARIABLES

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Abstract. The convergence properties of extended negatively dependent sequences under some conditions of uniform integrability are studied. Some sufficient conditions of the weak law of large numbers, the p-mean convergence and the complete convergence for extended negatively dependent sequences are obtained, which extend and enrich the known results in the literature.

1. Introduction and preliminaries

The concept of negatively dependent (ND) random variables was introduced by Ebrahimi and Ghosh ([4]).

Definition 1.1. The random variables $X_1, \ldots, X_k$ are said to be negatively upper dependent (NUD) if for all real $x_1, \ldots, x_k$,

$$P(X_i > x_i, i = 1, 2, \ldots, k) \leq \prod_{i=1}^{k} P(X_i > x_i), \tag{1.1}$$

and negatively lower dependent (NLD) if

$$P(X_i \leq x_i, i = 1, 2, \ldots, k) \leq \prod_{i=1}^{k} P(X_i \leq x_i). \tag{1.2}$$

Random variables $X_1, \ldots, X_k$ are said to be negatively dependent (ND) if they are both NUD and NLD.

Obviously sequences of ND random variables are a family of very wide scope, which contain sequences of independent random variables. Joag-Dev and Proschan ([6]) once pointed out that NA (negatively associated) implies ND, but neither NUD nor NLD implies NA. Since the paper of Joag-Dev and Proschan ([6]) appeared, the convergence properties of ND random sequences

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have been studied by Bozorgnia and Patterson ([2]), Taylor et al. ([13], [14]), Amini and Bozorgnia ([1]), Mi-Hwa Ko et al. ([7], [8]).

Liu ([10]) extended the negatively dependent structure. She introduced the concept of extended negatively dependent (END) random variables.

Definition 1.2. We call random variables $\{X_i, i \geq 1\}$ END if there exists a constant $M > 0$ such that both

$$P(X_i \leq x_i, i = 1, 2, \ldots, n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i) \tag{1.3}$$

and

$$P(X_i > x_i, i = 1, 2, \ldots, n) \leq M \prod_{i=1}^{n} P(X_i > x_i), \tag{1.4}$$

hold for each $n = 1, 2, \ldots$ and all $x_1, \ldots, x_n$.

Liu ([10]) pointed out the END structure is substantially more comprehensive than the ND structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent. So it is very significant to study probabilistic properties of this wider END class.

The following examples were provided in Liu ([10]) to illustrate that the extended negative dependence indeed allows a wide range of dependence structures.

Example 1.1. If $\{X_i, i = 1, 2\}$ and $\{X_i, i \geq 3\}$ are independent of each other, where X_1 is possibly valued at $x_{1_1} \leq x_{1_2} \leq \cdots \leq x_{1_N}$ and $\{X_i, i \geq 3\}$ is a sequence of mutually independent random variables, then the random variables $\{X_i, i \geq 1\}$ are END. In fact, for any x_1 and x_2 such that

$$P(X_1 \leq x_1)P(X_2 \leq x_2) = 0 \quad \text{or} \quad P(X_1 > x_1)P(X_2 > x_2) = 0,$$

both (1.3) and (1.4) hold trivially. Additionally, for any x_1 and x_2 such that

$$P(X_1 \leq x_1)P(X_2 \leq x_2) \neq 0 \quad \text{and} \quad P(X_1 > x_1)P(X_2 > x_2) \neq 0,$$

take

$$M = 1/\min\{P(X_1 = x_{1_1}), P(X_1 = x_{1_N})\},$$

then both (1.3) and (1.4) still hold. Note that there are no dependence restrictions between random variables X_1 and X_2.

Example 1.2. For any $n = 1, 2, \ldots$, let $X_1, \ldots, X_n$ be dependent according to a copula function $C(u_1, \ldots, u_n)$ with absolutely continuous dfs $F_1, \ldots, F_n$. Assume that the joint copula density

$$C_{1,\ldots,n}(u_1, \ldots, u_n) = \frac{\partial^n}{\partial u_1 \cdots \partial u_n} C(u_1, \ldots, u_n)$$

exists and is uniformly bounded in the whole domain. The random variables $\{X_i, i \geq 1\}$ are then END. As noted in Remark 3.1 of Ko and Tang ([9]), for example, copulas in the Frank family of the form

$$C_\alpha(u_1, \ldots, u_n) = \frac{1}{\alpha} \ln\left(1 + \frac{(\mathrm{e}^{\alpha u_1} - 1) \cdots (\mathrm{e}^{\alpha u_n} - 1)}{(\mathrm{e}^{\alpha} - 1)^{n-1}}\right), \quad \alpha < 0$$

belong to this category.

