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SINGLY-PERIODIC MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. We construct three kinds of complete embedded singly-periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The first one is a 1-parameter family of minimal surfaces which is asymptotic to a horizontal plane and a vertical plane; the second one is a 2-parameter family of minimal surfaces which has a fundamental piece of finite total curvature and is asymptotic to a finite number of vertical planes; the last one is a 2-parameter family of minimal surfaces which fill $\mathbb{H}^2 \times \mathbb{R}$ by finite Scherk's towers.

1. Introduction

In this paper we construct complete embedded singly-periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The study of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ was initiated by Nelli and Rosenberg [10, 12]. By several mathematicians, some interesting examples are found as follows: the catenoid that is a surface of revolution about the vertical $\mathbb{R}$-axis; the helicoid that is ruled by the horizontal geodesic; the Riemann type minimal surface that is foliated by horizontal circles and geodesics; the Jenkins-Serrin type minimal surface that is a minimal graph over a domain bounded by an ideal polygon and is asymptotic to vertical planes; the k-noid that is a finite total curvature minimal surface which is conformal to $\mathbb{S}^2 \setminus \{p_1, \ldots, p_k\}, k \geq 2$ (see [3, 7, 8, 10, 11, 13]).

Suppose that W is a mean convex domain bounded by minimal surfaces and $\{\Gamma_n\}$ is a sequence of embedded 1-cycles in W converging to Γ. By Morrey's results [9], every Γ_n bounds a compact least-area surface Σ_n lying inside of W. Since ∂W is barrier of $\{\Sigma_n\}$, there exists a convergent subsequence of $\{\Sigma_n\}$ whose limit surface Σ is bounded by Γ. This method is called *the barrier method*.

Using the barrier method, we construct a 1-parameter family of complete embedded singly-periodic minimal surfaces which are asymptotic to a horizontal plane and a vertical plane. We call these surfaces *horizontal Scherk's towers*. A horizontal Scherk's tower is invariant under a horizontal translation. This surface can be viewed as the desingularization of the union of a horizontal

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plane and a vertical plane, and it gives an affirmative answer of Rosenberg's suggestion (see [12]).

Similarly, we construct complete embedded singly-periodic minimal surfaces which are asymptotic to a finite number of vertical planes that share a vertical line, which we call as *Scherk's towers*. Scherk's tower is invariant under a vertical translation and the fundamental piece of the surface has finite total curvature $4(1-k)\pi$. Recently, Morabito and Rodriguez [8] construct the same surfaces by using the conjugation method in $\mathbb{H}^2 \times \mathbb{R}$. The conjugate method is developed by Daniel [2] and Hauswirth, Sa Earp and Toubiana [4].

There are infinitely many regular tessellations of $\mathbb{H}^2$. Using each regular tessellation of hyperbolic plane, we construct a 2-parameter family of complete embedded singly-periodic minimal surfaces which fill $\mathbb{H}^2 \times \mathbb{R}$ by *finite Scherk's towers*. This is similar to a fence of Scherk's towers in $\mathbb{R}^3$ (see [6]). So we call it a *fence of Scherk's towers*.

This paper is organized as follows. In Section 2, we give a model for $\mathbb{H}^2 \times \mathbb{R}$ and state well-known theorems. In Section 3, we use Douglas-Morrey solutions (the smallest area surface among surfaces of the disk type) of Plateau's problem and barrier method to construct horizontal Scherk's towers. In Section 4, we construct Scherk's towers. In Section 5, we construct a fence of Scherk's towers in $\mathbb{H}^2 \times \mathbb{R}$.

