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Estimating the reliability and distribution of ratio in two independent variables with different distributions

Sang-Un Yun[1] · Chang-Soo Lee[2]

[1]Department of Finance and Quantitative Analysis, Georgia Southern University
[2]Department of Flight Operation, Kyungwoon University
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Abstract

We consider estimations for the reliability in two independent variables with Pareto and uniform or exponential distributions. And then we compare the mean squared errors of two reliability estimators for each case. We also observe the skewness of densities of the ratio for each case.

Keywords: E-function, exponential, hypergeometric function, Pareto, ratio, reliability, skewness, uniform.

1. Introduction

For two independent random variables X and Y, and a real number c, the probability $P(X < cY)$ is as given in Woo (2006) : (i) it is reliability when $c = 1$, (ii) it is distribution of ratio $X/(X + Y)$ when $c = t/(1 - t)$ for $0 < t < 1$.

The reliability will increase the need for the industry to perform systematic study for the identifications and reduction of causes of failures. These reliability studies must be performed by persons who can identify and quantify the modes of failures, know how to obtain and analyze the statistics of failure occurrences, and can construct mathematical models of the failure that depend on, for example, the parameters of material strength or design quality, fatigue or wear resistance, and the stochastic nature of the anticipated duty cycle in Saunders (2007).

Ali and Woo (2005) studied an inference on the reliability in a power function distribution, and Woo (2007) also studied the reliability in two independent half-triangle distributions. Moon and Lee (2009) studied an inference on the reliability $P(Y < X)$ in the Gamma case. Moon *et al.* (2009) studied inferences for the reliability and the ratio in an exponentiated complementary power function distribution. Lee and Lee (2010) studied inference on the reliability and the ratio in a right truncated Rayleigh distribution. Ali *et al.* (2010) studied estimations of $P(Y < X)$ when X and Y belong to different distribution families.

In this paper, we consider the estimation of the reliability in two independent random variables with Pareto and uniform or exponential distributions. And then we compare mean squared errors of two reliability estimators for each case. We also observe skewness of densities of ratio for each case.

[1] Visiting professor, Department of Finance and Quantitative Analysis, Georgia Southern University, Statesboro, GA 30460-8151, U.S.A.
[2] Corresponding author: Associate professor, Department of Flight Operation, Kyungwoon University, Gumi 730-850, Korea. E-mail: cslee@ikw.ac.kr

2. Pareto-uniform distributions

2.1. Estimating the reliability

Let X and Y be independent random variables where X has a Pareto density given by

$$f(x) = \alpha\beta^{\alpha}/x^{\alpha+1},\ x > \beta > 0,\ \alpha\ (>0) : \text{known} \tag{2.1}$$

and Y has a uniform density over $(0, \theta)$.

In this section, we consider the estimation of the reliability $P(Y < X)$. As an application of this case Y, representing the time to sustain a fixed level of the radioactivity, is a uniform random variable and X, representing the life time of a white rat which is exposed to the radioactivity, is a Pareto random variable.

Now, we consider the reliability $R(\rho) = P(Y < X)$ as following :

Proposition 2.1 Let X and Y be two independent random variables where X has a Pareto density (2.1) with $\beta > 0$ and known α ($\neq 1$) and Y has a uniform density over $(0, \theta)$, respectively for $\beta < \theta$. Then(a) The reliability is given by

$$R(\rho) = \frac{1}{1-\alpha}\rho^{\alpha} - \frac{\alpha}{1-\alpha}\rho, \quad \alpha \neq 1,$$

where $\rho = \beta/\theta$.
(b) $R(\rho)$ is a monotone function of ρ.

Proof. (a) For $\beta < \theta$, that is, $0 < \rho < 1$

$$\begin{aligned} R(\rho) = P(Y < X) &= \int_{\beta}^{\infty} P(Y < x) f_X(x)dx = \int_{\beta}^{\theta} \frac{x}{\theta} f_X(x)dx + \int_{\theta}^{\infty} 1 \cdot f_X(x)dx \\ &= \frac{1}{1-\alpha}(\beta/\theta)^{\alpha} - \frac{\alpha}{1-\alpha}\cdot(\beta/\theta) = \frac{1}{1-\alpha}\cdot\rho^{\alpha} - \frac{\alpha}{1-\alpha}\cdot\rho. \end{aligned}$$

(b) Since $\frac{d}{d\rho}R(\rho)\begin{cases} > 0, & \text{if } \alpha > 1 \\ < 0, & \text{if } 0 < \alpha < 1 \end{cases}$, $R(\rho)$ is a monotone function of $0 < \rho < 1$.
This completes the proof. □

Therefore, the inference on $R(\rho)$ is equivalent to that on ρ (McCool, 1991). And hence it is sufficient for us to consider estimation of ρ instead of estimating $R(\rho)$.