Definition 1.3 (Chandra, [9]). Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $p > 0$. The sequence $\{X_n, n \geq 1\}$ is said to be uniform integrability in the Cesàro sense if

$$\lim_{x\to\infty} \sup_{n\in N} n^{-1} \sum_{k=1}^{n} E|X_k|^p I_{(|X_k|\geq x)} = 0. \tag{1.5}$$

Since

$$\begin{aligned} E|X_k|^p I_{(|X_k|\geq x)} &= \Big(\int_0^{x^p} + \int_{x^p}^{\infty}\Big) P(|X_k|^p I_{(|X_k|\geq x)} > t)\mathrm{d}t \\ &= \int_0^{x^p} P(|X_k| \geq x)\mathrm{d}t + \int_{x^p}^{\infty} P(|X_k|^p > t)\mathrm{d}t \\ &= x^p P(|X_k| \geq x) + \int_{x^p}^{\infty} P(|X_k|^p > t)\mathrm{d}t, \end{aligned}$$

we know (1.5) is equivalent to

$$\lim_{x\to\infty} \sup_{n\in N} n^{-1} \sum_{k=1}^{n} x^p P(|X_k| \geq x) = 0 \tag{1.6}$$

and

$$\lim_{x\to\infty} \sup_{n\in N} n^{-1} \sum_{k=1}^{n} \int_{x^p}^{\infty} P(|X_k|^p > t)\mathrm{d}t = 0. \tag{1.7}$$

Wu et al. ([15]) studied the weak law of large numbers and the p-mean convergence for a sequence of NA random variables under the conditions of (1.5) and (1.6). S. H. Sung et al. ([12]) studied the weak law of large numbers for an array of dependent random variables under some conditions of uniform integrability. The goal of this paper is to study the weak law of large numbers, the p-mean convergence and the complete convergence for END sequences under some conditions of uniform integrability in the Cesàro sense. For this goal we need the following lemmas.

Lemma 1.1 (Liu, [10]). *If random variables $\{X_i, i \geq 1\}$ are END, then*

(1) *for any $n = 1, 2, \ldots$, there exists a constant $M > 0$ such that*

$$E\Big(\prod_{i=1}^{n} X_i^+\Big) \leq M \prod_{i=1}^{n} EX_i^+; \tag{1.8}$$

(2) $\{g_i(X_i), i = 1, 2, \ldots\}$ *are still END, where* $\{g_i(\cdot), i = 1, 2, \ldots\}$ *are either all monotone increasing or all monotone decreasing.*

Lemma 1.2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of END random variables with mean zero and* $0 < B_n = \sum_{k=1}^{n} EX_k^2 < \infty$. *Let* $S_n = \sum_{k=1}^{n} X_k$. *Then there exists a constant* $M > 0$ *such that*

$$P(|S_n| \geq x) \leq \sum_{k=1}^{n} P(|X_k| \geq y) + 2M\exp\Big(\frac{x}{y} - \frac{x}{y}\log\big(1 + \frac{xy}{B_n}\big)\Big) \tag{1.9}$$

for $\forall x > 0, y > 0$.

Proof. The proof is similar to the proof of Theorem 2 in Fuk and Nagaev ([5]). Let $y > 0$, $Y_i = \min(X_i, y)$ and $U_n = \sum_{i=1}^{n} Y_i$. Clearly $EY_i \leq 0$, $EY_i^2 \leq EX_i^2$. By Lemma 1.1(2) for $h > 0$, $\{\mathrm{e}^{hY_i}, 1 \leq i \leq n\}$ is nonnegative END. Thus, by Lemma 1.1(1), there exists a constant $M > 0$ such that

$$E\mathrm{e}^{hU_n} = E\prod_{i=1}^{n} \mathrm{e}^{hY_i} \leq M\prod_{i=1}^{n} E\mathrm{e}^{hY_i}. \tag{1.10}$$

Denoting $F_i(x) = P(X_i < x)$, then

$$\begin{aligned} E\mathrm{e}^{hY_i} &= \int_{-\infty}^{y} \mathrm{e}^{hx}\mathrm{d}F_i(x) + \mathrm{e}^{hy}P(X_i \geq y) \\ &= 1 + hEY_i + \int_{-\infty}^{y} (\mathrm{e}^{hx} - 1 - hx)\mathrm{d}F_i(x) + (\mathrm{e}^{hy} - 1 - hy)P(X_i \geq y) \\ &\leq 1 + \int_{-\infty}^{y} (\mathrm{e}^{hx} - 1 - hx)\mathrm{d}F_i(x) + (\mathrm{e}^{hy} - 1 - hy)P(X_i \geq y). \end{aligned}$$

