Bull. Korean Math. Soc. **49** (2012), No. 5, pp. 1067–1079
http://dx.doi.org/10.4134/BKMS.2012.49.5.1067

UNIFORM AND COUNIFORM DIMENSION OF GENERALIZED INVERSE POLYNOMIAL MODULES

Renyu Zhao

Abstract. Let M be a right R-module, $(S,\leq)$ a strictly totally ordered monoid which is also artinian and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism, and let $[M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$ denote the generalized inverse polynomial module over the skew generalized power series ring $[[R^{S,\leq},\omega]]$. In this paper, we prove that $[M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$ has the same uniform dimension as its coefficient module M_R, and that if, in addition, R is a right perfect ring and S is a chain monoid, then $[M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$ has the same couniform dimension as its coefficient module M_R.

1. Introduction

Throughout this paper, R denotes a ring with identity and modules are unitary right R-modules. The uniform dimension (resp. couniform dimension) of a module M_R will be denoted by $\mathrm{u.dim}(M_R)$ (resp. $\mathrm{corank}(M_R)$). We will denote by $\mathrm{End}(R)$ the monoid of ring endomorphisms of R, and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R.

The behavior of the uniform dimension and the couniform dimension of a ring (resp. a module) under various polynomial extensions have been studied by many researchers, such as Shock [20], Varadarajan [19, 22, 23], Grzeszczuk [4], Matczuk [12] and Annin [1, 2]. In particular, Annin in [1, 2] obtained results how the uniform dimension and the couniform dimension of a module behaves on inverse polynomial modules. It was proved that for any right R-module M, $\mathrm{u.dim}\left(M[x^{-1}]_{R[x;\sigma]}\right) = \mathrm{u.dim}\left(M_R\right)$, and that if, in addition, R is a right perfect ring, then $\mathrm{corank}\left(M[x^{-1}]_{R[x;\sigma]}\right) = \mathrm{corank}\left(M_R\right)$, where $\sigma \in \mathrm{Aut}(R)$. In [10], as a generalization of inverse polynomial modules, Liu and Cheng introduced the notion of generalized inverse polynomial modules. Many properties of generalized inverse polynomial modules have been explored in recent years, see for example [6, 7, 8, 10, 11] and [24]. Motivated by these facts, in this paper, we will generalize Annin's work to generalized inverse

Received June 12, 2011.
2010 *Mathematics Subject Classification.* Primary 16W60.
Key words and phrases. skew generalized power series ring, generalized inverse polynomial module, uniform dimension, couniform dimension.

polynomial modules over skew generalized power series rings. We will show that, if $(S,\leq)$ is a strictly totally ordered monoid which is also artinian and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism, then for any right R-module M, $\mathrm{u.dim}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \mathrm{u.dim}\,(M_R)$, and that if, in addition, R is a right perfect ring and S is a chain monoid, then $\mathrm{corank}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \mathrm{corank}\,(M_R)$.

Let $(S,\leq)$ be a partially ordered set. Recall that $(S,\leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Unless stated otherwise, in this paper, S will always be a commutative monoid, the operation of S shall be denoted additively and the neutral element by 0. The following definition is due to [9, 13, 18].

Let R be a ring, $(S,\leq)$ a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid such that if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. For any $s \in S$, let ω_s denote the image of s under ω, that is $\omega_s = \omega(s)$. Consider the set A of all maps $f : S \longrightarrow R$ whose support $\mathrm{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f,g) = \{(u,v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$$

is finite. This fact allows to define the operation of convolution as follows:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u\left(g(v)\right), \qquad \text{if} \quad X_s(f,g) \neq \emptyset$$

and $(fg)(s) = 0$ if $X_s(f,g) = \emptyset$. With this operation and pointwise addition, A becomes a ring, which is called *the ring of skew generalized power series* with coefficients in R and exponents in S, and we denote it by $[[R^{S,\leq},\omega]]$.

