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On Convergence of Weighted Sums of LNQD Random

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Abstract

We discuss the strong convergence for weighted sums of linearly negative quadrant dependent(LNQD) random variables under suitable conditions and the central limit theorem for weighted sums of an LNQD case is also considered. In addition, we derive some corollaries in LNQD setting.

Keywords: Complete convergence, almost sure convergence, arrays, negative associated random variables, linearly negative quadrant random variables.

1. Introduction

Let $\{X_n|n \geq 1\}$ be a sequence of random variables. Hsu and Robbins (1947) introduced the concept of complete convergence of $\{X_n\}$. A sequence $\{X_n\}$ of random variables converges completely to the constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \epsilon) < \infty, \quad \text{for every } \epsilon > 0.$$

If $X_n \to c$ completely, then the Borel-Cantelli lemma implies that $X_n \to c$ is almost sure, but the converse is not true in general.

It was proved that the sequence of arithmetic means of independent identically distributed(i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. This result has been generalized and extended in several directions and carefully studied by many authors (see, Pruitt, 1966; Rohatgi, 1971; Gut, 1992; Wang *et al.*, 1993; Kuczmaszewska and Szynal, 1994; Magda and Sergey, 1997; Ghosal and Chandra, 1998; Hu *et al.*, 1999, 2001; Antonini *et al.*, 2001; Ahmed *et al.*, 2002; Liang *et al.*, 2004; Baek *et al.*, 2005).

Antonini *et al.* (2001) obtained result of the following theorem on complete and they had established some results for independent and identically distributed random variables.

Theorem 1. *Let $\{X_n|n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$ and $E(e^{t|X_1|}) < \infty$ for all $t > 0$. Let $\{a_{ni}|1 \leq i \leq m_n,\ n \geq 1\}$ is an array of real numbers satisfying the following conditions, where $\{m_n|n \geq 1\}$ is a sequence of positive integers.*

(a) $\max_{1\leq i\leq m_n} |a_{ni}| = O((\log n)^{-1})$.

(b) $\sum_{i=1}^{m_n} a_{ni}^2 = o((\log n)^{-1})$.

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Then

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{m_n} a_{ni}X_i\right|\right) > \varepsilon < \infty, \quad \text{for all } \varepsilon > 0.$$

In this paper, we discuss the strong law of large numbers for weighted sums of rowwise LNQD random variables. This paper extends and generalizes Theorem 1 for the i.i.d. random variables above to the case of LNQD random variables, which contains independent random variables and negatively associated random variables as special cases. In Section 2, we first give some complete convergence and almost sure convergence for LNQD random variables by using the exponential inequalities under some conditions. This result improves the theorem of Antonini *et al.* and in addition, we obtain some corollaries. Finally, in Section 3, we obtain the central limit theorem for partial sums of a LNQD random variables. We first recall the some definitions and lemmas of negatively associated, negative quadrant dependent, and linearly negative quadrant dependent random variables.

Definition 1. (Joag-Dev & Proschan, 1983) *A finite sequence $\{X_i|1 \leq i \leq n,\ n \geq 1\}$ of random variables is said to be negatively associated(NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$,*

$$Cov\left\{f(X_i : i \in A_1),\ g(X_j : j \in A_2)\right\} \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n|\ n \geq 1\}$ is NA if every finite subcollection is NA.

Definition 2. (Lehmann, 1966) *Two random variables X and Y are said to be negative quadrant dependent(NQD) if for any $x, y \in \mathbb{R}$,*

$$P(X < x, Y < y) \leq P(X < x)P(Y < y).$$

A sequence $\{X_n|n \geq 1\}$ of random variables is said to be pairwise NQD if X_i and X_j are NQD for all $i, j \in \mathbb{N}^+$ and $i \neq j$.

