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Interval-valued Fuzzy Quasi-ideals in a Semigroups

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Abstract

We initiate the study of interval-valued fuzzy quasi-ideal of a semigroup. In Section 2, we list some basic definitions in the later sections. In Section 3, we investigate interval-valued fuzzy subsemigroups and in Section 4, we define interval-valued fuzzy quasi-ideals and establish some of their basic properties. In Section 5, we obtain characterizations of regular and intraregular semigroups using the machinery developed in the preceding sections.

Key Words: interval-valued fuzzy set, interval-valued fuzzy left(right) ideal, interval-valued fuzzy bi-ideal, interval-valued fuzzy quasi-ideal, regular semigroup, intraregular semigroup

1. Introduction

The theory of fuzzy sets proposed by Zadeh [11] in 1965 has achieved a great success in various fields. Since then, Ahsan and Latif [1] investigated fuzzy quasi-ideals in a semigroup. With the research of fuzzy sets, in 1975, Zadeh [12] introduced the notion of interval-valued fuzzy sets as a generalization of fuzzy sets. After then, Biswas [3] applied it to group theory, Mondal and Samanto [9] to topology. Recently, Kang and Hur [6] studied interval-valued fuzzy subgroups and subrings and Choi et al [5] introduced the concept of interval-valued smooth topological spaces and investigated some of its properties. In particular, Cheong and Hur [4] studied interval-valued fuzzy generalized bi-ideals of a semigroup, and Lee et al [8] investigated interval-valued fuzzy ideals and bi-ideals in a semigroup.

In this paper, we initiate the study of interval-valued fuzzy quasi-ideal of a semigroup. In Section 2, we list some basic definitions in the later sections. In Section 3, we investigate interval-valued fuzzy subsemigroups and in Section 4, we define interval-valued fuzzy quasi-ideals and establish some of their basic properties. In Section 5, we obtain characterizations of regular and intraregular semigroups using the machinery developed in the preceding sections.

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2. Preliminaries

We will list some concepts and one result needed in the later sections.
Throughout this paper, we will denote the unit interval [0,1] as I and for an ordinary subset of a set X, we will denote the characteristic function of A as χ_A.
Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0,1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\tilde{0} = [0,0]$, $\tilde{1} = [1,1]$, and $\mathbf{a}$=$[a,a]$ for every $a \in (0,1)$. We also note that
(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,
(ii) $(\forall M, N \in D(I))$ $(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.
For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$.

Definition 2.1. [12]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, IVS) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$ [resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point* of x

to A. For any $[a, b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively. We will denote the set of all IVSs in X as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2. [9, 12]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(a) $A \subset B$ iff $A^L \le B^L$ and $A^U \le B^U$.
(b) $A = B$ iff $A \subset B$ and $B \subset A$.
(c) $A^c = [1 - A^U, 1 - A^L]$.
(d) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.
$(d)'$ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.
(e) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.
$(e)'$ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

3. Interval-valued fuzzy subsemigroups

Definition 3.1. [6]. Let $(X, \cdot)$ be a groupoid and let $A, B \in D(I)^X$. Then the *interval-valued fuzzy product* of A and B, denoted by $A \circ B$, is an IVS in X defined as follows : For each $x \in X$,

$$(A \circ B)(x) = \begin{cases} [a, b], & \text{if } yz = x; \\ [0, 0], & \text{otherwise.} \end{cases}$$

where $a = \bigvee_{yz=x} (A^L(y) \wedge B^L(z))$, $b = \bigvee_{yz=x} (A^U(y) \wedge B^U(z))$. It is clear that for any $A, B, C \in D(I)^X$, if $B \subset C$, then $A \circ B \subset A \circ C$ and $B \circ A \subset C \circ A$.

Result 3.A. [6, *Proposition* 3.3]. Let $(S, \cdot)$ be a groupoid.
(a) If " $\cdot$ " is associative[resp. commutative], then so is $\circ$ in $D(I)^S$.
(b) If " $\cdot$ " has an identity $e \in S$, then $e_1 \in IVF_p(X)$ is an identity of $\circ$ in $D(I)^S$.

Proposition 3.2. Let S be a groupoid and let $A, B, C \in D(I)^S$. Then

(a) $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$, $(B \cup C) \circ A = (B \circ A) \cup (C \circ A)$.
(b) $A \circ (B \cap C) \subset (A \circ B) \cap (A \circ C)$, $(B \cap C) \circ A \subset (B \circ A) \cap (C \circ A)$.

Proof. (a) Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $(A \circ (B \cup C))(x) = \tilde{0} = ((A \circ B) \cup (A \circ C))(x)$. Suppose x is expressible as $x = yz$. Then

$$\begin{aligned}
(A \circ (B \cup C))^L(x) &= \bigvee_{x=yz} (A^L(y) \wedge (B \cup C)^L(z) \\
&= \bigvee_{x=yz} (A^L(y) \wedge (B^L(z) \vee C^L(z)) \\
&= \bigvee_{x=yz} ((A^L(y) \wedge B^L(z)) \vee (A^L(y) \wedge C^L(z))) \\
&= \bigvee_{x=yz} (A^L(y) \wedge B^L(z)) \vee \bigvee_{x=yz} (A^L(y) \wedge C^L(z)) \\
&= (A \circ B)^L(x) \vee (A \circ C)^L(x) \\
&= ((A \circ B) \cup (A \circ C))^L(x).
\end{aligned}$$

Thus $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$. By the similar arguments, we have $(B \cup C) \circ A = (B \circ A) \cup (C \circ A)$.

(b) Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $(A \circ (B \cap C))(x) = \tilde{0} = ((A \circ B) \cap (A \circ C))(x)$. Suppose x is expressible as $x = yz$. Then

$$\begin{aligned}
(A \circ (B \cap C))^L(x) &= \bigvee_{x=yz} (A^L(y) \wedge (B \cap C)^L(z)) \\
&= \bigvee_{x=yz} (A^L(y) \wedge (B^L(z) \wedge C^L(z))) \\
&= \bigvee_{x=yz} ((A^L(y) \wedge B^L(z)) \wedge (A^L(y) \wedge C^L(z))) \\
&\le \bigvee_{x=yz} (A^L(y) \wedge B^L(z)) \wedge \bigvee_{x=yz} (A^L(y) \wedge C^L(z)) \\
&= (A \circ B)^L(x) \wedge (A \circ C)^L(x) \\
&= ((A \circ B) \cap (A \circ C))^L(x).
\end{aligned}$$

Similarly, we have that $(A \circ (B \cap C))^U(x) \le ((A \circ B) \cap (A \circ C))^U(x)$. Thus $A \circ (B \cap C) \subset (A \circ B) \cap (A \circ C)$. By the similar arguments, we have $(B \cap C) \circ A \subset (B \circ A) \cap (C \circ A)$. This completes the proof. □

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset of A such that $A^2 \subset A$ and by a *left* [resp. *right*] *ideal* of S we mean a non-empty subset A of S such that $SA \subset A$ [resp. $AS \subset A$]. By *two-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. We will denote the set of all left ideals[resp. right ideals and ideals] of S as LI(S)[resp. RI(S) and I(S)].