2. Preliminaries

For $\mathbb{H}^2 \times \mathbb{R}$, we use the Poincaré disk model for $\mathbb{H}^2$, hence the whole space is topologically a solid cylinder. Let $z = x + iy$ denote the coordinate of $\mathbb{H}^2$ and t denote the coordinate of $\mathbb{R}$. In this paper, we use interchangingly two coordinates (z, t) and (x, y, t) for a point $p \in \mathbb{H}^2 \times \{t\}$. Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain. In $\overline{\mathbb{H}}^2 \times \{0\}$, let $\partial\overline{\Omega} = \partial\Omega \cup \partial_\infty\Omega$, where the boundary part is $\partial\Omega \subset \mathbb{H}^2 \times \{0\}$ and the ideal boundary part is $\partial_\infty\Omega \subset \partial_\infty\mathbb{H}^2 \times \{0\}$. In $\mathbb{H}^2 \times \mathbb{R}$, we consider a *vertical graph* over a domain in the horizontal plane, say, $\mathbb{H}^2 \times \{0\}$. For a $\mathcal{C}^2$ function $t = u(x, y)$, the vertical minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$ is

$$\operatorname{div}_{\mathbb{H}}\Big(\frac{\nabla_{\mathbb{H}} u}{W_u}\Big) = 0, \tag{2.1}$$

where $\operatorname{div}_{\mathbb{H}}$ and $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, $|\cdot|_{\mathbb{H}}$ denotes the norm in $\mathbb{H}^2$.

We may also consider a *horizontal graph* in $\mathbb{H}^2 \times \mathbb{R}$. Let V be a vertical plane, say, $V = \{(x, y, t) \mid x = 0\}$. For every point $p \in V$, there exists a unique oriented geodesic $\gamma(s)$ with $\gamma(0) = p$, $\gamma'(0)$ is orthogonal to V, and $\gamma(s)$ is moving in the positive x-direction. A subset $\Delta \subset \mathbb{H}^2 \times \mathbb{R}$ is a horizontal graph over a domain in V if Δ intersects $\{\gamma(s)\}$ at most one point for $\gamma(s)$ with $\gamma(0)$ lying the domain.

We recall the maximum principles for minimal surfaces.

Theorem 2.1 (Maximum principles, [14]).

(1) (*Interior maximum principle*) *Let* Σ_1 *and* Σ_2 *be minimal surfaces in* $\mathbb{H}^2 \times \mathbb{R}$*. Suppose* p *is an interior point of both* Σ_1 *and* Σ_2 *respectively, and that* $T_p(\Sigma_1) = T_p(\Sigma_2)$*. If* Σ_1 *lies in one side of* Σ_2 *near* p*, then* $\Sigma_1 = \Sigma_2$;

(2) (*Boundary point maximum principle*) *Suppose* Σ_1, Σ_2 *have* $\mathcal{C}^2$*-boundaries* C_1, C_2 *and the tangent spaces of both* Σ_1, Σ_2 *and* C_1, C_2 *agree at* p*, i.e.,* $T_p(\Sigma_1) = T_p(\Sigma_2)$*,* $T_p(C_1) = T_p(C_2)$*. If* Σ_1 *lies in one side of* Σ_2 *near* p*, then* $\Sigma_1 = \Sigma_2$.

For a minimal surface with special boundary, we can reflect the minimal surface across the special boundary.

Theorem 2.2 (Schwarz reflection principle, [12]). *Let* $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ *be a minimal surface, and* $\Upsilon \subset \partial\Sigma$.

(1) *If* Υ *is a horizontal or vertical geodesic line, then* Σ *can be reflected smoothly across* Υ *by* $180°$*-rotation about* Υ.

(2) Υ *lies in a totally geodesic plane. If* Υ *is a geodesic of* Σ *and it is not a horizontal or vertical geodesic line, and* Σ *meets orthogonal to the totally geodesic plane along* Υ *then* Σ *can be extended smoothly across* Υ *by reflection through the totally geodesic plane containing* Υ.