For fixed $h > 0$, the function $f(x) = (\mathrm{e}^{hx} - 1 - hx)/x^2$ is increasing for all x. Note that $1 + u \leq \mathrm{e}^u$, $\forall u \in R$. Hence

$$\begin{aligned} E\mathrm{e}^{hY_i} &\leq 1 + \frac{\mathrm{e}^{hy} - 1 - hy}{y^2}\Big(\int_{-\infty}^{y} x^2\mathrm{d}F_i(x) + y^2P(X_i \geq y)\Big) \\ &\leq 1 + \frac{\mathrm{e}^{hy} - 1 - hy}{y^2}EX_i^2 \leq \exp\Big(\frac{\mathrm{e}^{hy} - 1 - hy}{y^2}EX_i^2\Big). \end{aligned}$$

Therefore, by (1.10), for $\forall x > 0$, $\forall h > 0$,

$$\mathrm{e}^{-hx}E\mathrm{e}^{hU_n} \leq M\exp\Big(-hx + B_n\frac{\mathrm{e}^{hy} - 1 - hy}{y^2}\Big).$$

Letting $h = \log(1 + \frac{xy}{B_n})/y$, we have

$$\begin{aligned} \mathrm{e}^{-hx}E\mathrm{e}^{hU_n} &\leq M\exp\Big(\frac{x}{y} - \frac{x}{y}\log(1 + \frac{xy}{B_n}) - \frac{B_n}{y^2}\log(1 + \frac{xy}{B_n})\Big) \\ &\leq M\exp\Big(\frac{x}{y} - \frac{x}{y}\log(1 + \frac{xy}{B_n})\Big). \end{aligned}$$

Therefore

$$\begin{aligned} P(S_n \geq x) &\leq P(S_n \neq U_n) + P(U_n \geq x) \\ &\leq \sum_{k=1}^{n} P(X_k \geq y) + \mathrm{e}^{-hx} E\mathrm{e}^{hU_n} \\ &\leq \sum_{k=1}^{n} P(X_k \geq y) + M\exp\Big(\frac{x}{y} - \frac{x}{y}\log(1 + \frac{xy}{B_n})\Big). \end{aligned}$$

Similarly, when X_i is replaced by $-X_i$, we have

$$P(-S_n \geq x) \leq \sum_{k=1}^{n} P(-X_k \geq y) + M\exp\Big(\frac{x}{y} - \frac{x}{y}\log(1 + \frac{xy}{B_n})\Big).$$

Therefore, for $\forall x > 0$, $\forall y > 0$, we have

$$\begin{aligned} P(|S_n| \geq x) &\leq P(S_n \geq x) + P(-S_n \geq x) \\ &\leq \sum_{k=1}^{n} P(|X_k| \geq y) + 2M\exp\Big(\frac{x}{y} - \frac{x}{y}\log(1 + \frac{xy}{B_n})\Big). \end{aligned}$$

The proof is complete. □

Lemma 1.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables satisfying* (1.6) *for some real number* $p > 0$. *Then*

$$\lim_{n\to\infty} n^{-\beta/p} \sum_{k=1}^{n} E|X_k|^\beta I_{(|X_k| \leq n^{1/p})} = 0, \quad \forall \beta > p. \tag{1.11}$$

Proof. Put $I = n^{-\beta/p} \sum_{k=1}^{n} E|X_k|^\beta I_{(|X_k| \leq n^{1/p})}$. Then

$$\begin{aligned} I &= n^{-\beta/p} \sum_{k=1}^{n} \int_0^\infty P\big(|X_k|^\beta I_{(|X_k| \leq n^{1/p})} \geq t\big)\mathrm{d}t \\ &= n^{-\beta/p} \sum_{k=1}^{n} \int_0^{n^{\beta/p}} P\big(|X_k|^\beta I_{(|X_k| \leq n^{1/p})} \geq t\big)\mathrm{d}t \\ &\leq n^{-\beta/p} \sum_{k=1}^{n} \int_0^{n^{\beta/p}} P\big(|X_k|^\beta \geq t\big)\mathrm{d}t. \end{aligned}$$

Let $t = y^\beta$. Then

$$\begin{aligned} I &\leq \beta n^{-\beta/p} \sum_{k=1}^{n} \int_0^{n^{1/p}} y^{\beta-1} P\big(|X_k| \geq y\big)\mathrm{d}y \\ &\leq \beta\, n^{-\beta/p+1} \int_0^{n^{1/p}} y^{\beta-1}\, n^{-1} \sum_{k=1}^{n} P\big(|X_k| \geq y\big)\mathrm{d}y. \end{aligned}$$