Definition 3.3. [8]. Let S be a semigroup and let $\widetilde{0} \ne A \in D(I)^S$. Then A is called an:

(i) *interval-valued fuzzy semigroup*(in short, $IVSG$) of S if $A^L(xy) \ge A^L(x) \wedge A^L(y)$ and $A^U(xy) \ge A^U(x) \wedge A^U(y)$ for any $x, y \in S$.
(ii) *interval-valued fuzzy left ideal*(in short, $IVLI$) of S if $A^L(xy) \ge A^L(y)$ and $A^U(xy) \ge A^U(y)$ for any $x, y \in S$.
(iii) *interval-valued fuzzy right ideal*(in short, $IVRI$) of S if $A^L(xy) \ge A^L(x)$ and $A^U(xy) \ge A^U(x)$ for any $x, y \in S$.
(iv) *interval-valued fuzzy*$($*two-sided*$)$ *ideal* (in short, IVI) of S if it is both an IVLI and an IVRI of S.

We will denote the set of all IVSGs [resp.IVLIs, IVRIs and IVIs] of S as IVSG(S) [resp.IVLI(S), IVRI(S) and IVI(S)]. It is clear that $A \in$ IVI(S) if and only if $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in S$, and if $A \in$ IVLI(S)[resp. IVRI(S) and IVI(S)], then $A \in$ IVSG(S).

The following is the immediate result of Definitions 3.1 and 3.3(i).

Proposition 3.4. Let S be a semigroup and let $\tilde{0} \neq A \in D(I)^S$. Then $A \in$ IVSG(S) if and only if $A \circ A \subset A$.

Result 3.B. $[3, Proposition\ 3.4]$. Let A be a non-empty subset of a semigroup S.
(a) A is a subsemigroup of S if and only if $[\chi_A, \chi_A] \in$ IVSG(S).
(b) $A \in$ LI(S)[resp. RI(S) and I(S)] if and only if $[\chi_A, \chi_A] \in$ IVLI(S) [resp. IVRI(S) and IVI(S)].

Result 3.C. $[4, Lemmas\ 2.3\ and\ 2.4]$. Let S be a semigroup and let $\tilde{0} \neq A \in D(I)^S$. Then $A \in$ IVLI(S)[resp. IVRI(S)] if and only if $\tilde{1} \circ A \subset A$[resp. $A \circ \tilde{1} \subset A$].

Proposition 3.5. Let S be a semigroup and let $A, B, C \in D(I)^S$. If $A \subset B$, then $A \circ C \subset B \circ C$ and $C \circ A \subset C \circ B$.

Proof. Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $(A \circ C)(x) = \tilde{0} = (B \circ C)(x)$. Suppose x is expressible as $x = yz$. Then

$$\begin{aligned} (A \circ C)^L(x) &= \bigvee_{x=yz} (A^L(y) \wedge C^L(z)) \\ &\leq \bigvee_{x=yz} (B^L(y) \wedge C^L(z)) \\ &\quad \text{(Since } A \subset B) \\ &= (B \circ C)^L(x). \end{aligned}$$

Similarly, we have that $(A \circ C)^U(x) \leq (B \circ C)^U(x)$. Hence $A \circ C \subset B \circ C$. By the similar arguments, we have $C \circ A \subset C \circ B$. This completes the proof. □

Proposition 3.6. Let S be a semigroup and $\tilde{0} \neq A \in D(I)^S$. Then $\tilde{1} \circ A \in$ IVLI(S)[resp. $A \circ \tilde{1} \in$ IVRI(S)].

Proof. $\tilde{1} \circ (\tilde{1} \circ A) = (\tilde{1} \circ \tilde{1}) \circ A \subset \tilde{1} \circ A$, by Results 3.A and Proposition 3.5, respectively. Hence, by Result 3.C, $\tilde{1} \circ A \in$ IVLI(S). Similarly, we can see that $A \circ \tilde{1} \in$ IVRI(S). This completes the proof. □

Proposition 3.7. Let S be a semigroup and $\tilde{0} \neq A \in D(I)^S$. Then $A \cup (\tilde{1} \circ A) \in$ IVLI(S)[resp. $A \cup (A \circ \tilde{1}) \in$ IVRI(S)].

Proof.

$$\begin{aligned} \tilde{1} \ \circ \ & (A \cup (\tilde{1} \circ A)) \\ &= (\tilde{1} \circ A) \cup (\tilde{1} \circ (\tilde{1} \circ A)) \\ &\quad \text{(By Proposition 3.2 } (a)) \\ &= (\tilde{1} \circ A) \cup (\tilde{1} \circ \tilde{1} \circ A) \subset (\tilde{1} \circ A) \cup (\tilde{1} \circ A) \\ &\quad \text{(By Result 3.A)} \\ &= \tilde{1} \circ A \subset A \cup (\tilde{1} \circ A). \end{aligned}$$

Hence, by Result 3. C, $A \cup (\tilde{1} \circ A) \in$ IVLI(S). By the similar arguments, we can see that $A \cup (A \circ \tilde{1}) \in$ IVRI(S). This completes the proof. □

Proposition 3.8. Let S be a semigroup and let $\tilde{0} \neq A \in D(I)^S$. If $A \in$ IVRI(S)[resp. IVLI(S)], then $A \cup (\tilde{1} \circ A) \in$ IVLI(S)[resp. $A \cup (A \circ \tilde{1})$ is an IVI of S.

Proof. Suppose $A \in$ IVRI(S).

$$\begin{aligned} (A \cup (\tilde{1} \circ A)) \ \circ \ & \tilde{1} \\ &= (A \circ \tilde{1}) \cup ((\tilde{1} \circ A)) \circ \tilde{1} \\ &\quad \text{(By Proposition 3.2 } (a)) \\ &= (A \circ \tilde{1}) \cup (\tilde{1} \circ (A \circ \tilde{1})) \\ &\quad \text{(By Result 3.A)} \\ &\subset (A \cup (\tilde{1} \circ A) \\ &\quad \text{(By Result 3.C and Proposition 3.5)} \end{aligned}$$

Thus, by Result 3.C, $A \cup (\tilde{1} \circ A) \in$ IVRI(S). From Proposition 3.7, it is clear that $(A \cup (\tilde{1} \circ A)) \in$ IVLI(S). So $(A \cup (\tilde{1} \circ A)) \in$ IVI(S). Similarly, we can see that if $A \in$ IVLI(S), then $(A \cup (A \circ \tilde{1})) \in$ IVI(S). This completes the proof. □

4. Interval-valued fuzzy quasi-ideals

A nonempty subset A of a semigroup S is called a *quasi-ideal* of S(See [10]) if $AS \cap SA \subset A$. We will denote the set of all quasi-ideals of S as QI(S).

Definition 4.1. Let S be a semigroup and let $\tilde{0} \neq A \in D(I)^S$. Then A is called an *interval-valued fuzzy quasi-ideal*(in short, *IVQI*) of S if $(\tilde{1} \circ A) \cap (A \circ \tilde{1}) \subset A$.

We will denote the set of all IVQIs of S as IVQI(S).

Example 4.1. Let $S = \{a, b, c\}$ be any semigroup with the following multiplication table:
We define a mapping $A : S \to D(I)$ as follows:
$A(a) = [0.1, 0.8], A(b) = [0.1, 0.8], A(c) = [0.3, 0.6]$.
Then we can see that $A \in$ IVQI(S). □

Theorem 4.2. Let A be a nonempty subset of a semigroup S. Then $A \in$ QI(S) if and only if $[\chi_A, \chi_A] \in$ IVQI(S).

	a	b	c
a	a	a	a
b	a	b	b
c	a	a	b

Proof. ($\Rightarrow$): Suppose $A \in$ QI(S) and let $x \in S$. Suppose $x \in A$. Then clearly

$$\chi_A(x) = 1 \geq ((\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}))^L(x).$$

Thus $(\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}) \subset [\chi_A, \chi_A]$. Suppose $x \notin A$. Then either x is expressible as $x = yz$ or not.