3. Horizontal Scherk's towers in $\mathbb{H}^2 \times \mathbb{R}$

We consider a Jordan curve $\mathcal{H}_{(\rho,\alpha)}$ which consists of two vertical geodesic segments, three horizontal geodesic segments and one horizontal curve in $\mathbb{H}^2 \times \mathbb{R}$ as follows. First we fix a horizontal geodesic in $\mathbb{H}^2 \times \{0\}$ through the origin as x-axis, and fix p a point on x-axis with distance α from the origin. Let h_1 be a horizontal geodesic segment from the origin, length ρ and orthogonal to x-axis, h_2 be a horizontal geodesic segment of length ρ from p, orthogonal to x-axis. Let v_1 be a vertical geodesic segment from the origin, length ρ, v_2 a vertical geodesic segment from the p, and length ρ. Let c_1 be a horizontal geodesic segment connecting the other end of v_1 and the other end of v_2, c_2 be a horizontal curve connecting the end of h_1 (different from the origin) and the end of h_2 (different from p) such that the vertical projection of the contour onto $\mathbb{H}^2 \times \{0\}$ bounds a convex domain in $\mathbb{H}^2 \times \{0\}$ (see, Left of Figure 1). In addition, as $\rho \to \infty$ the projected domains onto $\mathbb{H}^2 \times \{0\}$ exhaust the domain which is bounded by $\mathcal{H}_1$ be a horizontal geodesic ray from the origin and orthogonal to x-axis, $\mathcal{H}_2$ be a horizontal geodesic ray from the p and orthogonal to x-axis, and horizontal geodesic segment connecting the origin and p (see, Right of Figure 1).

Let $\Sigma_{(\rho,\alpha)}$ be a Douglas-Morrey solution for $\mathcal{H}_{(\rho,\alpha)}$, that is, a least area compact minimal surface of disk type bounded by $\mathcal{H}_{(\rho,\alpha)}$. We apply a theorem of Morrey to see that there is a smooth branched minimal immersion of the disk into $\mathbb{H}^2 \times \mathbb{R}$ bounded by $\mathcal{H}_{(\rho,\alpha)}$, which has the smallest area among surfaces of disk type. Morrey's result [9] requires the ambient manifold $\mathcal{M}^3$ to be complete and homogeneously regular. Recall that homogeneous regularity is

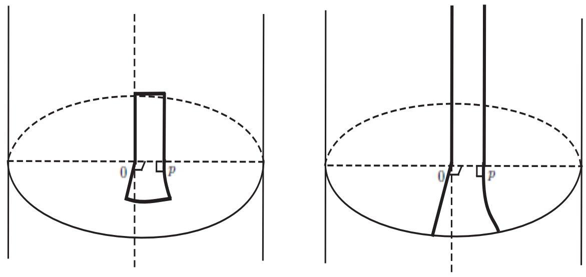

FIGURE 1. Left: the hexagonal polygon $\mathcal{H}_{(\rho,\alpha)}$; Right: the ideal polygon $\mathcal{H}_{(\alpha)}$.

an appropriately weak version of bounded geometry (see [1, 9]). In order to apply Morrey's result to our case, we construct a complete and homogeneously regular manifold $\mathcal{M}^3$ instead of $\mathbb{H}^2 \times \mathbb{R}$. Since $\mathcal{H}_{(\rho,\alpha)}$ is compact, it lies in a closed solid cylinder $D_R \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ of radius $R < 1$ with $0 \times \mathbb{R}$ as axis of revolution of $D_R \times \mathbb{R}$. Here, R is the polar coordinate of disk model for $\mathbb{H}^2$. We extend $D_R \times \mathbb{R}$ isometrically to a Riemannian manifold $\mathcal{M}^3$ diffeomorphic to $\mathbb{R}^3 = \{(x_1, x_2, x_3)\}$ with a rotationally symmetric metric along x_3-axis so that $\mathcal{M}^3$ is complete and homogeneously regular. For example, inclusion $D_R \times \mathbb{R}$ into $\mathbb{R}^3$ and a metric g which is $g = g_{\mathbb{H}^2 \times \mathbb{R}}$ on $D_R \times \mathbb{R}$ and $g = dx_1^2 + dx_2^2 + dx_3^2$ on $\sqrt{x_1^2 + x_2^2} > (R+1)/2$. Morrey's result shows that there is a smooth branched immersion of the disk into $\mathcal{M}^3$ with boundary $\mathcal{H}_{(\rho,\alpha)}$, having the smallest area among surfaces of disk type. Write its closed image as $\Sigma_{(\rho,\alpha)}$. Since $\Sigma_{(\rho,\alpha)}$ is compact, $\Sigma_{(\rho,\alpha)}$ lies in $D_R \times \mathbb{R}$ by maximum principle. Therefore $\Sigma_{(\rho,\alpha)} \subset \mathbb{H}^2 \times \mathbb{R}$.