Lemma 1. (Lehmann, 1966) *Let X and Y be NQD random variables, then (a) $EXY \leq EXEY$, (b) $P(X < x,\ Y < y) \leq P(X < x)P(Y < y)$, and (c) If f and g are both nondecreasing (or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.*

Definition 3. (Newman, 1984) *A sequence $\{X_n|n \geq 1\}$ of random variables is said to be linearly negative quadrant dependent(LNQD) if for any disjoint subsets $A, B \subset \mathbb{N}^+$ and positive r_j's,*

$$\sum_{k\in A} r_k X_k \quad \text{and} \quad \sum_{j\in B} r_j X_j \quad \text{are NQD}.$$

Lemma 2. *Let $\{X_n|n \geq 1\}$ be a sequence of LNQD random variables with $EX_n = 0$ for each $n \geq 1$, then for any $t > 0$,*

$$Ee^{t\sum_{i=1}^{n} X_i} \leq \prod_{i=1}^{n} Ee^{tX_i} \leq e^{\frac{t^2}{l}2\sum_{i=1}^{n} EX_i^2 e^{t|X_i|}}.$$

Proof: Noticing that tX_i and $\sum_{j=i+1}^n tX_j$ are LNQD, we know by Definition 3, e^{tX_i} and $e^{t\sum_{j=i+1}^n X_j}$ are also NQD for $i = 1, 2, \ldots, n-1$. We will prove the first inequality by mathematical induction that

$$Ee^{t\sum_{i=1}^n X_i} \leq \prod_{i=1}^n Ee^{tX_i}. \tag{1.1}$$

First, we observe that

$$\begin{aligned} Ee^{t(X_1+X_2)} &\leq Ee^{tX_1} Ee^{tX_2} \\ &= \prod_{i=1}^2 Ee^{tX_i}, \end{aligned}$$

where the inequality follows from Lemma 1. Thus (1.1) is true for $i = 2$. Assume now that the statement is true for $i = k$. We will show that it is true for $i = k+1$.

$$\begin{aligned} Ee^{t\sum_{i=1}^{k+1} X_i} &= E\left(e^{t\sum_{i=1}^k X_i} e^{tX_{k+1}}\right) \\ &\leq Ee^{t\sum_{i=1}^k X_i} Ee^{tX_{k+1}} \\ &\leq \prod_{i=1}^k Ee^{tX_i} Ee^{tX_{k+1}} \\ &= \prod_{i=1}^{k+1} Ee^{tX_i}. \end{aligned}$$

Next, we will prove the second inequality that

$$\prod_{i=1}^n Ee^{tX_i} \leq e^{\frac{t^2}{2}\sum_{i=1}^n EX_i^2 e^{t|X_i|}}.$$

For all $x \in \mathbb{R}$, taking $e^x \leq 1 + x + x^2/2e^{|x|}$ and $EX_i = 0$, we have

$$\begin{aligned} Ee^{tX_i} &\leq 1 + tEX_i + \frac{t^2}{2} EX_i^2 e^{t|X_i|} \\ &= 1 + \frac{t^2}{2} EX_i^2 e^{t|X_i|} \\ &\leq e^{\frac{t^2}{2} EX_i^2 e^{t|X_i|}}, \quad \text{by } 1 + x \leq e^x. \end{aligned}$$

Thus, we obtain

$$\prod_{i=1}^n Ee^{tX_i} \leq e^{\frac{t^2}{2}\sum_{i=1}^n EX_i^2 e^{t|X_i|}}.$$

□

Lemma 3. (Newman, 1984) *Suppose that $X_1, X_2, \ldots, X_n$ are LNQD random variables with finite variance. Then for any real $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

$$\left| Ee^{i\sum_{k=1}^n \lambda_k X_k} - \prod_{k=1}^n e^{i\lambda_k X_k} \right| \leq \sum_{k=1, j>k}^n \left|\lambda_k\right|\left|\lambda_j\right|\left|Cov(X_k, X_j)\right|.$$

Newman (1984) introduced the concepts of LNQD r.v.'s. Many authors derived several important properties about LNQD random variables and also discussed some applications in several areas (see Cai and Roussas, 1997; Wang and Zhang, 2006; Ko *et al.*, 2007 among others). Throughout this paper, $a = O(b)$ means that there exists some constant $C > 0$ such that $a \le Cb$.