Case (i): Suppose x is not expressible as $x = yz$. Then

$$((\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}))(x) = \tilde{0} = [\chi_A, \chi_A](x).$$

Case (ii): Suppose x is expressible as $x = yz$. Since $x \notin A$, either $y \in A$ or $z \notin A$. If $y \in A$ and $z \notin A$, then there cannot be another expression of the form $x = ab$, where $a \notin A$ and $b \in A$ (Assume that there exist $a \notin A$ and $b \in A$ such that $x = ab$. Then $x \in SA \cap AS \subset A$. Thus $x \in A$. This contradicts the fact that $x \notin A$). Thus either $(\tilde{1} \circ [\chi_A, \chi_A])(x) = \tilde{0}$ or $([\chi_A, \chi_A] \circ \tilde{1})(x) = \tilde{0}$. So $(\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}))(x) = \tilde{0}$. Then $(\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}) \subset [\chi_A, \chi_A]$. Hence, in all, $[\chi_A, \chi_A] \in$ IVQI(S).

($\Leftarrow$): Suppose the necessary condition holds. Let $x \in SA \cap AS$. Then $x \in SA$ and $x \in AS$. Thus there exist $a, a' \in A$ and $s, s' \in S$ such that $x = sa$ and $x = a's'$. So

$$\begin{aligned}
&((\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}))^L(x) \\
&\quad = (\tilde{1} \circ [\chi_A, \chi_A])^L(x) \wedge ([\chi_A, \chi_A] \circ \tilde{1})^L(x) \\
&\quad = \bigvee_{x=yz} (\tilde{1}^L(y) \wedge \chi_A(z)) \wedge \bigvee_{x=pq} (\chi_A(p) \wedge \chi_S(q)) \\
&\quad \geq (\chi_S(s) \wedge \chi_A(a)) \wedge (\chi_A(a') \wedge \tilde{1}^L(s')) \\
&\qquad\qquad (\text{Since } x = sa \text{ and } x = a's') \\
&\quad = 1.
\end{aligned}$$

Similarly, we have that $((\tilde{1} \circ [\chi_A, \chi_A]) \cap ([\chi_A, \chi_A] \circ \tilde{1}))^U(x) \geq 1$. Then, by the hypothesis, $\chi_A(x) \geq 1$. Thus $x \in A$. So $SA \cap AS \subset A$. Hence $A \in$ QI(S). This completes the proof. □

Definition 4.3. [1] A nonempty fuzzy set A of a semigroup S is called a *fuzzy quasi-ideal* of S if $(\chi_S \circ A) \wedge (A \circ \chi_S) \leq A$, where χ_S is the whole fuzzy set defined by $\chi_S(x) = 1$ for each $x \in S$.

Remark 4.3 Let S be a semigroup.
(a) If A is a fuzzy quasi-ideal of S, then $[A, A] \in$ IVQI(S).
(b) If $A \in$ IVQI(S), then A^L and A^U are fuzzy quasi-ideals of S.

Proposition 4.4. Let S be a semigroup. Then IVQI(S) $\subset$ IVSG(S).

Proof. Let $A \in$ IVQI(S). Since $A \subset \tilde{1}$, by Proposition 3.5, $A \circ A \subset \tilde{1} \circ A$ and $A \circ A \subset A \circ \tilde{1}$. Then $A \circ A \subset [\tilde{1} \circ A] \cap [A \circ \tilde{1}]$. Since $A \in$ IVQI(S), $(\tilde{1} \circ A) \cap (A \circ \tilde{1}) \subset A$. Thus $A \circ A \subset A$. Hence, by Proposition 3.4, $A \in$ IVSG(S). □

Proposition 4.5. Let S be a semigroup. Then IVLI(S) $\subset$ IVQI(S) and IVRI(S) $\subset$ IVQI(S).

Proof. Let $A \in$ IVLI(S). Then, by Result 3.C, $\tilde{1} \circ A \subset A$. Thus $(\tilde{1} \circ A) \cap [A \circ \tilde{1}] \subset \tilde{1} \circ A \subset A$. Hence $A \in$ IVQI(S). Similarly, we can see that IVRI(S) $\subset$ IVQI(S). □

Proposition 4.6. Let S be a semigroup, let $A \in$ IVLI(S) and let $B \in$ IVRI(S). Then $A \cap B \in$ IVQI(S).

Proof. Let $A \in$ IVLI(S) and let $B \in$ IVRI(S). Then, by Result 3.C, $A \circ \tilde{1} \subset A$ and $\tilde{1} \circ B \subset B$. Thus

$$\begin{aligned}
&(\tilde{1} \circ (A \cap B)) \cap ((A \cap B) \circ \tilde{1}) \\
&\quad \subset ((\tilde{1} \circ A) \cap (\tilde{1} \circ B)) \cap ((A \circ \tilde{1}) \cap (B \circ \tilde{1})) \\
&\qquad\qquad (\text{By Proposition 3.2 (b), 3.5 and Result 3.C}) \\
&\quad \subset ((\tilde{1} \circ A) \cap B) \cap (A \circ \cap (B \circ \tilde{1})) \\
&\quad = ((\tilde{1} \circ A) \cap (B \circ \tilde{1})) \cap (A \cap B) \\
&\quad \subset A \cap B.
\end{aligned}$$

Hence $A \cap B \in$ IVQI(S). □

The following is the immediate result of Propositions 3.5 and 4.6.

Corollary 4.6 Let S be a semigroup and let $\tilde{0} \neq A \in D(I)^S$. Then $(A \cup (\tilde{1} \circ A)) \cap (A \cup (A \circ \tilde{1})) \in$ IVQI(S).

Proposition 4.7. Let S be a semigroup and let $A \in$ IVQI(S). Then

$$A = (A \cup (\tilde{1} \circ A)) \cap (A \cup (A \circ \tilde{1})).$$

Proof. It is clear that $(A \cup (\tilde{1} \circ A)) \in$ IVLI(S) and $(A \cup (A \circ \tilde{1})) \in$ IVRI(S), from Proposition 3.7. Also from Proposition 4.6, it is clear that $(A \cup (\tilde{1} \circ A)) \cap (A \cup (A \circ \tilde{1})) \in$ IVQI(S). Then it is sufficient to show that the equality holds.

$$\begin{aligned}
A &\subset (A \cup (\tilde{1} \circ A)) \cap (A \cup (A \circ \tilde{1})) \\
&\qquad (\text{Since } A \subset A \cup (\tilde{1} \circ A) \text{ and } A \subset A \cup (A \circ \tilde{1})) \\
&= ((A \cup (\tilde{1} \circ A)) \cap A) \cup ((A \cup (\tilde{1} \circ A)) \cap (A \circ \tilde{1})) \\
&\qquad (\text{By Proposition 3.2}) \\
&\subset A \cup ((A \cup (\tilde{1} \circ A)) \cap (A \circ \tilde{1})) \\
&\qquad (\text{Since } (A \cup (\tilde{1} \circ A)) \cap A \subset A) \\
&= A \cup (A \cap (A \circ \tilde{1})) \cup ((\tilde{1} \circ A) \cap (A \circ \tilde{1})) \\
&\qquad (\text{By Proposition 3.2}) \\
&\subset A \cup (A \cap (A \circ \tilde{1})) \cup A \\
&\qquad (\text{Since } (\tilde{1} \circ A) \cap (A \circ \tilde{1}) \subset A)
\end{aligned}$$

$$\begin{aligned}&\subset A\cup A\cup A\\&\qquad \text{(Since } A\cap(A\circ\tilde{1})\subset A)\\&= A.\end{aligned}$$

Hence, the equality holds. □

The following is the immediate result of Propositions 4.6 and 4.7.