Assume $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be independent random samples from the Pareto density (2.1) having $\beta > 0$, known α ($\neq 1$) and a uniform density over $(0, \theta)$ for $\beta < \theta$, respectively. Then the estimators of β and θ are given as follows (Johnson *et al.*, 1994) :

$$\widehat{\beta} = X_{(1;n)} = X_{(1)} \text{ and } \widehat{\theta} = Y_{(m;m)} = Y_{(m)}. \tag{2.2}$$

From (2.2), the MLE of ρ is given by :

$$\widehat{\rho} = X_{(1)}/Y_{(m)}. \tag{2.3}$$

The densities of $X_{(1)}$ and $Y_{(m)}$ are given as in Rotatgi (1976) by

$$f_{X_{(1)}}(x) = n\alpha\beta^{n\alpha}x^{-n\alpha-1}, \quad x > \beta,$$
$$f_{Y_{(m)}}(x) = mx^{m-1}/\theta^{m}, \quad 0 < x < \theta. \tag{2.4}$$

From the densities in (2.4) and formula 3.381(4) in Gradshteyn and Ryzhik (1965), we obtain the followings :

$$E(\widehat{\rho}) = \frac{nm\alpha}{(n\alpha-1)(m-1)}\cdot\rho,$$
$$Var(\widehat{\rho}) = \left[\frac{nm\alpha}{(n\alpha-2)(m-2)} - \frac{n^2m^2\alpha^2}{(n\alpha-1)^2(m-1)^2}\right]\rho^2, \ m>2, \ n\alpha>2. \tag{2.5}$$

From the expectation in (2.5), an unbiased estimator $\tilde{\rho}$ of ρ can be obtained as :

$$\widetilde{\rho} = \frac{(n\alpha-1)(m-1)}{nm\alpha}\cdot\frac{X_{(1)}}{Y_{(m)}},$$

which has variance :

$$Var(\tilde{\rho}) = \left[\frac{(n\alpha-1)^2(m-1)^2}{nm\alpha(n\alpha-2)(m-2)} - 1\right]\rho^2, \ m>2, \ n\alpha>2 \tag{2.6}$$

From (2.5) and (2.6), we can calculate mean squared errors (MSEs) of the MLE $\widehat{\rho}$ and the unbiased estimator $\widetilde{\rho}$ as in Table 2.1.

Table 2.1 MSEs of the MLE $\widetilde{\rho}$ and an unbiased estimator $\widehat{\rho}$ (units: ρ^2)

n	m	ρ	α =1/4	α =1/2	α =2	α =4
10	10	$\widehat{\rho}$	3.54630	.30556	.20403	.18279
		$\widetilde{\rho}$	.82250	.08000	.12819	.12574
	20	$\widehat{\rho}$	3.04678	.22027	.08348	.07190
		$\widetilde{\rho}$	.80500	.06963	.05849	.05625
	30	$\widehat{\rho}$	2.90887	.19951	.05367	.04469
		$\widetilde{\rho}$	.80214	.06794	.03859	.03639
20	10	$\widehat{\rho}$	.30556	.09337	.18279	.17415
		$\widetilde{\rho}$	.08000	.02516	.12574	.12518
	20	$\widehat{\rho}$	.22027	.04971	.07190	.06767
		$\widetilde{\rho}$	.06963	.01531	.05625	.05572
	30	$\widehat{\rho}$	.19951	.04044	.04469	.04164
		$\widetilde{\rho}$	.06974	.01371	.03639	.03588
30	10	$\widehat{\rho}$	.14044	.06135	.17689	.17153
		$\widetilde{\rho}$	.03705	.01769	.12532	.12508
	20	$\widehat{\rho}$	.08600	.02641	.06897	.06646
		$\widetilde{\rho}$	.02709	.00792	.05586	.05563
	30	$\widehat{\rho}$	.07377	.01952	.04256	.04081
		$\widetilde{\rho}$	.02546	.00633	.03601	.03579

Because the inference on $R(\rho)$ is equivalent to that on ρ by McCool (1991), we obtain the following :

Fact 2.1 Assume $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be independent random samples from Pareto density (2.1) having $\beta > 0$ and known α ($\neq 1$) and a uniform density over $(0, \theta)$ for

$\beta < \theta$, respectively. Then for $\rho \equiv \beta/\theta <1$, an estimator $R(\widetilde{\rho})$ of the reliability $R(\rho) = P(Y < X)$ performs better than MLE $R(\widehat{\rho})$ in the sense of MSE.