By Rado's theorem, the minimal surface $\Sigma_{(\rho,\alpha)}$ is a graph for every ρ. Clearly it is embedded and has no branched point. From maximum principle, we see that $\Sigma_{(\rho_1,\alpha)}$ is above $\Sigma_{(\rho_2,\alpha)}$ whenever $\rho_1 > \rho_2$.

To show that the existence of the limit surface of a sequence $\{\Sigma_{(n,\alpha)}\}_{n\geq 1}$ as n increases to ∞, we need to find a suitable barrier. Consider

$$v(x, y) = \log\left(\frac{\sqrt{x^2 + y^2} + y}{x}\right), \ x > 0, \ y > 0,$$

in the upper-half plane model for $\mathbb{H}^2$ (see [10]). The graph of v is a minimal surface, which is ∞ on the positive y-axis and vanishes identically on the positive x-axis. Applying a suitable bilinear transformation, we get a minimal graph on the upper half disk with infinity value on the x-axis and vanishes identically on the upper semi-circle. So v is a suitable barrier for the sequence $\{\Sigma_{(n,\alpha)}\}_{n\geq 1}$. So, the limit surface $\Sigma_{(\alpha)}$ is embedded and bounded by a pair of vertical geodesic rays and a pair of horizontal geodesic rays which are orthogonal to the x-axis.

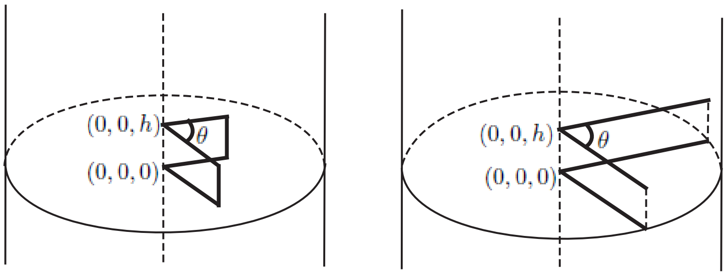

FIGURE 2. Left: the hexagonal polygon $\mathcal{H}_{(\rho,\theta,h)}$; Right: the ideal polygon $\mathcal{H}_{(\theta,h)}$.

We reflect consecutively the minimal surface $\Sigma_{(\alpha)}$ about its boundary, to get a complete embedded singly periodic minimal surface $M(\alpha)$.

Theorem 3.1. *For positive number α, there exists a complete, embedded, singly-periodic minimal surface $M(\alpha)$ in $\mathbb{H}^2\times\mathbb{R}$ satisfying the following properties*:

(a) *The fundamental piece of $M(\alpha)$ has infinite total curvature*;

(b) *It is invariant under a horizontal translation by length 2α.*

Proof. Since the Gaussian curvature of $\Sigma_{(\alpha)}$ is asymptotic to -1 near $\partial_\infty\mathbb{H}^2\times\mathbb{R}$, we can choose $p \in \Sigma_{(\alpha)}$ such that the Gaussian curvature $K(p) < -1/2$ and a neighborhood $\Delta_p \subset \Sigma_{(\alpha)}$ of p such that $K(q) < -1/2$ for every $q \in \Delta_p$. Since the intrinsic area of Δ_p is infinite, $\Sigma_{(\alpha)}$ has infinite total (Gaussian) curvature. This proves conclusion (a). (b) is clear by the construction. □

Remark 3.2. We may regard $M(\alpha)$ as desingularization of a horizontal plane and a vertical plane adding genus along the intersection line. This is an affirmative geometric answer of Rosenberg's suggestion (see page 1194 in [12]).

4. Scherk's towers in $\mathbb{H}^2 \times \mathbb{R}$

Given a triple $(\rho,\theta,h) \in (0,\infty]\times(0,\pi)\times(0,\infty)$ of positive real numbers, and $r = \tanh\left(\frac{\rho}{2}\right) \in (0,1]$, $c = \cos\left(\frac{\theta}{2}\right)$ and $s = \sin\left(\frac{\theta}{2}\right)$, we consider the hexagonal polygon $\mathcal{H}_{(\rho,\theta,h)}$ which consists of geodesics in $\mathbb{H}^2 \times \mathbb{R}$ (see Figure 2). The Jordan curve $\mathcal{H}_{(\rho,\theta,h)}$ is obtained from the following contour:

$$O \to (rc,-rs,0) \to (rc,-rs,h) \to (0,0,h) \to (rc,rs,h) \to (rc,rs,0) \to O.$$