Theorem 4.8. Let S be a semigroup and let $\tilde{0}\neq A\in D(I)^S$. Then $A\in$ IVQI(S) if and only if there exist $B\in$ IVRI(S) and $C\in$ IVLI(S) such that $A=B\cap C$.

Proposition 4.9. Let S be a semigroup and let $\{A_\alpha\}_{\alpha\in\Gamma}\subset$ IVQI(S). Then either $\bigcap_{\alpha\in\Gamma}A_\alpha=\tilde{0}$ or $\bigcap_{\alpha\in\Gamma}A_\alpha\in$ IVQI(S).

Proof. Let $\{A_\alpha\}_{\alpha\in\Gamma}\subset$ IVQI(S) and let $A=\bigcap_{\alpha\in\Gamma}A_\alpha$. Suppose $A\neq\tilde{0}$. Then

$$\begin{aligned}(\tilde{1}\circ A)\cap(A\circ\tilde{1})&=(\tilde{1}\circ\bigcap_{\alpha\in\Gamma}A_\alpha)\cap(\bigcap_{\alpha\in\Gamma}A_\alpha\circ\tilde{1})\\&\subset\bigcap_{\alpha\in\Gamma}(\tilde{1}\circ A_\alpha)\cap(\bigcap_{\alpha\in\Gamma}(A_\alpha\circ\tilde{1}))\\&\subset\bigcap_{\alpha\in\Gamma}((\tilde{1}\circ A_\alpha)\cap(A_\alpha\circ\tilde{1}))\\&\subset\bigcap_{\alpha\in\Gamma}A_\alpha\\&=A.\end{aligned}$$

Hence $A=\bigcap_{\alpha\in\Gamma}A_\alpha\in$ IVQI(S). □

A nonempty subset A of a semigroup S is called a *bi-ideal* [7] of S if $A^2\subset A$ and $ASA\subset A$. We will denote the set of all bi-ideals of S as BI(S).

Definition 4.10. [8]. Let S be a semigroup and let $\tilde{0}\neq A\in D(I)^S$. Then A is called an *interval-valued fuzzy bi-ideal* (in short, $IVBI$) of S if it satisfies the following conditions: for any $x,y,z\in S$.

(a) $A^L(xy)\geq A^L(x)\wedge A^L(y)$ and $A^U(xy)\geq A^U(x)\wedge A^U(y)$

(b) $A^L(xyz)\geq A^L(x)\wedge A^L(z)$ and $A^U(xyz)\geq A^U(x)\wedge A^U(z)$.

We will denote the set of all IVBIs of S as IVBI(S).

Result 4.A. [8, $Theorem$ 2.8]. Let A be a nonempty subset of a semigroup. Then $A\in$ BI(S) if and only if $[\chi_A,\chi_A]\in$ IVBI(S).

Theorem 4.11. Let S be a semigroup and let $\tilde{0}\neq A\in D(I)^S$. Then $A\in$ IVBI(S) if and only if $A\circ A\subset A$ and $A\circ\tilde{1}\circ A\subset A$.

Proof. $(\Rightarrow)$: Suppose $A\in$ IVBI(S). From Proposition 3.4, $A\circ A\subset A$. Let $x\in S$. Suppose x is not expressible as $x=yz$. Then clearly $(A\circ\tilde{1}\circ A)(x)=\tilde{0}$. Thus $A\circ\tilde{1}\circ A\subset A$. Suppose x is expressible as $x=yz$. Then $(A\circ\tilde{1}\circ A)(x)\neq\tilde{0}$. Thus

$$(A\circ\tilde{1}\circ A)^L(x)=\bigvee_{x=yz}(A^L(y)\wedge(\tilde{1}\circ A)^L(z))>0$$

and

$$(A\circ\tilde{1}\circ A)^U(x)=\bigvee_{x=yz}(A^U(y)\wedge(\tilde{1}\circ A)^U(z))>0.$$

So $(\tilde{1}\circ A)^L(z)>0$ and $(\tilde{1}\circ A)^U(z)>0$. Then there exist $p,q\in S$ with $z=pq$ such that

$$(\tilde{1}\circ A)^L(z)=\bigvee_{z=pq}(\tilde{1}^L(p)\wedge A^L(q))=\bigvee_{z=pq}A^L(q)$$

and

$$(\tilde{1}\circ A)^U(z)=\bigvee_{z=pq}(\tilde{1}^U(p)\wedge A^U(q))=\bigvee_{z=pq}A^U(q).$$

Since $A\in$ IVBI(S),

$$A^L(x)=A^L(ypq)\geq A^L(y)\wedge A^L(q)$$

and

$$A^U(x)=A^U(ypq)\geq A^U(y)\wedge A^U(q).$$

Then

$$A^L(x)\geq\bigvee_{x=yz}(A^L(y)\wedge(\bigvee_{z=pq}A^L(q)))=(A\circ\tilde{1}\circ A)^L(x)$$

and

$$A^U(x)\leq\bigvee_{x=yz}(A^U(y)\wedge(\bigvee_{z=pq}A^U(q)))=(A\circ\tilde{1}\circ A)^U(x).$$

Hence, in all, $A\circ\tilde{1}\circ A\subset A$.

$(\Leftarrow)$: Suppose the necessary condition holds. Since $A\circ A\subset A$, it is clear that the following hold: $A^L(xy)\geq A^L(x)\wedge A^L(y)$ and $A^U(xy)\geq A^U(x)\wedge A^U(y)$ for any $x,y\in S$. Let $x,y,z\in S$ and let $p=xyz$. Then

$$\begin{aligned}A^L(xyz)&=A^L(p)\geq(A\circ\tilde{1}\circ A)^L(p)\\&\qquad\text{(By the hypothesis)}\\&=\bigvee_{p=st}(A^L(s)\wedge(\tilde{1}\circ A)^L(t))\\&\geq A^L(x)\wedge(\tilde{1}\circ A)^L(yz)\quad\text{(Since } p=x(yz))\\&=A^L(x)\wedge(\bigvee_{yz=ab}(\tilde{1}^L(a)\wedge A^L(b)))\\&\geq A^L(x)\wedge\tilde{1}^L(y)\wedge A^L(z)\\&=A^L(x)\wedge A^L(z).\end{aligned}$$

Similarly, we have that $A^U(xyz)\geq A^U(x)\wedge A^U(z)$. Hence, $A\in$ IVBI(S). This completes the proof. □

Proposition 4.12. Let S be a semigroup. Then IVQI(S) $\subset$ IVBI(S).

Proof. Let $A \in$ IVQI(S). Then, by Proposition 4.4, $A \in$ IVSG(S). Thus $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in S$. So, by Proposition 3.4, $A \circ A \subset A$. It is clear that $A \circ \tilde{1} \subset \tilde{1}$ and $\tilde{1} \circ A \subset \tilde{1}$. Then, by Proposition 3.5, $A \circ \tilde{1} \circ A \subset \tilde{1} \circ A$ and $A \circ \tilde{1} \circ A \subset A \circ \tilde{1}$. Thus $A \circ \tilde{1} \circ A \subset [\tilde{1} \circ A] \cap [A \circ \tilde{1}] \subset A$ (Since $A \in$ IVQI(S)). Hence, by Theorem 4.11, $A \in$ IVBI(S). □

The converse inclusion of Proposition 4.12 is not generally true.

