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INTERPOLATION PROBLEMS FOR OPERATORS WITH CORANK IN ALG$\mathcal{L}$

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Abstract. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. And let X and Y be operators acting on a Hilbert space $\mathcal{H}$. Let $sp(x) = \{\alpha x : \alpha \in \mathbb{C}\}$ for any $x \in \mathcal{H}$. Assume that $\mathcal{H} = \overline{range\ X} \oplus sp(h)$ for some $h \in \mathcal{H}$ and $< h, E^{\perp}Xf >= 0$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$. Then there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ if and only if

$\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty$. Moreover, if the necessary condition holds, then we may choose an operator A such that $AX = Y$ and $\|A\| = K$.

1. Introduction

On the process of solving operator equation $AX = Y$ for two given operators X and Y in the algebra $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on a Hilbert space $\mathcal{H}$, many mathematicians have applied the problem on their fields. What is a condition for the operator A to be a member of $\mathcal{A}$ which is a specified subalgebra of $\mathcal{B}(\mathcal{H})$? The subalgebras in this problem were given in various forms and accordingly the solution to the problem has been different.

Douglas[2] used the range inclusion property of operators to show necessary and sufficient conditions for the existence of an operator A satisfying $AX = Y$. Kadison[10] has done research on C*-algebras, Lance[12] on nest-algebras, Hopenwasser[3] on CSL-algebras, Munch for Hilbert-Schmidt operators on nest-algebras, and Hopenwasser[4] for

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Hilbert-Schmidt operators on CSL-algebras, Moore and Trent[13] on CSL-algebra Alg$\mathcal{L}$.

Authors[6] obtained a necessary and sufficient condition that there exists an interpolation operator A in Alg$\mathcal{L}$ when every E in $\mathcal{L}$ reduces A. And authors[7] showed that the necessary and sufficient condition on [13] is satisfied in Alg$\mathcal{L}$ when $\mathcal{L}$ is a subspace lattice. Again authors[9] proved that the condition is a condition for interpolating operator when $PE = EP$ for each E in $\mathcal{L}$ where P is the projection onto the $\overline{rangeX}$. In this paper author investigate an interpolation problem for operators with corank-one in Alg$\mathcal{L}$.

Let $\mathcal{H}$ be a Hilbert space. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$ containing the trivial projections 0 and I. The symbol Alg$\mathcal{L}$ denotes the algebra of bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathcal{L}$; Alg$\mathcal{L}$ is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. Let $x_1, \cdots, x_n$ be vectors of $\mathcal{H}$. Then $sp(\{x_1, \cdots, x_n\}) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \mid \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}\ \}$. Let M be a subset of $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{C}$ be the set of complex numbers.

2. The Equation $AX = Y$ in Alg$\mathcal{L}$

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Then Alg$\mathcal{L}$ is the algebra of all bounded linear operators acting on $\mathcal{H}$ which leave invariant each projection E in $\mathcal{L}$. Assume that X and Y are operators in $\mathcal{B}(\mathcal{H})$ and A is an operator in Alg$\mathcal{L}$ such that $AX = Y$. Then $\|E^{\perp}Yf\| = \|E^{\perp}AXf\| = \|E^{\perp}AE^{\perp}Xf\| \leq \|A\|\|E^{\perp}Xf\|$, for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequality above may be stated in the form

$$\sup_{E \in \mathcal{L}} \frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} \leq \|A\|.$$

Theorem A [R. G. Douglas][2]. *Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) rangeY^* $\subseteq$ range X^*

(2) $Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$

(3) there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.
Moreover, if (1), (2), and (3) are valid, then there exists a unique operator A so that
(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$
(b) $kerY^* = kerA^*$ and
(c) $rangeA^* \subseteq rangeX^-$.

Theorem 2.1. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. And let X and Y be operators acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{H} = \overline{range\ X} \oplus sp(h)$ for some $h \in \mathcal{H}$. If $< h, E^{\perp}Xf >= 0$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*
(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.