2. Main Results

Theorem 2. *Let $\{X_{ni}|1 \le i \le n,\ n \ge 1\}$ be an array of rowwise LNQD random variables such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $i \ge 0$ and $x > 0$ and let $\{a_{ni}|1 \le i \le n,\ n \ge 1\}$ be an array of real numbers satisfying the following conditions;*

(a) $1/|a_{nn}| = O((\log n)^{-1})$.

(b) $a_{nn}X_{ni} \to 0$ *in probability* .

(c) $\sum_{i=1}^{n} a_{ni}X_{ni} \to 0$ *almost surely.*

Then $E(e^{t|X|}) < \infty$ for all $t > 0$.

Proof: Let $Y_n = \sum_{i=1}^{n-1} a_{ni}X_{ni}$ and $Z_n = a_{nn}X_{nn}$. Then Y_n and $\{Z_n, Z_{n+1}, \ldots\}$ are LNQD by Definition 3 and noticing (a) and (b) imply $Y_n \to 0$ in probability as $n \to \infty$ and (c) imply $Y_n + Z_n \to 0$ almost surely as $n \to \infty$, and hence $Z_n \to 0$ almost surely as $n \to \infty$. Since Z_n are LNQD by Definition 3, it follows by the Borel-Cantelli lemma that for $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(e^{\frac{c|X|}{\varepsilon}} > n\right) &= \sum_{n=1}^{\infty} P\left(\log e^{\frac{c|X|}{\varepsilon}} > \log n\right)\\ &\le O(1)\sum_{n=1}^{\infty} P\left(|a_{nn}X| > \varepsilon\right)\\ &= O(1)\sum_{n=1}^{\infty} P\left(|Z_n| > \varepsilon\right) < \infty.\end{aligned}$$

Hence $E(e^{t|X|}) < \infty$ for all $t > 0$. □

From the above theorem, we can obtain the following Corollary 1.

Corollary 1. *Let $\{X_{ni}|1 \le i \le m_n,\ n \ge 1\}$ be an array of rowwise LNQD random variables such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $x \ge 0$ and $1 \le i \le m_n$, $n \ge 1$, where $\{m_n|n \ge 1\}$ is a strictly increasing sequence of positive integers, and let $\{a_{ni}|1 \le i \le m_n,\ n \ge 1\}$ be an array of real numbers. Suppose the following conditions satisfying;*

(a) f is a positive nondecreasing function such that $f(g(n)) \le n$ for all $n \ge 1$.

(b) $g(n) \to \infty$ as $n \to \infty$.

(c) $\sum_{i=1}^{m_n} a_{ni}X_{ni} \to 0$ completely, where $\{m_n|n \ge 1\}$ is a strictly increasing sequence of integers. If $g(n) = 1/\max_{1\le i\le m_n}|a_{ni}|$, then $E(f(t|X|)) < \infty$ for all $t > 0$.

Proof: Suppose l_n be such that $|a_{nl_n}| = 1/g(n)$, $n \ge 1$. Let $Y_n = \sum_{i=1}^{m_n} a_{ni}X_{ni} - a_{nl_n}X_{nl_n} + a_{nl_n}X_{nl_n} - a_{nm_n}X_{nm_n}$ and $Z_n = a_{nl_n}X_{nm_n}$. Then Y_n and $\{Z_n|Z_{n+1}, \ldots\}$ are LNQD. Hence, we can obtain the rest of the result from Theorem 2. □

From Theorem 2, we state and prove one of our important results.

Theorem 3. *Let $\{X_{ni}|1 \leq i \leq m_n, n \geq 1\}$ be an array of rowwise LNQD random variables with $EX_{ni} = 0$. Suppose that there is a random variable X such that $P(|X_{ni}| > x) \leq O(1)P(|X| > x)$ for all $1 \leq i \leq m_n$, $n \geq 1$ and $x > 0$, which $\{m_n|n \geq 1\}$ is a sequence of positive integers. Assume that $\{a_{ni}|1 \leq i \leq m_n, n \geq 1\}$ is an array of real numbers satisfying the following conditions;*

(a) $\max_{1\leq i\leq m_n} |a_{ni}| = O((\log n)^{-1})$.

(b) $\sum_{i=1}^{m_n} a_{ni}^2 = o((\log n)^{-1})$.