Example 4.2. Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

We define a mapping $A : S \to D(I)$ as follows:

$$A(a) = [1, 1], A(b) = [0.3, 0.6], A(c) = [0.2, 0.7]$$

and

$$A(d) = [0.5, 0.5].$$

Then we can see that $A \notin$ IVQI(S) but $A \in$ IVBI(S). □

The product of two quasi-ideals need not be a quasi-ideal. So the interval-valued fuzzy product of two IVQIs need not be an IVQI.

Proposition 4.13. Let S be a semigroup, let $A \in$ IVQI(S) and let $\tilde{0} \neq B \in D(I)^S$. Then $A \circ B, B \circ A \in$ IVBI(S).

Proof. Let $A \in$ IVQI(S) and let $\tilde{0} \neq B \in D(I)^S$. Thus, by Proposition 4.12, $A \in$ IVBI(S). Then, by Theorem 4.11, $A \circ \tilde{1} \circ A \subset A$ and $A \circ A \subset A$. So

$$\begin{aligned}(A \circ B) \circ (A \circ B) &\subset (A \circ \tilde{1}) \circ (A \circ B) \\ &\quad (\text{ Since } A \circ B \subset A \circ \tilde{1}) \\ &= (A \circ \tilde{1} \circ A) \circ B \\ &\quad (\text{ By Result 4.A}) \\ &\subset A \circ B. \\ &\quad (\text{Since } A \circ \tilde{1} \circ A \subset A)\end{aligned}$$

On the other hand,

$$\begin{aligned}&(A \circ B) \circ \tilde{1} \circ (A \circ B) \\ &\subset (A \circ \tilde{1}) \circ \tilde{1} \circ (A \circ B) \\ &\quad (\text{Since } A \circ B \subset A \circ \tilde{1}) \\ &= A \circ (\tilde{1} \circ \tilde{1}) \circ (A \circ B) \\ &\quad (\text{By Result 3.A}) \\ &\subset A \circ \tilde{1} \circ (A \circ B) \\ &= (A \circ \tilde{1} \circ A) \circ B \\ &\quad (\text{By Result 3.A}) \\ &\subset A \circ B. \quad (\text{Since } A \circ \tilde{1} \circ A \subset A)\end{aligned}$$

Hence, by Theorem 4.11, $A \circ B \in$ IVBI(S). Similarly, we can see that $B \circ A \in$ IVBI(S). This completes the proof. □

The following is the immediate result of Proposition 4.13.

Corollary 4.13 Let S be a semigroup and let $A, B \in$ IVQI(S). Then $A \circ B \in$ IVBI(S).

5. Regular semigroups

A semigroup S is said to be *regular* if for each $a \in S$ there exists $x \in S$ such that $a = axa$.

Theorem 5.1. Let S be a semigroup. Then the following are equivalent :

(a) S is regular.

(b) For each $A \in$ IVRI(S) and each $B \in$ IVLI(S), $A \circ B = A \cap B$.

(c) For each $A \in$ IVRI(S) and each $B \in$ IVLI(S),

(1) $A^2 = A \circ A = A$.

(2) $B^2 = B \circ B = B$.

(3) $A \circ B \in$ IVQI(S).

(d) (IVQI(S), $\circ$) is a regular semigroup.

(e) Every IVQI A of S has the form $A = A \circ \tilde{1} \circ A$.

Proof. (a)⇒(b) : Suppose S is regular. Let $A \in$IVRI(S) and let $B \in$IVLI(S). Then, by Result 3.C, $A \circ B \subset A \circ \tilde{1} \subset A$ and $A \circ B \subset \tilde{1} \circ B \subset B$. Thus $A \circ B \subset A \cap B$.

Now let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then

$$\begin{aligned}(A \circ B)^L(a) &= \bigvee_{a=yz} [A^L(y) \wedge B^L(z)] \geq A^L(ax) \wedge B^L(a) \\ &\quad (\text{Since } a = axa) \\ &\geq A^L(a) \wedge B^L(a) \ (\text{Since } A \in \text{IVRI(S)}) \\ &= (A \cap B)^L(a)\end{aligned}$$

Similarly, we have that

$$(A \circ B)^U(a) \geq (A \cap B)^U(a)$$

Thus $A \circ B \supset A \cap B$. Hence $A \circ B = A \cap B$.

(b)⇒(c) : Suppose the condition (b) holds.

(1) Let $A \in$ IVRI(S). Then, by Proposition 3.8, $A \cup (\tilde{1} \circ A) \in$ IVI(S). By the hypothesis,

$$\begin{aligned} A &= A \cap (A \cup (\tilde{1} \circ A)) = A \circ (A \cup (\tilde{1} \circ A)) \\ &= (A \circ A) \cup (A \circ (\tilde{1} \circ A)) \\ &\qquad \text{(By Proposition 3.2 (a))} \\ &= (A \circ A) \cup ((A \circ \tilde{1}) \circ A) \quad \text{(By Result 3.A)} \\ &\subset (A \circ A) \cup (A \circ A) \quad \text{(Since } A \in \text{IVRI(S))} \\ &= (A \circ A). \end{aligned}$$

So $A \subset A \circ A$. On the other hand, $A \circ A \subset A \circ \tilde{1} \subset A$. Hence $A \circ A = A$.

(2) Let $B \in$ IVLI(S). Then, by the similar arguments of the proof of (1), we can see that $B \circ B = B$.

(3) Let $A \in$ IVRI(S) and let $B \in$ IVLI(S). Then, by the hypothesis, $A \circ B = A \cap B$. By Proposition 4.6, $A \cap B \in$ IVQI(S). Hence $A \circ B \in$ IVQI(S).

(c)⇒(d): Suppose the condition (c) holds. Let $A \in$ IVQI(S). Then, by Proposition 3.7, $A \cup (\tilde{1} \circ A) \in$ IVLI(S). Thus

$$\begin{aligned} A &\subset A \cup (\tilde{1} \circ A) \\ &= (A \cup (\tilde{1} \circ A)) \circ (A \cup (\tilde{1} \circ A)) \\ &\qquad \text{(By the condition (c) (2))} \\ &= ((A \cup (\tilde{1} \circ A) \circ A) \cup ((A \cup (\tilde{1} \circ A)) \circ (\tilde{1} \circ A)) \\ &\qquad \text{(By Proposition 3.2(a))} \\ &= ((A \circ A) \cup \{(\tilde{1} \circ A) \circ A\}) \cup ((A \circ (\tilde{1} \circ A)) \cup (\tilde{1} \circ A)^2 \\ &\qquad \text{(By Proposition 3.2(a))} \\ &= ((A \circ A) \cup \{\tilde{1} \circ (A \circ A)\}) \cup ((A \circ (\tilde{1} \circ A)) \cup (\tilde{1} \circ A)^2) \\ &\qquad \text{(By Result 3. A)} \\ &\subset ((\tilde{1} \circ A) \cup (\tilde{1} \circ A)) \cup ((\tilde{1} \circ (\tilde{1} \circ A)) \cup (\tilde{1} \circ A)^2 \\ &\qquad\qquad \text{(Since } A \circ A \subset A \text{ and } A \subset \tilde{1}) \\ &= ((\tilde{1} \circ A) \cup (\tilde{1} \circ A)) \cup ((\tilde{1} \circ (\tilde{1} \circ A)) \cup (\tilde{1} \circ A) \\ &\qquad\qquad \text{(By the condition (c) (2))} \\ &\subset (\tilde{1} \circ A) \cup (\tilde{1} \circ A) \cup (\tilde{1} \circ A) \\ &\qquad \text{(Since } \tilde{1} \circ A \in \text{IVLI(S))} \\ &= \tilde{1} \circ A. \end{aligned}$$