(2) $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Proof. Assume that $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H}, E \in \mathcal{L}\right\} = K < \infty$. Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^{\perp}X) = E^{\perp}Y$ and $\|A_E\| \leq K$ by Theorem A. In particular, if $E = 0$, then we have an operator A_0 in $\mathcal{B}(\mathcal{H})$ such that $A_0X = Y$ and $\|A_0\| \leq K$. So $A_E(E^{\perp}X) = E^{\perp}Y = E^{\perp}A_0X$. Hence $A_EE^{\perp} = E^{\perp}A_0$ on $\overline{range\ X}$ for each E in $\mathcal{L}$. Since $< h, E^{\perp}Xf >= 0 =< E^{\perp}h, E^{\perp}Xf >$ for any f in $\mathcal{H}$, $E^{\perp}h \in \overline{range\ E^{\perp}X}^{\perp}$. By the definitions of A_E and A_0, $A_EE^{\perp}h = 0$ and $A_0h = 0$. So $A_EE^{\perp}x = E^{\perp}A_0x$ for x in $\overline{range\ X}^{\perp}(= sp(h))$. Therefore $A_EE^{\perp} = E^{\perp}A_0$ on $\mathcal{H}$.

For each E in $\mathcal{L}$,

$$E^{\perp}A_0E^{\perp} = A_EE^{\perp}E^{\perp} = A_EE^{\perp} = E^{\perp}A_0\ .$$

So A_0 is an operator in Alg$\mathcal{L}$. □

Theorem 2.2. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. And let X and Y be operators acting on a Hilbert space $\mathcal{H}$. Let n be a natural number($n \geq 2$) and let $\{h_1, \cdots, h_n\}$ be an orthonormal set of vectors in $\mathcal{H}$ such that $\mathcal{H} = \overline{range\ X} \oplus sp(\{h_1, \cdots, h_n\})$. If $< h_i, E^{\perp}Xf >= 0(i = 1, \cdots, n)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*
(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.

(2) $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Proof. Assume that $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H}, E \in \mathcal{L}\right\} = K < \infty$. Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^{\perp}X) = E^{\perp}Y$ and $\|A_E\| \leq K$ by Theorem A. In particular, if $E = 0$, then we have an operator A_0 in $\mathcal{B}(\mathcal{H})$ such that $A_0X = Y$ and $\|A_0\| \leq K$. So $A_E(E^{\perp}X) = E^{\perp}Y = E^{\perp}A_0X$. Hence $A_EE^{\perp} = E^{\perp}A_0$ on $\overline{range\ X}$ for each E in $\mathcal{L}$. Since $< h_i, E^{\perp}Xf >= 0 =< E^{\perp}h_i, E^{\perp}Xf > (i = 1, \cdots, n)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, $E^{\perp}h_i \in \overline{range\ E^{\perp}X}^{\perp}$ for each $i = 1, 2, cdots, n$. By the definitions of A_E and A_0, $A_EE^{\perp}h_i = 0$ and $A_0h_i = 0$ for each $i = 1, 2, \cdots, n$. Hence $A_EE^{\perp} = E^{\perp}A_0$ on $\overline{range\ X}^{\perp}(= sp(\{h_1, \cdots, h_n\})$. Therefore $A_EE^{\perp} = E^{\perp}A_0$ on $\mathcal{H}$.

For each E in $\mathcal{L}$,

$$E^{\perp}A_0E^{\perp} = A_EE^{\perp}E^{\perp} = A_EE^{\perp} = E^{\perp}A_0 .$$

So A_0 is an operator in Alg$\mathcal{L}$. □

We can generalize the above theorem for the countable case.

Theorem 2.3. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. And let X and Y be operators acting on a Hilbert space $\mathcal{H}$. Let $\{h_1, h_2, \cdots\}$ be an orthonormal set of vectors h_i in $\mathcal{H}$ such that $\mathcal{H} = \overline{range\ X} \oplus \overline{sp(\{h_1, h_2, \cdots\})}$. If $< h_i, E^{\perp}Xf >= 0 (i = 1, 2, \cdots)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.

(2) $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Corollary 2.4. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. And let X and Y be operators acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{B}$ be a basis of $\overline{range\ X}^{\perp}$. If $< h, E^{\perp}Xf >= 0$ for each $h \in \mathcal{B}$, $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.

(2) $\sup\left\{\frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Then Alg$\mathcal{L}$ is the algebra of all bounded linear operators acting on $\mathcal{H}$ which leave invariant each projection E in $\mathcal{L}$. Assume that $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are operators in $\mathcal{B}(\mathcal{H})$ and A is an operator in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for each $i = 1, \cdots, n$. Then $E^{\perp}Y_if_i = E^{\perp}AX_if_i = E^{\perp}AE^{\perp}X_if_i$ for each $i = 1, \cdots, n$ and $E \in \mathcal{L}$. Hence

$$\begin{aligned}\|\sum_{i=1}^{n} E^{\perp}Y_if_i\| &= \|\sum_{i=1}^{n} E^{\perp}AX_if_i\| \\ &= \|\sum_{i=1}^{n} E^{\perp}AE^{\perp}X_if_i\| \\ &\leq \|A\|\|\sum_{i=1}^{n} E^{\perp}X_if_i\|\end{aligned}$$

for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequality above may be stated in the form

$$\sup_{E\in\mathcal{L}} \frac{\|\sum_{i=1}^{n} E^{\perp}Y_if_i\|}{\|\sum_{i=1}^{n} E^{\perp}X_if_i\|} \leq \|A\|.$$