By (1.6), for $\forall\varepsilon > 0$, $\exists M > 0$ such that when $y > M$, we have

$$\sup_{n\in N} n^{-1}\sum_{k=1}^{n} P(|X_k| \geq y) \leq \varepsilon y^{-p}.$$

Hence when $n^{1/p} > M$, we have

$$I \leq \beta\, n^{-\beta/p+1}\Big(\int_0^M y^{\beta-1}\, n^{-1}\sum_{k=1}^{n} P\big(|X_k| \geq y\big)\mathrm{d}y + \varepsilon \int_M^{n^{1/p}} y^{\beta-p-1}\,\mathrm{d}y\Big)$$
$$\leq \beta\, n^{-\beta/p+1}\Big(C + \frac{\varepsilon}{\beta-p} n^{\beta/p-1}\Big) = C\beta\, n^{-\beta/p+1} + \frac{\beta\varepsilon}{\beta-p}\,.$$

Since $p < \beta$ and $\varepsilon > 0$ is arbitrary, $I \to 0$ as $n \to \infty$. □

Here in after, the symbol C stands for a generic positive constant which may differ from one place to another. Let $S_n = \sum_{k=1}^{n} X_k$.

2. Main results

Theorem 2.1. *Let $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$. Then condition* (1.6) *implies*

$$n^{-1/p} S_n \xrightarrow{P} 0, \quad n \to \infty. \tag{2.1}$$

Theorem 2.2. *Let $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$. Then condition* (1.5) *implies*

$$n^{-1/p} S_n \xrightarrow{L_p} 0, \quad n \to \infty. \tag{2.2}$$

Corollary 2.1. *Let $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with common distribution. Then $E|X|^p < \infty$ implies* (2.2).

Remark 2.1. Pyke and Root ([11]) obtained the p-mean convergence for a sequence of i.i.d. random variables under the same condition of Corollary 2.1. Therefore, Theorem 2.2 extends the result of Pyke and Root ([11]).

Remark 2.2. Wu et al. ([15]) obtained the weak law of large numbers and the p-mean convergence for a sequence of NA random variables under the same conditions of Theorems 2.1 and 2.2. Since NA implies ND or ND implies END, Theorems 2.1 and 2.2 extend the results of Wu et al. ([15]).

Theorem 2.3. *Let $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$. For $\delta > 2/p - 1$, $\alpha p \geq 1$, suppose*

$$\lim_{x\to\infty} \sup_{n\in N} n^{-1}\sum_{k=1}^{n} x^{1+\delta} P\big(|X_k|^p \geq x\big) = 0. \tag{2.3}$$

Then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\big(|S_n| > n^{\alpha}\varepsilon\big) < \infty, \quad \forall\varepsilon > 0. \tag{2.4}$$

Proof of Theorem 2.1. For any $1 \le k \le n$, let

$$X_k^{'} = -n^{1/p} I_{(X_k \le -n^{1/p})} + X_k I_{(|X_k| < n^{1/p})} + n^{1/p} I_{(X_k \ge n^{1/p})},$$
$$X_k^{''} = X_k - X_k^{'} = (X_k + n^{1/p}) I_{(X_k \le -n^{1/p})} + (X_k - n^{1/p}) I_{(X_k \ge n^{1/p})},$$
$$S_n^{'} \widehat{=} \sum_{k=1}^{n} X_k^{'}, \quad S_n^{''} \ \widehat{=} \ \sum_{k=1}^{n} X_k^{''}.$$

By Lemma 1.1(2), $X_k^{'}$ and $X_k^{''}$ are still END. For $\forall \varepsilon > 0$, we have

$$\begin{aligned} P\big(n^{-1/p}|S_n| \ge \varepsilon\big) &\le P\big(|S_n^{'} - ES_n^{'}| \ge n^{1/p}\varepsilon/2\big) + P\big(|S_n^{''} - ES_n^{''}| \ge n^{1/p}\varepsilon/2\big) \\ &\widehat{=} I_1 + I_2. \end{aligned}$$