So $A \subset \tilde{1} \circ A$. By the similar arguments, we can see that $A \subset A \circ \tilde{1}$. Then $A \subset (\tilde{1} \circ A) \cap (A \circ \tilde{1})$. Since $A \in$ IVQI(S), $(\tilde{1} \circ A) \cap (A \circ \tilde{1}) \subset A$. So

$$A = (\tilde{1} \circ A) \cap (A \circ \tilde{1}). \tag{5.1}$$

Let $C \in$ IVRI(S) and let $D \in$ IVLI(S). Then, by the condition (c)(3), $C \circ D \in$ IVQI(S). Thus, by (5.1),

$$C \circ D = (\tilde{1} \circ (C \circ D)) \cap ((C \circ D) \circ \tilde{1}). \tag{5.2}$$

Now let $A, B \in$ IVQI(S). Then, by Proposition 3.6, $\tilde{1} \circ A \circ B \in$ IVLI(S) and $A \circ B \circ \tilde{1} \in$ IVRI(S). By Proposition 4.5, $\tilde{1} \circ A \circ B$, $A \circ B \circ \tilde{1} \in$ IVQI(S). Thus

$$\begin{aligned} \tilde{1} \circ A \circ B &= (\tilde{1} \circ (\tilde{1} \circ A \circ B)) \cap ((\tilde{1} \circ A \circ B) \circ \tilde{1}) \\ &\qquad \text{(By (5.1))} \\ &= (\tilde{1} \circ A \circ B) \circ (\tilde{1} \circ A \circ B) \\ &\qquad \text{(By (5.2))} \\ &= ((\tilde{1} \circ A \circ B) \circ \tilde{1}) \circ (\tilde{1} \circ (\tilde{1} \circ A \circ B)) \\ &= (\tilde{1} \circ A \circ B) \circ (\tilde{1} \circ \tilde{1}) \circ (\tilde{1} \circ A \circ B) \\ &= (\tilde{1} \circ A \circ B) \circ \tilde{1} \circ (\tilde{1} \circ A \circ B). \\ &\qquad \text{(By the condition (c)(2))} \end{aligned}$$

So $\tilde{1} \circ A \circ B = (\tilde{1} \circ A \circ B) \circ \tilde{1} \circ (\tilde{1} \circ A \circ B)$. Similarly, we have that

$$A \circ B \circ \tilde{1} = (A \circ B \circ \tilde{1}) \circ \tilde{1} \circ (A \circ B \circ \tilde{1}).$$

Then

$$\begin{aligned} &(\tilde{1} \circ A \circ B) \cap (A \circ B \circ \tilde{1}) \\ &= ((\tilde{1} \circ A \circ B) \circ \tilde{1} \circ (\tilde{1} \circ A \circ B)) \cap ((A \circ B \circ \tilde{1}) \\ &\quad \circ \tilde{1} \circ (A \circ B \circ \tilde{1})) \\ &= (\tilde{1} \circ (A \circ B \circ \tilde{1}) \circ (\tilde{1} \circ A \circ B)) \cap ((A \circ B \circ \tilde{1}) \\ &\quad \circ (\tilde{1} \circ A \circ B) \circ \tilde{1})) \\ &= (A \circ B \circ \tilde{1}) \circ (\tilde{1} \circ A \circ B) \subset (A \circ B \circ \tilde{1}) \circ \tilde{1} \circ (\tilde{1} \circ B) \\ &\qquad \text{(By (5.2) and } A \subset \tilde{1}) \\ &= (A \circ B \circ \tilde{1}) \circ (\tilde{1} \circ \tilde{1}) \circ B \\ &= (A \circ B \circ \tilde{1}) \circ \tilde{1} \circ B \\ &\qquad \text{(Since } \tilde{1} \circ \tilde{1} = \tilde{1}) \\ &= (A \circ B) \circ (\tilde{1} \circ \tilde{1}) \circ B \\ &= A \circ (B \circ \tilde{1} \circ B) \\ &\subset A \circ B. \\ &\qquad \text{(Since } B \circ \tilde{1} \circ B \subset B) \end{aligned}$$

So $A \circ B \in$ IVQI(S). Since " $\circ$ " is associative , (IVQI(S), $\circ$) is a semigroup. Let $A \in$ IVQI(S). Then

$$\begin{aligned} A &= (A \circ \tilde{1}) \cap (\tilde{1} \circ A) \quad \text{(By (5.1))} \\ &= (A \circ \tilde{1}) \circ (\tilde{1} \circ A) \quad \text{(By the condition (3))} \\ &= A \circ \tilde{1} \circ A. \end{aligned}$$

It is clear that $\tilde{1} \in$ IVQI(S). So A is a regular element of IVQI(S). Hence (IVQI(S),$\circ$) is a regular semigroup.

(d)⇒(e): Suppose the condition (d) holds. Let $A \in$ IVQI(S). Then, by the hypothesis, there exists $B \in$ IVQI(S) such that $A = ABA$. Thus

$$\begin{aligned} A &= ABA = A \circ \tilde{1} \circ A \\ &\qquad \text{(Since } B \subset \tilde{1}) \\ &\subset (A \circ \tilde{1}) \cap (\tilde{1} \circ A) \\ &\subset A. \end{aligned}$$

Hence $A = A \circ \tilde{1} \circ A$.

(e)⇒(a) : Suppose the condition (e) holds. Let $x \in S$ and let $A = \{x\} \cup (Sx \cap xS)$ be the quasi-ideal of S generated by x. Then, by Theorem 4.2, $[\chi_A, \chi_A] \in$ IVQI(S). Thus, by the hypothesis, $[\chi_A, \chi_A] = \tilde{1} \circ \tilde{1} \circ [\chi_A, \chi_A]$. So

$$\begin{aligned} 1 &= \chi_A(x) = ([\chi_A, \chi_A] \circ \tilde{1} \circ [\chi_A, \chi_A])^L(x) \\ &= \bigvee_{x=yz} ((A^L(y) \wedge (\tilde{1} \circ [\chi_A, \chi_A])^L(z)). \end{aligned}$$

Then there exist $p, q \in S$ with $x = pq$ such that

$$\chi_A(p) = 1 \quad \text{and} \quad (\tilde{1} \circ [\chi_A, \chi_A)])^L(q) = 1.$$

Since $\chi_A(p) = 1, p \in A$, i.e., $p = x$ or $p = xs$ where $s \in S$. Since $(\tilde{1} \circ [\chi_A, \chi_A])(q) = [1, 1]$,

$$\bigvee_{q=st} (\tilde{1}^L(s) \wedge \chi_A(t)) = 1 \text{ and } \bigvee_{q=st} (\tilde{1}^L(s) \wedge \chi_A(t)) = 1.$$

Then there exist $a, b \in S$ with $q = ab$ such that $\chi_A(b) = 1$. So $b \in A$, i.e., either $b = x$ or $b = s_1 x$ where $s_1 \in S$. Hence $x = pq = xcx$ where $c \in S$. Therefore x is a regular element of S. Hence S is regular. This completes the proof. □

Theorem 5.2. Let S be a regular semigroup and let $\tilde{0} \neq A \in D(I)^S$. Then $A \in$ IVQI(S) if and only if there exist $B \in$ IVRI(S) and $C \in$ IVLI(S) such that $A = B \circ C$.