Next we consider an interval estimation of $\rho = \beta/\theta <1$, only when the density (2.1) has $\beta > 0$ and known α ($\neq 1$).

The density of a pivot quantity $P \equiv \beta/\theta \cdot Y_{(m)}/X_{(1)}$ is derived as :

$$f_P(x) = \frac{mn\alpha}{m - n\alpha}(x^{n\alpha-1} - x^{m-1}), \quad 0 < x < 1, \quad m \neq n\alpha.$$

And then, for given $p_i \geq 0,\ i = 1, 2$ with $0 < 1 - p_1 - p_2 < 1$,

$$\left(l(p_1) \cdot \frac{X_{(1)}}{Y_{(m)}},\ u(p_2) \cdot \frac{X_{(1)}}{Y_{(m)}} \right)$$

is an $100(1 - p_1 - p_2)\%$ conservative confidence interval for $\rho = \beta/\theta$, where a lower limit $l(p_1)$ and an upper limit $u(p_2)$ satisfy

$$\int_0^{l(p_1)} f_P(t)dt = p_1 \text{ and } \int_{u(p_2)}^1 f_P(t)dt = p_2.$$

Remark 2.1 (Large-sample confidence interval)

Based on the MLE $\widehat{\rho} = X_{(1)}/Y_{(m)}$ of ρ and the variance of $\widehat{\rho}$ in (2.5), an approximate symmetric $100(1 - \gamma)\%$ confidence interval for ρ is given by :

$$\left(\widehat{\rho} - z_{\gamma/2}\sqrt{\widehat{var}(\widehat{\rho})},\ \widehat{\rho} + z_{\gamma/2}\sqrt{\widehat{var}(\widehat{\rho})} \right),$$

where $\widehat{Var}(\widehat{\rho}) = [nm\alpha/(n\alpha - 2)(m - 2) - n^2m^2\alpha^2/(n\alpha - 1)^2(m - 1)^2]\widehat{\rho}^2$.

2.2. Distribution of the ratio

In this section, we consider the ratio $R_Y = Y/(X + Y)$, when X has a Pareto density (2.1) with $\beta > 0$ and known α ($\neq 1$), and Y has a uniform random variable over $(0, \theta)$, respectively.

First, from the quotient density in Rohatgi (1976) and the integration, we can derive the quotient density $Q = X/Y$ as follows : For $\rho \equiv \beta/\theta$

$$f_Q(x) = \frac{\alpha}{\alpha - 1}\left[\rho x^{-2} - \rho^{\alpha} x^{-\alpha-1}\right], \quad \text{if } x > \rho. \tag{2.7}$$

From the quotient density (2.7), we can derive the density of the ratio R_Y as follows :

$$f_{R_Y}(r) = \frac{\alpha}{\alpha - 1}\left[\rho(1 - r)^{-2} - \rho^{\alpha} r^{\alpha-1}(1 - r)^{-\alpha-1}\right], \quad \text{if } 0 < r < (1 + \rho)^{-1}. \tag{2.8}$$

From (2.8) and formula 3.194(1) in Gradshteyn and Ryzhik (1965), we can obtain the k-th moments of R_Y as follows :

For $k = 1, 2, \cdots$,

$$E(R_Y^k) = \frac{\alpha}{\alpha - 1}\rho^{-k-2}\Big[\frac{1}{k + 1}{}_2F_1(k, k + 1; k + 2; -1/\rho) - \frac{1}{k + \alpha}{}_2F_1(k, k + \alpha; k + \alpha + 1; -1/\rho)\Big], \tag{2.9}$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function given in Gradshteyn and Ryzhik (1965).

From the k-th moments in (2.9) and recursion formula 15.2.13 and formula 15.1.8 of the hypergeometric function in Abramowitz and Stegun (1970), we can obtain mean, variance, and coefficients of the skewness of the density (2.7) as in Table 2.2.