Here, O denotes the origin $(0,0,0) \in \mathbb{H}^2 \times \mathbb{R}$. As $\rho \to \infty$, we obtain the ideal polygon $\mathcal{H}_{(\theta,h)} := \mathcal{H}_{(\infty,\theta,h)}$ that runs the following route:

$$O \to (c,-s,0) \to (c,-s,h) \to (0,0,h) \to (c,s,h) \to (c,s,0) \to O.$$

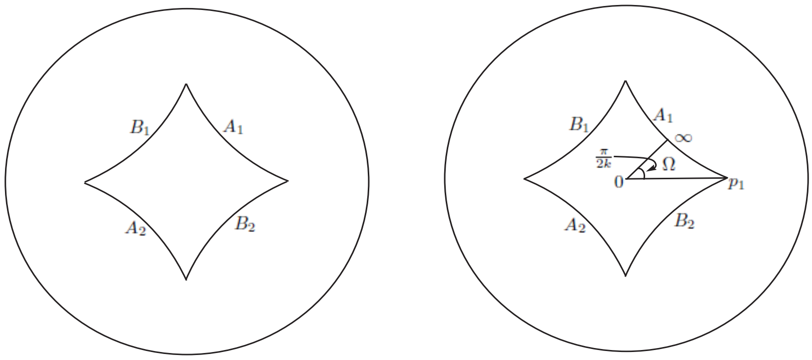

FIGURE 3. Left: Geodesic 4-gon; Right: Fundamental piece of $\Delta_{\alpha,2}$ of $S_{\alpha,2}$.

Let $\Sigma_{(\rho,\theta,h)}$ be a Douglas-Morrey solution of Plateau's problem for $\mathcal{H}_{(\rho,\theta,h)}$. Let Π_1 be the vertical plane containing the horizontal geodesic from O to $(c,-s,0)$, Π_2 the vertical plane containing the horizontal geodesic from O to $(c,s,0)$ and Π_0 the vertical plane containing the horizontal geodesic from $(c,-s,0)$ to $(c,s,0)$.

Since the sectional curvature of $\mathbb{H}^2\times\mathbb{R}$ is bounded above by zero and the (boundary) total curvature of $\mathcal{H}_{(\rho,\theta,h)}$ is $4\pi-2\theta$ (less than 4π). By Choe and Gulliver, $\Sigma_{(\rho,\theta,h)}$ is embedded (see Theorem 3 in [1]). Moreover, we see that $\Sigma_{(\rho,\theta,h)}$ is a horizontal graph over a domain in $V=\{(x,y,t)|x=0\}$ as follows. If $\Sigma_{(\rho,\theta,h)}$ is not a graph, then there exists a horizontal geodesic $\gamma_0(s)$ for which the number of the intersection points are at least two. Let us denote by $\Sigma^s_{(\rho,\theta,h)}$ the isometrically translated surface of $\Sigma_{(\rho,\theta,h)}$ along $\gamma_0(s)$ distance s. For sufficient large s_1, $\Sigma_{(\rho,\theta,h)}$ and $\Sigma^{s_1}_{(\rho,\theta,h)}$ are disjoint. If we decrease s_1 to 0, then there exists $s_t>0$ such that $\Sigma^{s_t}_{(\rho,\theta,h)}$ is the first touching with $\Sigma_{(\rho,\theta,h)}$. This is a contradiction by the maximum principle.