Proof. (⇒) : Suppose $A \in$ IVQI(S). Then

$$\begin{aligned} A &= A \circ \tilde{1} \circ A \quad \text{(By Theorem 5.1)} \\ &= A \circ (\tilde{1} \circ \tilde{1}) \circ A \\ &= (A \circ \tilde{1}) \circ (\tilde{1} \circ A). \end{aligned}$$

Let $A \circ \tilde{1} = B$ and let $C = \tilde{1} \circ A$. Then, by Proposition 3.6, $B \in$ IVRI(S) and $C \in$ IVLI(S). Hence the necessary condition holds.

(⇐) : Suppose the necessary condition holds. Let $A \in D(I)^S$. Then there exist $B \in$ IVRI(S) and $C \in$ IVLI(S) such that $A = B \circ C$. Since S is regular, by Theorem 5.1, $B \circ C \in$ IVQI(S). Hence $A \in$ IVQI(S). This completes the proof. □

Theorem 5.3. Let S be a regular semigroup. Then the following hold:

(a) If $A \in$ IVQI(S), then $A^2 = A^3$.

(b) $A \in$ IVQI(S) if and only if $A \in$ IVBI(S).

Proof. (a) Suppose $A \in$ IVQI(S). Since $A \in$ IVSG(S), $A \circ A \subset A$. Thus $A \circ (A \circ A) \subset A \circ A$. So $A^3 \subset A^2$. Since S is regular, by Theorem 5.1, $A \circ A \in$ IVQI(S). Since IVQI(S) is regular, there exists $B \in$ IVQI(S) such that $A^2 = A^2 \circ B \circ A^2$. On the other hand, $A^2 \circ B \circ A^2 \subset A^2 \circ \tilde{1} \circ A^2 = A \circ (A \circ \tilde{1} \circ A) \circ A = A \circ A \circ A = A^3$. Thus $A^2 \subset A^3$. Hence $A^2 = A^3$.

(b) (⇒) : It is clear from Proposition 4.12.

(⇐) : Suppose $A \in$ IVBI(S). Then $\tilde{1} \circ A \in$ IVLI(S) and $A \circ \tilde{1} \in$ IVRI(S). Thus

$$\begin{aligned} [A \circ \tilde{1}] \cap [\tilde{1} \circ A] &= [A \circ \tilde{1}] \circ [\tilde{1} \circ A] \\ &\qquad \text{(By Theorem 5.1)} \\ &= A \circ (\tilde{1} \circ \tilde{1}) \circ A \\ &\qquad \text{(By Result 3.A)} \\ &= A \circ \tilde{1} \circ A \subset A. \\ &\qquad \text{(By the hypothesis)} \end{aligned}$$

Hence $A \in$ IVQI(S). This completes the proof. □

A semigroup S is called a *band*(See [2]) if for each $a \in S$, $aa = a$.

Theorem 5.4. Let S be a semigroup. Then the following are equivalent :

(a) For each $A \in$ IVRI(S) and each $B \in$ IVLI(S), $A \circ B = A \cap B \subset B \circ A$.

(b) (IVQI(S), $\circ$) is a band.

(c) For each $A \in$ IVQI(S), $A \circ A = A$.

Proof. (a)⇒(b) : Suppose the condition(a) holds. Then, by Theorem 5.1, S is regular. Thus, by Theorem 5.1,(IVQI(S), $\circ$) is a regular semigroup. Let $A \in$ IVQI(S). Then,

$$\begin{aligned} A &= A \circ \tilde{1} \circ A \\ &\qquad \text{(By Theorem 5.1)} \\ &= (A \circ \tilde{1} \circ A) \circ \tilde{1} \circ (A \circ \tilde{1} \circ A) \\ &\qquad \text{(By theorem 5.1)} \\ &= (A \circ \tilde{1}) \circ (A \circ \tilde{1}) \circ (\tilde{1} \circ A) \circ (\tilde{1} \circ A) \\ &\qquad \text{(Since } \tilde{1} \circ \tilde{1} = \tilde{1}) \\ &\subset (A \circ \tilde{1}) \circ (\tilde{1} \circ A) \circ (A \circ \tilde{1}) \circ (\tilde{1} \circ A) \\ &\qquad \text{(By the hypothesis)} \\ &= A \circ (\tilde{1} \circ \tilde{1}) \circ A \circ A \circ (\tilde{1} \circ \tilde{1}) \circ A \\ &\qquad \text{(By Result 3.A)} \\ &= (A \circ \tilde{1} \circ A) \circ (A \circ \tilde{1} \circ A) \\ &\qquad \text{(Since } \tilde{1} \circ \tilde{1} = \tilde{1}) \\ &= A \circ A \quad \text{(Since } A \circ \tilde{1} \circ A = A) \\ &\subset A. \quad \text{(Since } A \circ A \subset A) \end{aligned}$$

Thus $A \circ A = A$. So A is an idempotent element of IVQI(S). Hence (IVQI(S), $\circ$) is a band.

(b)⇒(c) : It is clear.

(c)⇒(a) : Suppose the condition (c) holds. Let $A \in$ IVRI(S) and let $B \in$ IVLI(S). Then, by Proposition 4.6,

$A \cap B \in$ IVQI(S). Thus,

$$\begin{aligned} A \cap B &= (A \cap B) \circ (A \cap B) \quad \text{(By the hypothesis)} \\ &\subset (A \circ (A \cap B)) \cap (B \circ (A \cap B)) \\ &\quad \text{(By Proposition 3.2(b))} \\ &\subset (A \circ A) \cap (A \circ B) \cap (B \circ A) \cap (B \circ B). \\ &\quad \text{(By Proposition 3.2(b))} \end{aligned}$$

So $A \cap B \subset A \circ B$ and $(A \cap B) \subset B \circ A$. (5.3)
On the other hand, $A \circ B \subset A \circ \tilde{1} \subset A$ and $A \circ B \subset \tilde{1} \circ B \subset B$. Thus $A \circ B \subset A \cap B$. Hence, by (5.3), $A \circ B = A \cap B \subset B \circ A$. This completes the proof. □

A semigroup S is said to be *intra-regular* [10] if for each $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$.

Theorem 5.5. Let S be a semigroup. Then S is intra-regular if and only if for each $A \in$ IVRI(S) and each $B \in$ IVLI(S), $A \cap B \subset B \circ A$.

Proof. $(\Rightarrow)$: Suppose S is intra-regular. Let $A \in$ IVRI(S), let $B \in$ IVLI(S) and let $a \in S$. Then, by the hypothesis, there exist $x, y \in S$ such that $a = xa^2y$. Thus

$$\begin{aligned} (B \circ A)^L(a) &= \bigvee_{a=st} (B^L(s) \wedge A^L(t)) \\ &\geq B^L(xa) \wedge A^L(ay) \\ &\quad \text{(Since } a = (xa)(ay)) \\ &\geq B^L(a) \wedge A^L(a) \\ &\quad \text{(Since } B \in \text{IVLI(S) and } A \in \text{IVRI(S))} \\ &= (A \cap B)^L(a) \end{aligned}$$

Similarly, we have that $(B \circ A)^U(a) \geq (A \cap B)^U(a)$. Hence $A \cap B \subset B \circ A$.