Theorem 2.5. *Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be bounded operators acting on $\mathcal{H}$. Let $\mathcal{H} = \overline{range\ X_k} \oplus sp(h)$ for some k in $\{1, \cdots, n\}$ and some $h \in \mathcal{H}$. If $< h, E^{\perp}X_if >= 0 (i = 1, \cdots, n)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_if_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_if_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Proof. Assume that $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty$. Let E be in $\mathcal{L}$ and

$$\mathcal{M}_E = \left\{\sum_{i=1}^{n} E^{\perp} X_i f_i \ :\ f_i \in \mathcal{H}\right\}.$$

Define $A_E : \mathcal{M}_E \to \mathcal{H}$ by $A_E(\sum_{i=1}^{n} E^{\perp} X_i f_i) = \sum_{i=1}^{n} E^{\perp} Y_i f_i$. Then A_E is well-defined and bounded linear. Extend A_E on $\overline{\mathcal{M}_E}$ continuously. Define $A_E f = 0$ for each $f \in \mathcal{M}_E{}^{\perp}$. Then $A_E : \mathcal{H} \to \mathcal{H}$ is a bounded linear and $A_E E^{\perp} X_i = E^{\perp} Y_i$ for each $i = 1, \cdots, n$. If $E = 0$, then $A_0 X_i = Y_i$ for $i = 1, \cdots, n$. Hence $A_E(E^{\perp} X_i) = E^{\perp} Y_i = E^{\perp} A_0 X_i$ for each $i = 1, \cdots, n$. We will show that $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{H}$. Since $A_E(E^{\perp} X_k) = E^{\perp} Y_k = E^{\perp}(A_0 X_k)$, $A_E E^{\perp} = E^{\perp} A_0$ on $\overline{\mathit{range}\ X_k}$ for each E in $\mathcal{L}$. Since $< h, E^{\perp} X_i f >= 0 (i = 1, \cdots, n)$ for any f in $\mathcal{H}$, $< h, \sum_{i=1}^{n} E^{\perp} X_i f_i >= 0$. Hence $E^{\perp} h \in \overline{\mathcal{M}_E}^{\perp}$. So $A_E E^{\perp} h = 0$ and $A_0 h = 0$. Hence $A_E E^{\perp} = E^{\perp} A_0$ on $sp(h)$. Therefore $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{H}$.

For each E in $\mathcal{L}$,

$$E^{\perp} A_0 E^{\perp} = A_E E^{\perp} E^{\perp} = A_E E^{\perp} = E^{\perp} A_0 .$$

So A_0 is an operator in Alg$\mathcal{L}$ and $A_0 X_i = Y_i (i = 1, \cdots, n)$. □

Theorem 2.6. *Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be bounded operators acting on $\mathcal{H}$. Let m be a natural number($m \geq 2$) and let $\{h_1, \cdots, h_m\}$ be an orthonormal set of vectors h_j in $\mathcal{H}$ such that $\mathcal{H} = \overline{\mathit{range}\ X_k} \oplus sp(\{h_1, \cdots, h_m\})$ for some k in $\{1, 2, \cdots, n\}$. If $< h_j, E^{\perp} X_k f >= 0 (i = 1, \cdots, n,\ j = 1, \cdots, m)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Proof. Assume that $\sup\left\{\dfrac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K <$ ∞. Let E be in $\mathcal{L}$ and

$$\mathcal{M}_E = \left\{\sum_{i=1}^{n} E^{\perp} X_i f_i \mid f_i \in \mathcal{H}\right\}.$$