We now fix θ and h and consider a sequence of minimal surfaces $\{\Sigma_{(n,\theta,h)}\}_{n\geq 1}$. By the maximum principle, every $\Sigma_{(n,\theta,h)}$ is bounded by Π_1, Π_2, Π_0, $\mathbb{H}^2\times\{0\}$ and $\mathbb{H}^2\times\{h\}$. Here Π_1, Π_2 and Π_0 are defined as above. For each n_0, the previous surfaces $\{\Sigma_{(n,\theta,h)}\}$, $n<n_0$ are barriers for $\Sigma_{(n_0,\theta,h)}$. Hence the sequence $\{\Sigma_{(n,\theta,h)}\}$ is monotone increasing horizontal graphs and bounded above by Π_0. So, the limit surface

$$\Sigma_{(\theta,h)}=\lim_{n\to\infty}\Sigma_{(n,\theta,h)}$$

exists in $\mathbb{H}^2\times\mathbb{R}$. Since $\Sigma_{(\theta,h)}$ is trapped among Π_1, Π_2 and Π_0, and $\Sigma_{(\theta,h)}$ is asymptotic to Π_0.

Hence, the limit surface $\Sigma_{(\theta,h)}$ is embedded and bounded by a pair of two intersecting geodesic rays.

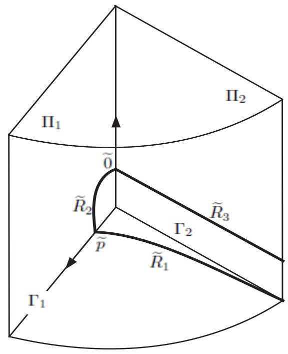

FIGURE 4. In case of $k = 2$, the conjugate surface $\Delta_{\frac{\pi}{2}}$.

Now, we describe $\Sigma_{(\theta,h)}$ by conjugate method (see [8] for more detail).

We briefly describe the Scherk type minimal surface $S_{\alpha,k}$. Let $\mathcal{P}$ be a regular $2k$-gon in $\mathbb{H}^2 \times \{0\}$ with (open) edges $A_1, B_1, \ldots, A_k, B_k$ and α be the distance from the center of the regular $2k$-gon to a vertex of the regular $2k$-gon. $S_{\alpha,k}$ is a minimal graph over the domain $\mathcal{D}$ bounded by $\mathcal{P}$ such that boundary values are alternatively $+\infty$ on the edges A_i and $-\infty$ on the edges B_i. It is a special case of Jenkins-Serrin type theorem.

We take the smallest fundamental piece $\Delta = \Delta_{\alpha,k}$ of $S_{\alpha,k}$. The fundamental piece Δ is the graph of u over Ω and is bounded by a vertical geodesic ray, a horizontal geodesic segment and a geodesic of $S_{\alpha,k}$ on a vertical plane. Let R_1 be a vertical geodesic ray from $(p_1, 0)$, R_2 a horizontal geodesic segment from $(p_1, 0)$ to the origin $O = (0, 0)$ of $\mathbb{H}^2 \times \{0\}$, and R_3 a geodesic of $S_{\alpha,k}$ on a vertical plane (see Figure 3).

Let $\Delta_{\frac{\pi}{2}}$ be a conjugate surface of Δ and $\widetilde{R}_i$ be the conjugate image of R_i, $i = 1, 2, 3$. Then, $\widetilde{R}_1$ lies on a horizontal plane, denoted by $\mathbb{H}^2 \times \{0\}$, and $\Delta_{\frac{\pi}{2}}$ is orthogonal to the horizontal plane along $\widetilde{R}_1$; $\widetilde{R}_2$ is contained in a vertical plane, denoted by, Π_1; $\widetilde{R}_3$ is a horizontal geodesic ray starting from $\widetilde{O}$, the image of O (see Figure 4).

Using Schwarz reflection principle, we can reflect $\Delta_{\frac{\pi}{2}}$ with respect to Π_1 and then extend with respect to $\mathbb{H}^2 \times \{0\}$. The extended surface $\Sigma_{\alpha,k}$ is bounded by a pair of two connected geodesic rays and asymptotic to a vertical plane (see Figure 5). Since conjugate surface is isometric to the original surface, the total curvature of $\Sigma_{\alpha,k}$ is $TC(\Sigma_{\alpha,k}) = 4(1-k)\pi/k$.