$(\Leftarrow)$: Suppose the necessary condition holds and let $x \in S$. Let $L = \{x\} \cup Sx$ and $R = \{x\} \cup xS$ be the left and right ideals of S generated by x, respectively. Then, by Result 3.B (b), $[\chi_L, \chi_L] \in$ IVLI(S) and $[\chi_R, \chi_R] \in$ IVRI(S). Thus, by the hypothesis,

$$[\chi_R, \chi_R] \cap [\chi_L, \chi_L] \subset [\chi_L, \chi_L] \circ [\chi_R, \chi_R].$$

So

$$\begin{aligned} ([\chi_L, \chi_L] \circ [\chi_R, \chi_R])^L \ (x) &\\ &= \bigvee_{x=yz} (\chi_L(y) \wedge \chi_R(z)) \\ &\geq \chi_L(x) \wedge \chi_R(x) = 1. \end{aligned}$$

Similarly, we have that $([\chi_L, \chi_L] \circ [\chi_R, \chi_R])^U(x) \geq 1$. Then there exist $p, q \in S$ with $x = pq$ such that $\chi_L(p) = 1$, $\chi_L(p) = 0$ and $\chi_R(q) = 1$, $\chi_R(q) = 0$. Thus $p \in L$ and $q \in R$. So $p = x$ or $p = sx$ and $q = x$ or $q = xs$ where $s \in S$. In any case, $x = pq = ax^2b$, where $a, b \in S$. Hence S is intra-regular. This completes the proof. □

The following is the immediate results of Theorems 5.4 and 5.5.

Theorem 5.6. Let S be a semigroup. Then the following are equivalent:

(a) S is regular and intra-regular.

(b) For each $A \in$ IVRI(S) and each $B \in$ IVLI(S), $A \circ B = A \cap B \subset B \circ A$.

(c) (IVQI(S), $\circ$) is a band.

(d) For each $A \in$ IVQI(S), $A \circ A = A$.

Example 5.7 Let $S = \{a, b, c, d, e\}$ be the semigroup with the following multiplication table:

	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

Then clearly S is a non-commutative semigroup which is not intra-regular. We define two mappings $A, B : S \to D(I)$ as follows, respectively :

$$A^L(a) \geq A^L(x), \quad A^U(a) \geq A^U(x)$$

for each $x \in S$,

$$A(b) = A(d), \quad A(c) = A(e)$$

and

$$B^L(a) \geq B^L(x), \quad B^U(a) \geq B^U(x)$$

for each $x \in S$,

$$B(b) = B(c), \quad B(d) = B(e).$$

Then we can easily see that $\tilde{0} \neq A \in$ IVLI(S) and $\tilde{0} \neq B \in$ IVRI(S). Moreover, we can check that $A \circ A = A$, $B \circ B = B$ and $B \circ A = B \cap A$. Now we define a mapping $C : S \to D(I)$ as follows:

$$C(a) = [1, 1], \ C(b) = C(c) = C(d) = [0.4, 0.5]$$

and

$$C(e) = [0.5, 0.5].$$

Then we can see that $A \in$ IVQI(S) and $A \neq A \circ A = A \circ A \circ A$. □

Example 5.8 Let $S = \{0, a, 1\}$ be the semigroup with the following multiplication table:

	0	a	1
0	a	a	a
a	a	a	a
1	a	a	b

Then S is a commutative semigroup which is regular and intra-regular. Let $A \in$ IVQI(S) and let $x \in S$. Then

$$\begin{aligned} A^L(x) &\geq (A \circ A)^L(x) \quad \text{(Since } A \in \text{IVSG(S))} \\ &= \bigvee_{x=yz} (A^L(y) \wedge A^L(z)) = \bigvee_{x=yz} A^L(y) \\ &\qquad \text{(Since } S \text{ is commutative)} \end{aligned}$$

and

$$\begin{aligned} A^U(x) &\geq (A \circ A)^U(x) = \bigvee_{x=yz} (A^U(y) \wedge A^U(z)) \\ &= \bigvee_{x=yz} A^U(y). \end{aligned}$$

Thus

$$\begin{aligned} A^L(0) &= \bigvee_{0=yz} A^L(y) \geq A^L(x), \quad A^U(0) \\ &= \bigvee_{0=yz} A^U(y) \geq A^U(x) \end{aligned}$$

for each $x \in S$ and

$$A^L(a) = A^L(a) \wedge A^L(1) \geq A^L(1),$$

$$A^U(a) = A^U(a) \wedge A^U(1) \geq A^U(1).$$

So $A^L(0) \geq A^L(a) \geq A^L(1)$ and $A^U(0) \geq A^U(a) \geq A^U(1)$. Then

$$\begin{aligned} (A \circ A)^L(0) &= \bigvee_{0=xy} (A^L(x) \wedge A^L(y)) = A^L(0), \\ (A \circ A)^U(0) &= \bigvee_{0=xy} (A^U(x) \wedge A^U(y))^U = A^U(0), \\ (A \circ A)^L(a) &= (A^L(a) \wedge A^L(a)) \vee (A^L(a) \wedge A^L(1)) \\ &= A^L(a), \\ (A \circ A)^U(a) &= (A^U(a) \wedge A^U(a)) \vee (A^U(a) \wedge A^U(1)) \\ &= A^U(a) \end{aligned}$$

and

$$\begin{aligned} (A \circ A)^L(1) &= A^L(1) \wedge A^L(1) = A^L(1), \\ (A \circ A)^U(1) &= A^U(1) \wedge A^U(1) = A^U(1). \end{aligned}$$

Hence $A^2 = A$. □

Example 5.9 Let $S = \{a, b, c\}$ be the semigroup with the following multiplication table:

	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then S is not commutative but it is regular and intra-regular. We can easily see that $A \in$ IVQI(S) for each $\tilde{0} \neq A \in D(I)^S$. Let $A \in$ IVQI(S) and let $a \in S$. Then

$$\begin{aligned} &(A \circ A)^L(a) \\ &\quad = \bigvee_{a=xy} (A^L(x) \wedge A^L(y)) \\ &\quad = (A^L(a) \wedge A^L(a)) \vee (A^L(a) \wedge A^L(b)) \\ &\qquad \vee (A^L(a) \wedge A^L(c)) \\ &\quad = A^L(a), \\ &(A \circ A)^U(a) = \bigvee_{a=xy} (A^U(x) \wedge A^U(y)) \\ &\quad = (A^U(a) \wedge A^U(a)) \vee (A^U(a) \wedge A^U(b)) \\ &\qquad \vee (A^U(a) \wedge A^U(c)) \\ &\quad = A^U(a), \\ &(A \circ A)^L(b) = (A^L(b) \wedge A^L(a)) \vee (A^L(b) \wedge A^L(b)) \\ &\qquad \vee (A^L(b) \wedge A^L(c)) \\ &\quad = A^L(b), \\ &(A \circ A)^U(b) = (A^U(b) \wedge A^U(a)) \vee (A^U(b) \wedge A^U(b)) \\ &\qquad \vee (A^U(b) \wedge A^U(c)) \\ &\quad = A^U(b), \end{aligned}$$

and

$$\begin{aligned} &(A \circ A)^L(c) \\ &\quad = (A^L(c) \wedge A^L(a)) \vee (A^L(c) \wedge A^L(b)) \\ &\qquad \vee (A^L(c) \wedge A^L(c)) \\ &\quad = A^L(c), \\ &(A \circ A)^U(c) \\ &\quad = (A^U(c) \wedge A^U(a)) \vee (A^U(c) \wedge A^U(b)) \\ &\qquad \vee (A^U(c) \wedge A^U(c)) \\ &\quad = A^U(c). \end{aligned}$$

So $A \circ A = A$. Hence each $A \in$ IVQI(S) is idempotent. □

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