Let $B_n^{'} = \sum_{k=1}^{n} E(X_k^{'} - EX_k^{'})^2$ and $x = y = n^{1/p}\varepsilon/2$. By Lemma 1.2 and the Markov inequality, we have

$$\begin{aligned} I_1 &\le \sum_{k=1}^{n} P\big(|X_k^{'} - EX_k^{'}| \ge n^{1/p}\varepsilon/2\big) + \frac{CB_n^{'}}{B_n^{'} + n^{2/p}\varepsilon^2/4} \\ &\le Cn^{-2/p} B_n^{'} \le Cn^{-2/p} \sum_{k=1}^{n} E(X_k^{'})^2 \\ &\le Cn^{-2/p} \sum_{k=1}^{n} n^{2/p} P(|X_k| \ge n^{1/p}) + Cn^{-2/p} \sum_{k=1}^{n} EX_k^2 I_{(|X_k| < n^{1/p})} \\ &= C \sum_{k=1}^{n} P(|X_k| \ge n^{1/p}) + Cn^{-2/p} \sum_{k=1}^{n} EX_k^2 I_{(|X_k| < n^{1/p})} \\ &\widehat{=} I_{11} + I_{12}. \end{aligned}$$

Clearly, (1.6) implies $I_{11} \to 0$ as $n \to \infty$. Take $\beta = 2$ in (1.11). Then by Lemma 1.3, $I_{12} \to 0$ as $n \to \infty$. Therefore, $I_1 \to 0$ as $n \to \infty$.

It remains to prove $I_2 \to 0$ as $n \to \infty$. From (1.6) and the definition of $X_k^{''}$, we have

$$\begin{aligned} I_2 &\le P\Big(\sum_{k=1}^{n} |X_k^{''} - EX_k^{''}| \ge n^{1/p}\varepsilon/2\Big) \\ &\le P\Big(\exists k;\ 1 \le k \le n, \text{ such that } |X_k| \ge n^{1/p}\Big) \\ &\le \sum_{k=1}^{n} P(|X_k| \ge n^{1/p}) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

The proof is complete. □

Proof of Theorem 2.2. For $\forall \varepsilon > 0$, we have

$$E\big|n^{-1/p} S_n\big|^p = n^{-1} \int_0^\infty P\big(|S_n| > t^{1/p}\big) \mathrm{d}t \le \varepsilon + n^{-1} \int_{\varepsilon n}^\infty P\big(|S_n| > t^{1/p}\big) \mathrm{d}t.$$

For $t \geq \varepsilon n$, let

$$Y_k = -t^{1/p} I_{(X_k \leq -t^{1/p})} + X_k I_{(|X_k| < t^{1/p})} + t^{1/p} I_{(X_k \geq t^{1/p})},$$
$$Z_k = X_k - Y_k = (X_k + t^{1/p}) I_{(X_k \leq -t^{1/p})} + (X_k - t^{1/p}) I_{(X_k \geq t^{1/p})}.$$

By Lemma 1.1(2), Y_k and Z_k are still END. Therefore, we have

$$\begin{aligned} E\big|n^{-1/p} S_n\big|^p &\leq \varepsilon + n^{-1} \int_{\varepsilon n}^{\infty} P\Big(\Big|\sum_{k=1}^{n} Z_k\Big| > t^{1/p}/2\Big) \mathrm{d}t \\ &\quad + n^{-1} \int_{\varepsilon n}^{\infty} P\Big(\Big|\sum_{k=1}^{n} Y_k\Big| > t^{1/p}/2\Big) \mathrm{d}t \\ &\widehat{=} \varepsilon + I_3 + I_4. \end{aligned}$$

To prove (2.2), it suffices to prove that $I_3 \to 0$ and $I_4 \to 0$ as $n \to \infty$. For I_3, we can get

$$\begin{aligned} I_3 &\leq n^{-1} \int_{\varepsilon n}^{\infty} P\Big(\exists k;\ 1 \leq k \leq n, \text{ such that } |X_k| > t^{1/p}\Big) \mathrm{d}t \\ &\leq n^{-1} \int_{\varepsilon n}^{\infty} \sum_{k=1}^{n} P(|X_k| \geq t^{1/p}) \mathrm{d}t \ \leq \ \sum_{k=1}^{n} n^{-1} E|X_k|^p I_{(|X_k| \geq (\varepsilon n)^{1/p})} \\ &\leq \sup_{m \in N} m^{-1} \sum_{k=1}^{m} E|X_k|^p I_{(|X_k| \geq (\varepsilon n)^{1/p})} \to 0 \quad \text{as } \ n \to \infty. \end{aligned}$$