Table 2.2 Mean, variance and coefficient of the skewness of the density (2.7) having $\alpha = 3$

ρ	mean	variance	skewness
1/8	.66477	.04112	-1.30625
1/6	.61037	.04188	-1.08848
1/4	.52793	.04010	-0.81648
1/2	.38203	.03009	0,01263
1	.25	.01694	0.02992
2	.14919	.00728	0.07935
4	.08292	.00253	0.21270

From Table 2.2, we observe the following trends :

Fact 2.3 When X and Y have independent Pareto density (2.1) with $\beta > 0$ and α ($\neq 1$), and a uniform density over $(0, \theta)$ for $\beta < \theta$, respectively. For $\alpha = 3$ and $\rho = \beta/\theta \leq 1/4$, the density of the ratio R_Y =Y/(X+Y) is left skewed, but elsewhere it is right skewed.

3. Pareto-exponential distributions

3.1. Estimating the reliability

Let X and Y be two independent random variables where X has a Pareto density (2.1) with known $\alpha > 0$ and Y has an exponential density with the mean $\sigma > 0$, respectively.

In this section, we consider the estimation of $P(Y < X)$. As an application of this case X, representing the time to repair a electronics when it broke down, is a Pareto random variable and Y, representing the life time of an used electronics, is exponential random variable.

Now, we consider the reliability $R(\eta) = P(Y < X)$ as following :

Proposition 3.1 Let X and Y be two independent random variables where X has a Pareto density (2.1) with known $\alpha > 0$ and Y has an exponential density with the mean $\sigma > 0$, respectively. Then (a) The reliability is given by

$$R(\eta) = 1 - \alpha\eta^{\alpha}\Gamma(-\alpha, \eta), \quad \eta \equiv \beta/\sigma, \tag{3.1}$$

where $\Gamma(a, x)$ is an incomplete gamma function.
(b) The reliability $R(\eta)$ is a monotone increasing function of η.

Proof. (a) $R(\eta) = P(Y < X) = \int_{\beta}^{\infty} P(Y < x) f_X(x)dx = \int_{\beta}^{\infty} (1 - e^{-x/\sigma}) f_X(x)dx$
$= 1 - \alpha(\beta/\sigma)^{\alpha}\Gamma(-\alpha, \beta/\sigma) = 1 - \alpha\eta^{\alpha}\Gamma(-\alpha, \eta), \eta \equiv \beta/\sigma.$
(b) Since $dR(\eta)/d\eta = -\alpha d\eta^{\alpha}\Gamma(-\alpha, \eta)/d\eta = \alpha\eta^{\alpha-1}\Gamma(1 - \alpha, \eta)$, we get $dR(\eta)/d\eta > 0$.
This completes the proof. □

Since $R(\eta)$ is a monotone function of η, the inference on $R(\eta)$ is equivalent to that on η (McCool, 1991). And hence it is sufficient for us to consider the estimation of η instead of estimating $R(\eta)$.

Assume $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ have independent random samples from the Pareto density (2.1) with known $\alpha > 0$ and an exponential density with the mean $\sigma > 0$, respectively.

Then the estimators of β and θ are given as follows (Johnson *et al.*, 1994) :

$$\widehat{\beta} = X_{(1;n)} = X_{(1)} \text{ and } \widehat{\sigma} = \sum_{i=1}^{m} Y_i/m. \tag{3.2}$$

From (3.2), the MLE of η is given by :

$$\widehat{\eta} = \widehat{\beta}/\widehat{\sigma} = X_{(1)}/\left(\sum_{i=1}^{m} Y_i/m\right). \tag{3.3}$$

It is well-known by Rohatgi (1976) that $\sum_{i=1}^{m} Y_i$ follows a gamma distribution with shape m and scale σ.

From the density of $X_{(1)}$ in (2.4), the distribution of $\sum_{i=1}^{m} Y_i$ and formula 3.381(4) in Gradshteyn and Ryzhik (1965), we obtain the followings :

$$E(\widehat{\eta}) = \frac{nm\alpha}{(n\alpha-1)(m-1)} \cdot \eta,$$
$$Var(\widehat{\eta}) = \left[\frac{nm^2\alpha}{(n\alpha-2)(m-1)(m-2)} - \frac{n^2m^2\alpha^2}{(n\alpha-1)^2(m-1)^2}\right]\eta^2, \;\; m>2, \;\; n\alpha > 2. \tag{3.4}$$