Define $A_E : \mathcal{M}_E \to \mathcal{H}$ by $A_E(\sum_{i=1}^{n} E^{\perp} X_i f_i) = \sum_{i=1}^{n} E^{\perp} Y_i f_i$ and $A_E f = 0$ for all $f \in \mathcal{M}_E{}^{\perp}$. Then A_E is well-defined and bounded linear. Extend A_E on $\overline{\mathcal{M}_E}$ continuously. Define $A_E f = 0$ for each $f \in \mathcal{M}_E{}^{\perp}$. Then $A_E : \mathcal{H} \to \mathcal{H}$ is a bounded linear and $A_E E^{\perp} X_i = E^{\perp} Y_i$ for each $i = 1, \cdots, n$. If $E = 0$, then $A_0 X_i = Y_i$ for $i = 1, \cdots, n$. Hence $A_E(E^{\perp} X_i) = E^{\perp} Y_i = E^{\perp} A_0 X_i$ for each $i = 1, \cdots, n$. We will show that $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{H}$. Since $A_E(E^{\perp} X_k) = E^{\perp} Y_k = E^{\perp}(A_0 X_k)$, $A_E E^{\perp} = E^{\perp} A_0$ on $\overline{range\ X_k}$ for each E in $\mathcal{L}$. Since $< h_j, E^{\perp} X_i f >= 0 (i = 1, \cdots, n,\ j = 1, \cdots, m)$ for any f in $\mathcal{H}$, $< h_j, \sum_{i=1}^{n} E^{\perp} X_i f >= 0$. Hence $E^{\perp} h_j \in \mathcal{M}_E{}^{\perp}$ for $j = 1, \cdots, m$. By the definition of A_E and A_0, $A_E E^{\perp} h_j = 0$ and $A_0 h_j = 0$ for each $j = 1, \cdots, m$. Hence $A_E E^{\perp} = E^{\perp} A_0$ on $\overline{range\ X_k}^{\perp} (= sp(\{h_1, \cdots, h_m\})\ )$. Therefore $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{H}$.

For each E in $\mathcal{L}$,

$$E^{\perp} A_0 E^{\perp} = A_E E^{\perp} E^{\perp} = A_E E^{\perp} = E^{\perp} A_0 .$$

So A_0 is an operator in Alg$\mathcal{L}$ and $A_0 X_i = Y_i (i = 1, \cdots, n)$. □

Theorem 2.7. *Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be bounded operators acting on $\mathcal{H}$. Let $\{h_1, h_2, \cdots\}$ be an orthonormal set of vectors h_j in $\mathcal{H}$ such that $\mathcal{H} = \overline{range\ X_k} \oplus \overline{sp(\{h_1, h_2, \cdots\})}$ for some k in $\{1, 2, \cdots, n\}$. If $< h_j, E^{\perp} X_i f >= 0 (i = 1, \cdots, n,\ j = 1, 2, \cdots)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup\left\{\dfrac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Corollary 2.8. *Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be bounded operators acting on $\mathcal{H}$. Let $\mathcal{B}$ be a basis of $\overline{range\ X_k}^{\perp}$ for some k in*

$\{1, 2, \cdots, n\}$. If $< h, E^{\perp}X_i f >= 0 (i = 1, \cdots, n)$ for each $h \in \mathcal{B}$, $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup\left\{\dfrac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

We can generalize above Theorems to the countable case easily.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Then Alg$\mathcal{L}$ is the algebra of all bounded linear operators acting on $\mathcal{H}$ which leave invariant each projection E in $\mathcal{L}$. Assume that $\{X_i\}$ and $\{Y_i\}$ are operators in $\mathcal{B}(\mathcal{H})$ and A is an operator in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for each $i = 1, 2, \cdots$. Then $E^{\perp}Y_i f_i = E^{\perp}AX_i f_i = E^{\perp}AE^{\perp}X_i f_i$ for each $i = 1, 2, \cdots$ and $E \in \mathcal{L}$. Hence

$$\begin{aligned}\|\sum_{i=1}^{n} E^{\perp}Y_i f_i\| &= \|\sum_{i=1}^{n} E^{\perp}AX_i f_i\| \\ &= \|\sum_{i=1}^{n} E^{\perp}AE^{\perp}X_i f_i\| \\ &\leq \|A\|\|\sum_{i=1}^{n} E^{\perp}X_i f_i\|\end{aligned}$$

for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequality above may be stated in the form

$$\sup_{E\in\mathcal{L}} \frac{\|\sum_{i=1}^{n} E^{\perp}Y_i f_i\|}{\|\sum_{i=1}^{n} E^{\perp}X_i f_i\|} \leq \|A\|.$$

Theorem 2.9. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. Let $\mathcal{H} = \overline{range\ X_k} \oplus sp(h)$ for some k in $\{1, \cdots, n\}$ and some $h \in \mathcal{H}$. If $< h, E^{\perp}X_k f >= 0$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