Then we prove $I_3 \to 0$ as $n \to \infty$. Note that $|Z_k| \leq |X_k| I_{(|X_k| \geq t^{1/p})}$. From $EX_k = 0$ and (1.5), we have

$$\begin{aligned} \max_{t \geq \varepsilon n} \Big| t^{-1/p} \sum_{k=1}^{n} EY_k \Big| &= \max_{t \geq \varepsilon n} \Big| t^{-1/p} \sum_{k=1}^{n} EZ_k \Big| \\ &\leq \max_{t \geq \varepsilon n} t^{-1/p} \sum_{k=1}^{n} E|X_k| I_{(|X_k| \geq t^{1/p})} \\ &\leq (\varepsilon n)^{-1/p} \sum_{k=1}^{n} E|X_k| I_{(|X_k| \geq (\varepsilon n)^{1/p})} \\ &\leq \varepsilon^{-1} \sup_{m \in N} m^{-1} \sum_{k=1}^{m} E|X_k|^p I_{(|X_k| \geq (\varepsilon n)^{1/p})} \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

Therefore, while n is sufficiently large, for $t \geq \varepsilon n$, we have

$$P\Big(\Big|\sum_{k=1}^{n} Y_k\Big| > t^{1/p}/2\Big) \leq P\Big(\Big|\sum_{k=1}^{n} (Y_k - EY_k)\Big| > t^{1/p}/4\Big).$$

Let $B_n^{''} = \sum_{k=1}^n E(Y_k - EY_k)^2$, $x = t^{1/p}/4$, $y = t^{1/p}/4\gamma$, $\gamma > p$. By Lemma 1.2, we have

$$\begin{aligned} I_4 &\le n^{-1}\int_{\varepsilon n}^{\infty} P\Big(\Big|\sum_{k=1}^n \big(Y_k - EY_k\big)\Big| > t^{1/p}/4\Big)\mathrm{d}t \\ &\le n^{-1}\int_{\varepsilon n}^{\infty} \sum_{k=1}^n P\Big(\big|Y_k - EY_k\big| > t^{1/p}/4\gamma\Big)\mathrm{d}t \\ &\quad + Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(\frac{B_n^{''}}{B_n^{''} + t^{2/p}/16\gamma}\Big)^{\gamma}\mathrm{d}t \ \widehat{=}\ I_5 + I_6. \end{aligned}$$

Since

$$\begin{aligned} \max_{t\ge\varepsilon n} t^{-1/p}|EY_k| &= \max_{t\ge\varepsilon n} t^{-1/p}|EZ_k| \\ &\le (\varepsilon n)^{-1/p}E|X_k|I_{(|X_k|\ge(\varepsilon n)^{1/p})} \\ &\le (\varepsilon n)^{-1/p}\sum_{k=1}^n E|X_k|I_{(|X_k|\ge(\varepsilon n)^{1/p})} \\ &\le \varepsilon^{-1}\sup_{m\in N} m^{-1}\sum_{k=1}^m E|X_k|^p I_{(|X_k|\ge(\varepsilon n)^{1/p})} \to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

Hence

$$\begin{aligned} I_5 &\le n^{-1}\sum_{k=1}^n\int_{\varepsilon n}^{\infty} P\Big(\big|Y_k\big| > t^{1/p}/8\gamma\Big)\mathrm{d}t \\ &= n^{-1}\sum_{k=1}^n\int_{\varepsilon n}^{\infty} P\Big(\big|X_k\big|I_{(|X_k|<t^{1/p})} > t^{1/p}/8\gamma\Big)\mathrm{d}t \\ &\quad + n^{-1}\sum_{k=1}^n\int_{\varepsilon n}^{\infty} P\Big(\big|X_k\big| \ge t^{1/p}\Big)\mathrm{d}t \\ &\widehat{=}\ I_{51} + I_{52}. \end{aligned}$$

By similar argument as in the proof of $I_3 \to 0$, we may prove $I_{52} \to 0$. For I_{51}, we have

$$\begin{aligned} I_{51} &= n^{-1}\sum_{k=1}^n\int_{\varepsilon n}^{\infty} P\Big(\big|X_k\big|I_{((\varepsilon n)^{1/p}/8\gamma<|X_k|<t^{1/p})} > t^{1/p}/8\gamma\Big)\mathrm{d}t \\ &\le n^{-1}\sum_{k=1}^n\int_{0}^{\infty} P\Big(\big|X_k\big|I_{(|X_k|>(\varepsilon n)^{1/p}/8\gamma)} > t^{1/p}/8\gamma\Big)\mathrm{d}t \\ &\le Cn^{-1}\sum_{k=1}^n E|X_k|^p I_{(|X_k|>(\varepsilon n)^{1/p}/8\gamma)} \end{aligned}$$

$$\leq C\sup_{m\in N} m^{-1}\sum_{k=1}^{m} E|X_k|^p I_{(|X_k|>(\varepsilon n)^{1/p}/8\gamma)} \to 0 \quad \text{as} \quad n\to\infty.$$