From the expectation in (3.4), an unbiased estimator $\widetilde{\eta}$ of η can be obtained as:

$$\widetilde{\eta} = \frac{(n\alpha-1)(m-1)}{n\alpha} \cdot X_{(1)}/\sum_{i=1}^{m} Y_i,$$

which has variance :

$$Var(\widetilde{\eta}) = [\frac{(n\alpha-1)^2(m-1)}{n\alpha(n\alpha-2)(m-2)} - 1]\eta^2, \; m>2, \; n\alpha>2. \tag{3.5}$$

From (3.4) and (3.5), we can calculate mean squared errors (MSEs) of the MLE $\widehat{\eta}$ and the unbiased estimator $\widetilde{\eta}$ as in Table 3.1

Because the inference on $R(\eta)$ is equivalent to that on η by McCool (1991), we obtain the following :

Fact 3.2 Assume $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be independent random samples from a Pareto density (2.1) with known $\alpha > 0$ and an exponential density with the mean $\sigma > 0$, respectively. Then for $\eta = \beta/\sigma$, an estimator $R(\widetilde{\eta})$ for the reliability $R(\eta) = P(Y < X)$ performs better than the MLE $R(\widehat{\eta})$ in the sense of MSE.

Table 3.1 MSEs of the MLE $\widehat{\eta}$ and an unbiased estimator $\widetilde{\eta}$ (units: η^2)

n	m	η	α =1/4	α =1/2	α =2	α =4
10	10	$\widehat{\eta}$	4.24074	.53704	.20403	.18279
		$\widetilde{\eta}$	1.02500	.20000	.12819	.12574
	20	$\widehat{\eta}$	3.33918	.31774	.08348	.07190
		$\widetilde{\eta}$	.90000	.12593	.05488	.05625
	30	$\widehat{\eta}$	3.09360	.26108	.05367	.04469
		$\widetilde{\eta}$	.86429	.10476	.03859	.03639
20	10	$\widehat{\eta}$	.53704	.26698	.18279	.17415
		$\widetilde{\eta}$	.20000	.13906	.12574	.12518
	20	$\widehat{\eta}$	.31774	.12281	.07190	.06767
		$\widetilde{\eta}$	.12593	.06875	.05625	.05573
	30	$\widehat{\eta}$	.26108	.08662	.04469	.04164
		$\widetilde{\eta}$	.10476	.04866	.03639	.03588
30	10	$\widehat{\eta}$	.32984	.22161	.17689	.17153
		$\widetilde{\eta}$	.15227	.13077	.12532	.12508
	20	$\widehat{\eta}$	.16575	.09389	.06898	.06646
		$\widetilde{\eta}$	.08114	.06097	.05586	.05563
	30	$\widehat{\eta}$	.12415	.06214	.04256	.04081
		$\widetilde{\eta}$	.06082	.04103	.03601	.03579

Next, we consider a confidence interval for $\eta = \beta/\sigma$ as follows :

From formula 3.381(3) in Gradshteyn and Ryzhik (1965), the density of a pivot quantity $T \equiv \eta \cdot \sum_{i=1}^{m} Y_i / X_{(1)}$ is obtained by the following :

$$f_T(x) = \frac{n\alpha}{\Gamma(m)} x^{n\alpha-1}\Gamma(m - n\alpha, x), \quad x > 0,$$

where $\int_0^\infty f_T(t)dt = 1$ is easily obtained from formula 13.39 in Oberhettinger (1974).

To find a lower and an upper limits of confidence interval for $\eta = \beta/\sigma$, we first obtain the following by changing the order of double integration.

Lemma 3.3 Let $f_T(x) = ax^{b-1} \cdot \Gamma(c, x), \quad x > 0, \ a > 0, \ b + c > 0$ and $c > 0$ be a density form, where $\Gamma(c, x) = \int_x^\infty e^{-t}t^{c-1}dt$. For given $0 < p < 1$, if there exists $U(p)$ satisfying $p = \int_{U(p)}^\infty f_T(t)dt$, then $U(p) = [(\Gamma(b + c) - pb/a)/\Gamma(c)]^{1/b}$.

From Lemma 3.3 and the density $f_T(x)$ of a pivot quantity T, an upper and a lower limits of the confidence interval for $\eta = \beta/\sigma$ can be obtained as :

$$U(p_2) = [\Gamma(m)(1 - p_2)/\Gamma(m - n\alpha)]^{1/n\alpha}, L(p_1) = [\Gamma(m)p_1/\Gamma(m - n\alpha)]^{1/n\alpha}, \quad \text{if } m > n\alpha.$$

And then, for given $p_i >= 0, \ i = 1, 2$, with $0 < 1 - p_1 - p_2 < 1$,

$$\left(L(p_1) \cdot X_{(1)} / \sum_{i=1}^{m} Y_i, \ U(p_2) \cdot X_{(1)} / \sum_{i=1}^{m} Y_i\right)$$

is an $100(1 - p_1 - p_2)\%$ conservative confidence interval for $\eta = \beta/\sigma$.