In fact, if we choose $\theta = \frac{\pi}{k}$ and suitable h, then $\Sigma_{\alpha,k}$ and $\Sigma_{(\theta,h)}$ are congruent, and $\partial\Sigma_{\alpha,k} = \partial\Sigma_{(\theta,h)}$. Put $\Sigma_{\alpha,k}$ at the same height of $\Sigma_{(\theta,h)}$ but disjoint. By horizontal translation (extension to $\mathbb{H}^2 \times \mathbb{R}$ of hyperbolic isometry for $\mathbb{H}^2$ along x-axis) of $\Sigma_{\alpha,k}$ toward $\Sigma_{(\theta,h)}$ from 'behind', until the boundaries coincide. By

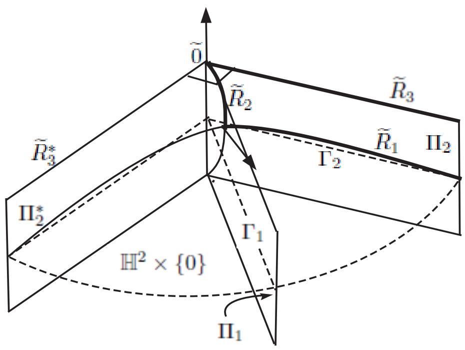

FIGURE 5. In case of $k = 2$, the extended surface $\Sigma_{\alpha,k}$ of $\Delta_{\frac{\pi}{2}}$.

the maximum principle, two surfaces are congruent or disjoint. If the surfaces are disjoint, horizontal translation of $\Sigma_{\alpha,k}$ toward $\Sigma_{(\theta,h)}$ from 'front', until the boundaries coincide. But by the maximum principle, the surfaces must be congruent.

Reflect consecutively $\Sigma_{(\theta,h)}$ (or $\Sigma_{\alpha,k}$) along its boundary (horizontal geodesic rays). We get a complete embedded singly periodic minimal surface $M(\alpha, k)$. We call it a Scherk's tower.

Theorem 4.1. *For positive integer $k \geq 2$, and $\alpha > 0$, there exists a complete, embedded, singly-periodic minimal surface $M(\alpha, k)$ in $\mathbb{H}^2 \times \mathbb{R}$ satisfying the following properties*:

(a) *The fundamental piece of $M(\alpha, k)$ has finite total curvature $4(1-k)\pi$*;

(b) *The fundamental piece of $M(\alpha, k)$ has genus zero and $2k$ ends which are asymptotic to vertical planes*;

(c) *$M(\alpha, k)$ is invariant under $\frac{\pi}{k}$-rotation of $M(\alpha, k)$ about the vertical $\mathbb{R}$-axis.*

Proof. Define $T = 2h\frac{\partial}{\partial t}$, where the h is the distance between the pair of two connected horizontal geodesic rays. The fundamental piece of $M(\alpha, k)$ is $M_T(\alpha, k)$. The $M_T(\alpha, k)$ is the k-times reflection of $\Sigma_{\alpha,k}$ along a horizontal geodesic ray bounded by a pair of k-horizontal geodesics with sharing a common point. Since $TC(\Sigma_{\alpha,k}) = 4(1-k)\pi/k$, the total curvature of $M_T(\alpha, k)$ is also $4(1-k)\pi$. This proves conclusion (a). Since each end of $\Sigma_{\alpha,k}$ converges to a vertical plane, (b) holds as in $\mathbb{R}^3$. And (c) is clear by the construction. $\square$

Remark 4.2. We may regard $M(\alpha, 2)$, extended surface of $\Sigma_{\alpha,2}$, as the desingularization of the two vertical planes by adding genus along the intersection line.

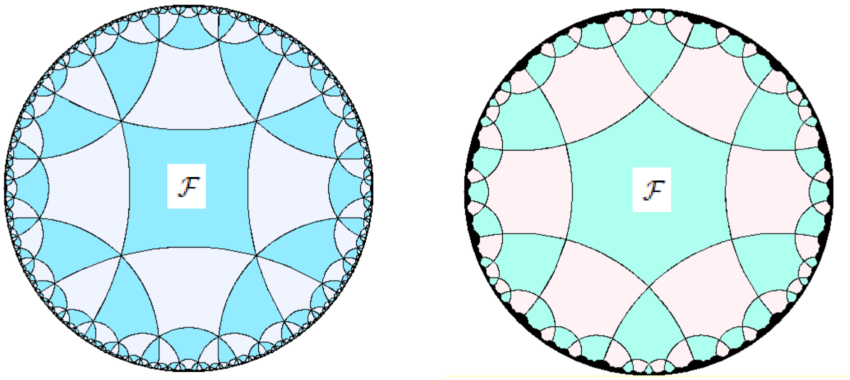

FIGURE 6. Left: a regular tessellation $\{4,6\}$ of $\mathbb{H}^2$; Right: a regular tessellation $\{6,4\}$ of $\mathbb{H}^2$.