Then we prove $I_6 \to 0$ as $n \to \infty$. Clearly, for $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\gamma > p \geq 1$, $(x+y+z)^\gamma \leq 3^{\gamma-1}(x^\gamma + y^\gamma + z^\gamma)$. Hence, by Cr-inequality, we have

$$\begin{aligned}
I_6 &\leq Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{n} EX_k^2 I_{(|X_k|<t^{1/p})} + \sum_{k=1}^{n} P(|X_k|\geq t^{1/p})\Big)^\gamma \mathrm{d}t\\
&= Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{n} EX_k^2 I_{(|X_k|<(\varepsilon n)^{1/p})}\\
&\quad +\ t^{-2/p}\sum_{k=1}^{n} EX_k^2 I_{((\varepsilon n)^{1/p}\leq|X_k|<t^{1/p})} + \sum_{k=1}^{n} P(|X_k|\geq t^{1/p})\Big)^\gamma \mathrm{d}t\\
&\leq Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{n} EX_k^2 I_{(|X_k|<(\varepsilon n)^{1/p})}\Big)^\gamma \mathrm{d}t\\
&\quad + Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(t^{-1/p}\sum_{k=1}^{n} E|X_k| I_{((\varepsilon n)^{1/p}\leq|X_k|<t^{1/p})}\Big)^\gamma \mathrm{d}t\\
&\quad + Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(\sum_{k=1}^{n} P(|X_k|\geq t^{1/p})\Big)^\gamma \mathrm{d}t\\
&\widehat{=}\ I_{61} + I_{62} + I_{63}.
\end{aligned}$$

Note that (1.5) implies (1.6). Take $\beta = 2$ in (1.11), by Lemma 1.3, $p < 2$ and $\gamma > p$, we have

$$\begin{aligned}
I_{61} &= Cn^{-1}\Big(\sum_{k=1}^{n} EX_k^2 I_{(|X_k|<(\varepsilon n)^{1/p})}\Big)^\gamma \int_{\varepsilon n}^{\infty} t^{-2\gamma/p}\mathrm{d}t\\
&\leq C\varepsilon\Big((\varepsilon n)^{-2/p}\sum_{k=1}^{n} EX_k^2 I_{(|X_k|<(\varepsilon n)^{1/p})}\Big)^\gamma \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

By $\gamma > p$, we have

$$\begin{aligned}
I_{62} &\leq Cn^{-1}\int_{\varepsilon n}^{\infty}\Big(t^{-1/p}\sum_{k=1}^{n} E|X_k| I_{(|X_k|\geq(\varepsilon n)^{1/p})}\Big)^\gamma \mathrm{d}t\\
&\leq Cn^{-1}\Big(\sum_{k=1}^{n} E|X_k| I_{(|X_k|\geq(\varepsilon n)^{1/p})}\Big)^\gamma \int_{\varepsilon n}^{\infty} t^{-\gamma/p}\mathrm{d}t\\
&\leq C\varepsilon\Big((\varepsilon n)^{-1/p}\sum_{k=1}^{n} E|X_k| I_{(|X_k|\geq(\varepsilon n)^{1/p})}\Big)^\gamma\\
&\leq C\varepsilon\Big((\varepsilon n)^{-1}\sum_{k=1}^{n} E|X_k|^p I_{(|X_k|\geq(\varepsilon n)^{1/p})}\Big)^\gamma
\end{aligned}$$

$$\leq C\varepsilon^{1-\gamma}\left(\sup_{m\in N} m^{-1}\sum_{k=1}^{m} E|X_k|^p I_{(|X_k|\geq(\varepsilon n)^{1/p})}\right)^{\gamma} \to 0 \quad \text{as } n\to\infty.$$

Finally, we prove $I_{63} \to 0$. Clearly,

$$\max_{t\geq\varepsilon n}\sum_{k=1}^{n} P(|X_k| > t^{1/p}) \leq \sum_{k=1}^{n} P(|X_k| > (\varepsilon n)^{1/p})$$
$$\leq \varepsilon^{-1}n^{-1}\sum_{k=1}^{n} E|X_k|^p I_{(|X_k|\geq(\varepsilon n)^{1/p})} \to 0 \quad \text{as } n\to\infty.$$

Therefore, while n is sufficiently large, $\sum_{k=1}^{n} P(|X_k| > t^{1/p}) < 1$ holds uniformly for $t \geq \varepsilon n$. By $\gamma > 1$ and similar argument as in the proof of $I_3 \to 0$, we can get

$$I_{63} \leq Cn^{-1}\int_{\varepsilon n}^{\infty}\sum_{k=1}^{n} P(|X_k| \geq t^{1/p})\mathrm{d}t$$
$$\leq C\sup_{m\in N} m^{-1}\sum_{k=1}^{m} E|X_k|^p I_{(|X_k|\geq(\varepsilon n)^{1/p})} \to 0 \quad \text{as } n\to\infty.$$