3.2. Distribution of the ratio

In this section, we consider the ratio $R_X = X/(X + Y)$, when X has a Pareto density (2.1) with known $\alpha > 0$ and Y has an exponential density with the mean $\sigma > 0$, respectively.

From the quotient density in Rohatgi (1976) and formula 3.1 in Oberhettinger and Badii (1973), we can derive the quotient density $V = Y/X$ as follows :

$$f_V(x) = \alpha\eta^\alpha x^{\alpha-1}\Gamma(1-\alpha,\ \eta x),\quad x>0,\ \eta=\beta/\sigma, \tag{3.6}$$

where $\Gamma(a,x)$ is an incomplete gamma function of x.

From the quotient density (3.6), we can derive the density function of the ratio R_X as follows :

$$f_{R_X}(x) = \alpha\eta^\alpha(1-r)^{\alpha-1}/r^{\alpha+1}\cdot\Gamma(1-\alpha,\ \eta\cdot(1-r)/r),\quad 0<r<1. \tag{3.7}$$

From the density (3.6) of the quotient V, the k-th moments of the ratio R_X is represented by MacRobert's E-function in Gradshteyn and Ryzhik (1965).

Proposition 3.4 For $k = 1, 2, \cdots$, k

$$E(R_X^k) = (\alpha/\Gamma(k))\cdot E(k,\alpha,3;\alpha+1;\eta)\text{ for }\eta=\beta/\sigma,$$

where $E(a, b, c; d; x)$ is MacRobert's E-function.

Proof. $E(R_X^k) = \alpha(\beta/\sigma)^\alpha\int_0^\infty(1+v)^{-k}v^{\alpha-1}dv\int_{\beta v/\sigma}^\infty e^{-t}t^{-\alpha}dt$
$= \alpha(\beta/\sigma)^\alpha\int_{t=0}^\infty e^{-t}t^{-\alpha}dt\int_o^{\sigma t/\beta}v^{\alpha-1}(1+v)^{-k}dv$
$= \int_0^\infty t^2e^{-t}{}_2F_1(k,\alpha;\alpha+;-\sigma t/\beta)dt$
$= (\beta/\sigma)^3\int_0^\infty x^{3-1}e^{-\beta x/\sigma}{}_2F_1(k,\alpha;\alpha+1;-x)dx$
$= (\alpha/\Gamma(k))E(k,\alpha,3;\alpha+1;\beta/\sigma).$
This completes the proof. □

From the density function of the ratio R_X, the k-th moment of the ratio R_X is represented by the following double integral (3.9)

$$E(R_X^k) = \alpha\eta^\alpha\int_{r=0}^1\int_{t=0}^{r/((1-r)\eta)} r^{k-\alpha-1}(1-r)^{\alpha-1}t^{\alpha-2}e^{-1/t}dtdr. \tag{3.8}$$

From approximate computations of double integral (3.9) of the k-th moments for the ratio R_X, approximate means, variances and coefficients of the skewness for $f_{R_X}(x)$ in (3.7) can be obtained as in Table 3.2.

Table 3.2 Approximate means, variances and coefficients of skewness of the density (3.7)

α	η	mean	variance	skewness
	1/4	.611338	.09039	-0.35672
	1/2	.70459	.07223	-0.81466
1/2	1	.77914	.06047	-1.53796
	2	.82990	.05912	-2.32454
	4	.85388	.07060	-2.61068
	1/4	.39507	.05892	0.70027
	1/2	.52536	.05658	-0.02147
3	1	.65613	.04413	-0.24592
	2	.77082	.02783	-0.75850
	4	.85799	.01527	-1.76913

From Table 3.2, we observe the following trends :

Fact 3.5 Let X and Y be two independent random variables where X has a Pareto density (2.1) with known $\alpha > 0$ and Y has an exponential density with the mean $\sigma > 0$, respectively. Then for $\eta = \beta/\sigma$, when $(\alpha, \eta) = (3, 1/4)$, the density of the ratio $R_X = X/(X+Y)$ is right skewed, but elsewhere it is left skewed.

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