5. A fence of Scherk's towers in $\mathbb{H}^2 \times \mathbb{R}$

A regular tessellation, or tiling, is a covering of the plane by regular polygons so that the same number of polygons meet at each vertex. The tessellations of the Euclidean plane are well-known. They are: $\{3,6\}$ in which 6 equilateral triangles meet at each vertex; $\{4,4\}$ in which 4 squares meet at each vertex; and $\{6,3\}$ in which 3 hexagons meet at each vertex. A notation like $\{3,6\}$ is called a *Schlafli symbol*.

There are infinitely many regular tessellations of the hyperbolic plane. We can determine whether $\{k,n\}$ will be a tessellation of the hyperbolic plane by looking at the sum $\frac{1}{k}+\frac{1}{n}$. If the sum equals $\frac{1}{2}$, as it does for the three tessellations mentioned above, then $\{k,n\}$ is a Euclidean tessellation. If the sum is less than $\frac{1}{2}$, then it is a tessellation for $\mathbb{H}^2$. For example, $\{4,6\}$ and $\{6,4\}$ are regular tessellation of the hyperbolic plane (see Figure 6 [5]).

Let $k \geq 4$ be integer and $n \geq 4$ even integer satisfying $\frac{1}{k}+\frac{1}{n}<\frac{1}{2}$, we have a regular tessellation of $\mathbb{H}^2$. Let $\mathcal{F}$ be a regular polygon of $\{k,n\}$ and $c_{\mathcal{F}}$ the center of $\mathcal{F}$. A Jordan curve $\mathcal{H}_{\left(\rho,\frac{2\pi}{n},h\right)}$ is defined as before where ρ is the intrinsic distance from $c_{\mathcal{F}}$ to a vertex of $\mathcal{F}$. The ρ is uniquely determined by k and n. Let $\Sigma_{\left(\rho,\frac{2\pi}{n},h\right)}$ be a Morrey's solution for $\mathcal{H}_{\left(\rho,\frac{2\pi}{n},h\right)}$. It is embedded as before.

Reflect consecutively $\Sigma_{\left(\rho,\frac{2\pi}{k},h\right)}$ along horizontal geodesic segments. We get an embedded singly-periodic minimal surface $\widetilde{\Sigma}_{\left(\rho,\frac{2\pi}{k},h\right)}$, which has k vertical geodesic lines. We call $\widetilde{\Sigma}_{\left(\rho,\frac{2\pi}{k},h\right)}$ a finite Scherk's tower. Finally, reflecting consecutively the finite Scherk's tower along its vertical geodesic lines, we have a complete embedded singly-periodic minimal surface $M(k,n,h)$.

Theorem 5.1. *For $k, n \geq 4$ even integers satisfying $\frac{1}{k} + \frac{1}{n} < \frac{1}{2}$, and $h > 0$, there exists a complete, embedded, singly-periodic minimal surface $M(k, n, h)$ in $\mathbb{H}^2 \times \mathbb{R}$.*

We call the $M(k, n, h)$ a fence of Scherk's tower in $\mathbb{H}^2 \times \mathbb{R}$.

Remark 5.2. For $k, n \geq 4$ be even integers satisfying $\frac{1}{k} + \frac{1}{n} < \frac{1}{2}$, the Scherk type minimal surface $S_{\rho,k/2}$ exists over a regular k-gon of a $\{k, n\}$-tessellation. Here, ρ is the intrinsic distance from the center to a vertex of the regular k-gon. The $S_{\rho,k/2}$ has k-vertical lines. We consecutively reflect $S_{\rho,k/2}$ along the vertical lines. We get a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. These are analogous to Scherk's doubly-periodic minimal surfaces. Rosenberg described only $n = 4$, that is, regular k-gon with vertex angle $\pi/2$.

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