If $E(e^{t|X|}) < \infty$ for all $t \geq 0$, then

$$\sum_{n=1}^{\infty} n^{\alpha} \left(\left| \sum_{i=1}^{m_n} a_{ni} X_{ni} \right| \right) > \varepsilon < \infty, \quad \textit{for all } \varepsilon > 0 \textit{ and } \alpha \geq 0.$$

Proof: Sine $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, it suffices to show that

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\left| \sum_{i=1}^{m_n} a_{ni}^{+} X_{ni} \right| > \varepsilon \right) < \infty, \quad \text{for all } \varepsilon > 0, \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\left| \sum_{i=1}^{m_n} a_{ni}^{-} X_{ni} \right| > \varepsilon \right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

Since the proof of (2.2) is similar to (2.1), we only prove (2.1). To prove (2.1), we need only to prove that

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\sum_{i=1}^{m_n} a_{ni}^{+} X_{ni} > \varepsilon \right) < \infty, \quad \text{for all } \varepsilon > 0, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\sum_{i=1}^{m_n} a_{ni}^{+} X_{ni} < -\varepsilon \right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.4}$$

By the definition of LNQD random variables, we know that $\{a_{ni}^{+} X_{ni}|1 \leq i \leq m_n, n \geq 1\}$ is still an array of rowwise LNQD random variables.

From an inequality $e^x \leq 1 + x + x^2/2e^{|x|}$ for all $x \in R$, using the first inequality of Lemma 2 and taking $t = \beta \log n/\varepsilon$, where β is a large constant and will be specified latter on, we have

$$\begin{aligned}
Ee^{t\sum_{i=1}^{m_n} a_{ni}X_{ni}} &\leq \prod_{i=1}^{m_n} Ee^{ta_{ni}X_{ni}} \\
&\leq \prod_{i=1}^{m_n} \left(1 + \frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 (\log n)^2 a_{ni}^2\right) E\left(X_{ni}^2 e^{\frac{\beta \log |a_{ni}X_{ni}|}{\varepsilon}}\right) \\
&\leq \prod_{i=1}^{m_n} \left(1 + \frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left(X_{ni}^2 e^{c|X_{ni}|}\right)\right) \\
&\leq \prod_{i=1}^{m_n} \left(1 + \frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left(X^2 e^{c|X|}\right)\right) \\
&\leq \prod_{i=1}^{m_n} \left(1 + \frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left(e^{(1+c)|X|}\right)\right).
\end{aligned} \tag{2.5}$$

Next, by using the second inequality of Lemma 2 and the result of above (2.5), we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha} P\left(\sum_{i=1}^{m_n} a_{ni}^{+} X_{ni} > \varepsilon\right) &\leq \sum_{n=1}^{\infty} n^{\alpha} e^{-\varepsilon t} E e^{t \sum_{i=1}^{m_n} a_{ni}^{+} X_{ni}} \\ &\leq \sum_{n=1}^{\infty} n^{\alpha-\beta} \prod_{i=1}^{m_n} \left(1 + \frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left(e^{(1+c)|X|}\right)\right) \\ &\leq \sum_{n=1}^{\infty} n^{\alpha-\beta} e^{\sum_{i=1}^{m_n} \frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left(e^{(1+c)|X|}\right)} \\ &\leq \sum_{n=1}^{\infty} n^{\alpha-\beta} e^{\frac{1}{2}\left(\frac{\beta}{\varepsilon}\right)^2 \sum_{i=1}^{m_n} (\log n)^2 a_{ni}^2 E\left(e^{(1+c)|X|}\right)} \\ &\leq \sum_{n=1}^{\infty} n^{\alpha-\beta+\varepsilon} < \infty\end{aligned}$$

provided $\beta > (\alpha + \varepsilon) + 1$, where c denote positive constant whose values are unimportant and may vary at different place.