(2) $\sup\left\{\dfrac{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L},\ m \in \mathbb{N}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Theorem 2.10. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. Let m be a natural number($m \geq 2$) and let $\{h_1, \cdots, h_m\}$ be an orthonormal set of vectors h_j in $\mathcal{H}$ such that $\mathcal{H} = \overline{range\ X_k} \oplus sp(\{h_1, \cdots, h_m\})$ for some k in $\{1, 2, \cdots, n\}$. If $< h_j, E^{\perp} X_i f >= 0 (i = 1, \cdots,\ j = 1, \cdots, m)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

(2) $\sup\left\{\dfrac{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L},\ m \in \mathbb{N}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Theorem 2.11. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. Let $\{h_1, h_2, \cdots\}$ be an orthonormal set of vectors h_j in $\mathcal{H}$ such that $\mathcal{H} = \overline{range\ X_k} \oplus \overline{sp(\{h_1, h_2, \cdots\})}$ for some k in $\{1, 2, \cdots, n\}$. If $< h_j, E^{\perp} X_i f >= 0 (i = 1, \cdots,\ j = 1, 2, \cdots)$ for each $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

(2) $\sup\left\{\dfrac{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L},\ m \in \mathbb{N}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Corollary 2.12. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for all*
$i = 1, 2, \cdots$. Let $\mathcal{B}$ be a basis of $\overline{range\ X_k}^{\perp}$ for some k in $\{1, 2, \cdots, n\}$. If $< h, E^{\perp} X_i f >= 0 (i = 1, \cdots)$ for each $h \in \mathcal{B}$, $f \in \mathcal{H}$ and $E \in \mathcal{L}$, then the following are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

(2) $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L},\ m \in \mathbb{N}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

3. The Equation $Ax = y$ in Alg$\mathcal{L}$

Let x and y be vectors in $\mathcal{H}$ and A be an operator in Alg$\mathcal{L}$ such that $Ax = y$. Then $\|E^{\perp}y\| = \|E^{\perp}Ax\| = \|E^{\perp}AE^{\perp}x\| \leq \|A\|\|E^{\perp}x\|$ for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the above inequality may be stated in the form

$$\sup_{E\in\mathcal{L}} \frac{\|E^{\perp}y\|}{\|E^{\perp}x\|} \leq \|A\|.$$

We consider the above fact when $\mathcal{L}$ is a subspace lattice without the commutative condition.

Let x, y and g be non-zero vectors in $\mathcal{H}$. Let $X = x\otimes g$ and $Y = y\otimes g$. Then we can obtain the following by Theorem 2.1 and Corollary 2.4.

Theorem 3.1. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let x and y be vectors in $\mathcal{H}$. If $< h, E^{\perp}x >= 0$ for each $h \in sp(x)^{\perp}$ and $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$.

(2) $\sup\left\{\frac{\|E^{\perp}y\|}{\|E^{\perp}x\|} :\ E \in \mathcal{L}\right\} = K_0 < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K_0$.

Proof. Assume that $\left\{\dfrac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} = K_0 < \infty$. Let g be non-zero vectors in $\mathcal{H}$ and $X = x \otimes g$ and $Y = y \otimes g$. Then

$$\begin{aligned}
\|E^{\perp}Yf\| &= \|E^{\perp}(y \otimes g)f\| \\
&= \|E^{\perp} < f, g > y\| \\
&= \| < f, g > E^{\perp}y\| \text{ and} \\
\|E^{\perp}Xf\| &= \|E^{\perp}(x \otimes g)f\| \\
&= \|E^{\perp} < f, g > x\| \\
&= \| < f, g > E^{\perp}x\| \ .
\end{aligned}$$

Hence $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H} \text{ and } E \in \mathcal{L}\right\} = \sup\left\{\dfrac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\}$.
Since
$\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H} \text{ and } E \in \mathcal{L}\right\} < \infty$, there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ by Theorem 2.1. Since $AX = A(x \otimes g) = (Ax) \otimes g = y \otimes g$, $Ax = y$. $\square$

Let $x_i, y_i (i = 1, \cdots, n)$ and g be non-zero vectors in $\mathcal{H}$. Let $X = x_i \otimes g$ and $Y = y_i \otimes g$. Then the next theorem is obtained by modifying the proof used in Theorem 2.5 and Corollary 2.8.

Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be vectors in $\mathcal{H}$ and A be an operator in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, \cdots, n)$. Then $E^{\perp}\alpha_i y_i = E^{\perp}\alpha_i Ax_i = \alpha_i E^{\perp}AE^{\perp}x_i = E^{\perp}AE^{\perp}\alpha_i x_i$ for all $E \in \mathcal{L}$. Hence

$$\begin{aligned}
\|\sum_{i=1}^{n} E^{\perp}\alpha_i y_i\| &= \|\sum_{i=1}^{n} E^{\perp}\alpha_i Ax_i\| \\
&= \|\sum_{i=1}^{n} E^{\perp}AE^{\perp}\alpha_i x_i\| \\
&\leq \|A\|\|\sum_{i=1}^{n} E^{\perp}\alpha_i x_i\|
\end{aligned}$$

for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the above inequality may be stated in the form

$$\sup_{E\in\mathcal{L}} \frac{\|\sum_{i=1}^n E^\perp \alpha_i y_i\|}{\|\sum_{i=1}^n E^\perp \alpha_i x\|} \le \|A\|.$$

Theorem 3.2. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be vectors in $\mathcal{H}$. If $< h, E^\perp x_i >= 0 (i = 1, \cdots, n)$ for each $h \in sp(x_k)^\perp$, $E \in \mathcal{L}$ and for some k in $\{1, 2, \cdots, n\}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup \left\{ \dfrac{\|E^\perp \sum_{i=1}^n \alpha_i y_i\|}{\|E^\perp \sum_{i=1}^n \alpha_i x_i\|} : \ E \in \mathcal{L}, \ \alpha_i \in \mathbb{C} \right\} = K_0 < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K_0$.

Proof. Assume that $\sup \left\{ \dfrac{\|E^\perp \sum_{i=1}^n \alpha_i y_i\|}{\|E^\perp \sum_{i=1}^n \alpha_i x_i\|} : \ E \in \mathcal{L}, \ \alpha_i \in \mathbb{C} \right\} = K_0 < \infty$. Let g be a non-zero vector in $\mathcal{H}$ and $X_i = x_i \otimes g$ and $Y_i = y_i \otimes g$ for $i = 1, \cdots, n$. Then

$$\begin{aligned} \|E^\perp(\sum_{i=1}^n Y_i f_i)\| &= \|\sum_{i=1}^n E^\perp (y_i \otimes g) f_i\| \\ &= \|\sum_{i=1}^n E^\perp < f_i, g > y_i\| \\ &= \|E^\perp \sum_{i=1}^n < f_i, g > y_i\| \text{ and} \\ \|E^\perp(\sum_{i=1}^n X_i f_i)\| &= \|\sum_{i=1}^n E^\perp (x_i \otimes g) f_i\| \\ &= \|\sum_{i=1}^n E^\perp < f_i, g > x_i\| \\ &= \|E^\perp \sum_{i=1}^n < f_i, g > x_i\| \ . \end{aligned}$$

Hence $\dfrac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} = \dfrac{\|\sum_{i=1}^n < f_i, g > E^\perp y\|}{\|\sum_{i=1}^n < f_i, g > E^\perp x\|}$ for each $E \in \mathcal{L}$. Since $\sup\left\{\dfrac{\|E^\perp \sum_{i=1}^n \alpha_i y_i\|}{\|E^\perp \sum_{i=1}^n \alpha_i x_i\|} : E \in \mathcal{L}, \ \alpha_i \in \mathbb{C}\right\} = K_0 < \infty$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i (i = 1, \cdots, n)$ by Theorem 2.5. Since $AX_i = A(x_i \otimes g) = (Ax_i) \otimes g = y_i \otimes g$, $y_i = Ax_i$ for each $i = 1, \cdots, n$. $\square$

We can extend Theorem 3.2 to countably infinite vectors and get the following theorem from Theorem 2.9 and Corollary 2.12.

Theorem 3.3. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let x_i and y_i be vectors in $\mathcal{H}$ for $i \in \mathbb{N}$. If $< h, E^\perp x_i >= 0$ for each $h \in sp(x_k)^\perp$, $E \in \mathcal{L}$ and for some k in $\{1, 2, \cdots\}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots$.

(2) $\sup\left\{\dfrac{\|E^\perp \sum_{i=1}^n \alpha_i y_i\|}{\|E^\perp \sum_{i=1}^n \alpha_i x_i\|} : E \in \mathcal{L}, \ \alpha_i \in \mathbb{C}, n \in \mathbb{N}\right\} = K_0 < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K_0$.

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