The proof is complete. □

Proof of Theorem 2.3. We follow the notations of $S_n^{'}$ and $S_n^{''}$ in the proof of Theorem 2.1. Let

$$X_k^{'} = -xI_{(X_k\leq -x)} + X_k I_{(|X_k|<x)} + xI_{(X_k\geq x)},$$
$$X_k^{''} = X_k - X_k^{'} = (X_k + x)I_{(X_k\leq -x)} + (X_k - x)I_{(X_k\geq x)}.$$

Here we take $x = n^{\alpha(2-p)/4}$. By Lemma 1.1(2), $X_k^{'}$ and $X_k^{''}$ are still END. For any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\big(|S_n| > n^{\alpha}\varepsilon\big)$$
$$\leq \sum_{n=1}^{\infty} n^{\alpha p-2} P\big(|S_n^{'} - ES_n^{'}| > n^{\alpha}\varepsilon/2\big) + \sum_{n=1}^{\infty} n^{\alpha p-2} P\big(|S_n^{''} - ES_n^{''}| > n^{\alpha}\varepsilon/2\big)$$
$$\widehat{=}\ I_7 + I_8.$$

To prove (2.4), it suffices to prove $I_7 < \infty$ and $I_8 < \infty$. Note that $|X_k^{'}| \leq n^{\alpha(2-p)/4}$. By similar argument as in the proof of $I_1 \to 0$, Lemma 1.2 and the Markov inequality, we have

$$I_7 \leq C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\sum_{k=1}^{n} E(X_k^{'})^2 \leq C\sum_{n=1}^{\infty} n^{-1-\alpha(2-p)/2} < \infty.$$

Note that $|X_k^{''}| \le |X_k| I_{(|X_k| \ge x)}$. By Lemma 1.2 and the Markov inequality, we also have

$$\begin{aligned} I_8 &\le C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{k=1}^{n} E|X_k|^2 I_{(|X_k| \ge x)} \\ &= C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{k=1}^{n} \Big(\int_0^{x^2} + \int_{x^2}^{\infty} \Big) P\big(|X_k|^2 I_{(|X_k| \ge x)} > t\big) \mathrm{d}t \\ &= C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{k=1}^{n} \Big(\int_0^{x^2} P\big(|X_k| \ge x\big) \mathrm{d}t + \int_{x^2}^{\infty} P\big(|X_k|^2 > t\big) \mathrm{d}t \Big) \\ &= C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{k=1}^{n} x^2 P\big(|X_k| \ge x\big) \\ &\quad + C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{k=1}^{n} \int_{x^2}^{\infty} P\big(|X_k|^2 > t\big) \mathrm{d}t \\ &\widehat{=} I_{81} + I_{82}. \end{aligned}$$

From (2.3), $\exists M > 0$, while $x > M$, we have

$$\sup_{n \in N} n^{-1} \sum_{k=1}^{n} P\big(|X_k|^p \ge x\big) \le x^{-(1+\delta)}. \tag{2.5}$$

By $x = n^{\alpha(2-p)/4}$, (2.5) and $1 + \delta - \frac{2}{p} > 0$ we have

$$\begin{aligned} I_{81} &= \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} \sum_{k=1}^{n} x^2 P\big(|X_k| \ge x\big) \\ &= \sum_{n=1}^{\infty} n^{\alpha p - 2\alpha - 1} n^{-1} \sum_{k=1}^{n} x^2 P\big(|X_k| \ge x\big) \\ &\le \sum_{n=1}^{\infty} n^{\alpha p - 2\alpha - 1} x^{-p(1+\delta)+2} \\ &= \sum_{n=1}^{\infty} n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\delta-\frac{2}{p})/4} < \infty \end{aligned}$$

and

$$\begin{aligned} I_{82} &= \sum_{n=1}^{\infty} n^{\alpha p - 2\alpha - 1} \int_{x^2}^{\infty} n^{-1} \sum_{k=1}^{n} P\big(|X_k|^2 > t\big) \mathrm{d}t \\ &\le \sum_{n=1}^{\infty} n^{\alpha p - 2\alpha - 1} \int_{x^2}^{\infty} t^{-\frac{p}{2}(1+\delta)} \mathrm{d}t \\ &\le C \sum_{n=1}^{\infty} n^{-1-\alpha(2-p)} x^{-p(1+\delta)+2} \end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\delta-\frac{2}{p})/4} < \infty.$$

The proof is complete. □

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