By replaying X_{ni} by $-X_{ni}$ from the above statement and noticing $\{a_{ni}^{+}(-X_{ni}) | 1 \leq i \leq m_n,\ n \geq 1\}$ is still an array of rowwise LNQD random variables, we obtain that

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\sum_{i=1}^{m_n} a_{ni}^{+} X_{ni} < -\varepsilon\right) < \infty, \quad \text{for any } \varepsilon > 0.$$

□

Corollary 2. *Let $\{X_{ni} | 1 \leq i \leq m_n,\ n \geq 1\}$ be an array of rowwise LNQD random variables with $EX_{ni} = 0$. Suppose that there is a random variable X such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $1 \leq i \leq m_n$, $n \geq 1$ and $x \geq 0$, which $\{a_{ni} | 1 \leq i \leq m_n,\ n \geq 1\}$ is an array of real numbers satisfying*

$$\limsup_{n \to 0} \sum_{i=1}^{m_n} c_{ni}^2 < \infty.$$

(a) If $Ee^{t|X|} < \infty$ for all $t > 0$, then

$$\sum_{n=1}^{\infty} n^{\alpha} P\left(\left|\sum_{i=1}^{m_n} c_{ni} X_{ni}\right| > \varepsilon \log n\right), \quad \textit{for all } \varepsilon > 0 \textit{ and } \alpha > 0.$$

(b) If $\sum_{n=1}^{\infty} n^{\alpha} P(|\sum_{i=1}^{m_n} b_{ni} X_{ni}| > \varepsilon \log n) < \infty$ for all $\epsilon > 0$ and $\alpha > 0$, then

$$Ee^{t|X|} < \infty, \quad \textit{for all } t > 0.$$

Proof of (a) and (b): Let $a_{ni} = c_{ni}/\log n$ and by using Theorem 3, we can obtain the result of (a),and Suppose that $a_{ni} = c_{ni}/\log n$ and $c_{ni} = 1/(n + 1 - i)$. Then, by using Corollary 1, we obtain that $Ee^{t|X|} < \infty$ for all $t > 0$. □

Corollary 3. *Let $\{X_{ni} | 1 \leq i \leq m_n,\ n \geq 1\}$ be an array of rowwise LNQD random variables with $EX_{ni} = 0$. Suppose that there is a random variable X such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for* *all* $1 \le i \le m_n,\ n \ge 1$ *and* $x \ge 0$, *which* $\{m_n | n \ge 1\}$ *is a sequence of positive integers. Assume that* $\{a_{ni} | 1 \le i \le m_n,\ n \ge 1\}$ *is an array of real numbers satisfying the following conditions;*

(a) $\max_{1\le i\le m_n} |a_{ni}| \log n = O((\log n)^{-1})$.

(b) $\sum_{i=1}^{m_n} a_{ni}^2 \ \log n = o((\log n)^{-1})$.

(c) $1/(|a_{nm_n}|) = O(\log n)$.

Then $\sum_{i=1}^{m_n} a_{ni}X_{ni} \to 0$ *a.s. if and only if* $\sum_{i=1}^{m_n} a_{ni}X_{ni} \to 0$ *completely.*

Proof: By using Theorem 2 and Theorem 3, we can obtain the result of Corollary 3. □

3. Central Limit Theorem

Theorem 4. *Let* $\{\xi_i | -\infty < i < \infty\}$ *be a LNQD sequence of random variables which* $E\xi_i = 0$ *is satisfying*

(a) $\sum_{j:|k-j|\ge u} |Cov(\xi_k, \xi_j)| \to 0$ *as* $u \to \infty$ *uniformly for* $k \ge 1$.

Assume that $\{a_{ni} | 1 \le i \le n,\ n \ge 1\}$ *is an array of real numbers such that*

(b) $\sup \sum_{i=1}^{n} a_{ni}^2 = O(1)$ *and* $\max_{1\le i\le n} |a_{ni}| \to 0$ *as* $n \to \infty$ *and* $Var(\sum_{i=1}^{n} a_{ni}\xi_i) = 1$.

If ξ_i *is uniformly integrable in* L_2, *then*

$$\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{D} N(0,1) \quad as\ \ n \to \infty.$$

Proof: By applying the Lemma 3, and using proof methods of Theorem 3.1 in Liang *et al.* (2004) and Theorem 4.2 in Billingsley (1968), we can get the result of Theorem 4. □

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