Let $(S,\leq)$ be a strictly totally ordered monoid which is also artinian, M a right R-module and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. We let B be the set of all maps $\varphi : S \longrightarrow M$ such that the set $\mathrm{supp}(\varphi) = \{s \in S \mid \varphi(s) \neq 0\}$ is finite. Now B can be turned into a right $[[R^{S,\leq},\omega]]$-module. The addition in B is componentwise and the scalar multiplication is defined as follows:

$$(\varphi f)(s) = \sum_{t\in S} \varphi(s+t)\omega_{s+t}^{-1}\left(f(t)\right) \qquad \text{for every} \;\; s \in S,$$

where $f \in [[R^{S,\leq},\omega]]$ and $\varphi \in B$. Then, by [10, Lemma 2.1], $\mathrm{supp}(\varphi f)$ is finite, and so φf belongs to B.

Suppose that $f, g \in [[R^{S,\leq},\omega]]$, $\varphi \in B$ and $s \in S$. Then

$$\begin{aligned}((\varphi f)g)(s) &= \sum_{v\in S} (\varphi f)(v+s)\omega_{v+s}^{-1}\left(g(v)\right) \\ &= \sum_{v\in S}\left(\sum_{u\in S}\varphi(u+v+s)\omega_{u+v+s}^{-1}\left(f(u)\right)\right)\omega_{v+s}^{-1}(g(v))\end{aligned}$$

$$= \sum_{u\in S}\sum_{v\in S}\varphi(u+v+s)\omega_{v+s}^{-1}\left(\omega_u^{-1}\left(f(u)\right)g(v)\right),$$

and

$$\begin{aligned}\big(\varphi(fg)\big)(s) &= \sum_{t\in S}\varphi(t+s)\omega_{t+s}^{-1}\left((fg)(t)\right)\\ &= \sum_{t\in S}\varphi(t+s)\omega_{t+s}^{-1}\left(\sum_{(u,v)\in X_t(f,g)} f(u)\omega_u\left(g(v)\right)\right)\\ &= \sum_{(u,v)\in X}\varphi(u+v+s)\omega_{u+v+s}^{-1}\left(f(u)\omega_u\left(g(v)\right)\right)\\ &= \sum_{(u,v)\in X}\varphi(u+v+s)\omega_{v+s}^{-1}\left(\omega_u^{-1}\left(f(u)\right)g(v)\right),\end{aligned}$$

where $X = \bigcup_{t\in S} X_t(f,g)$. Thus, $(\varphi f)g = \varphi(fg)$. Now, it is easy to see that B becomes a right $[[R^{S,\leq},\omega]]$-module, which we call *the generalized inverse polynomial module* over $[[R^{S,\leq},\omega]]$, and denote it by $[M^{S,\leq}]$. The elements of $[M^{S,\leq}]$ are called generalized inverse polynomials with coefficients in M and exponents in S.

For example, if $\omega_s = 1$, the identity map of R for every $s \in S$, then $[M^{S,\leq}]_{[[R^{S,\leq},\omega]]} = [M^{S,\leq}]_{[[R^{S,\leq}]]}$, the generalized inverse polynomial module in the sense of Liu [6, 7, 8, 10, 11]. In this situation, if we take $S = \mathbb{N}\cup\{0\}$, and $\leq$ the usual order, then $[M^{\mathbb{N}\cup\{0\},\leq}] \cong M[x^{-1}]$, the usual right $R[[x]]$-module introduced in [14, 15], which is also called the Macaulay-Northcott module in [16, 17]. Let α be a ring automorphism of R, $S = \mathbb{N}\cup\{0\}$ be endowed with the usual order and define $\omega : S \longrightarrow \mathrm{Aut}(R)$ via $\omega_k = \alpha^k$ for every $k \in \mathbb{N}\cup\{0\}$ (where $\alpha^0 = 1$, the identity map of R). Then $[[R^{S,\leq},\omega]] = R[[x;\alpha]]$, and $[M^{\mathbb{N}\cup\{0\},\leq}] = M[x^{-1}]$, the inverse polynomial modules over skew power series rings $R[[x;\alpha]]$.

We shall henceforth assume that $(S,\leq)$ is a strictly totally ordered monoid which is also artinian. In this situation, by [10], $0 \leq s$ for any $s \in S$. This fact will be often used in our discussions. Also, in this case, for any $0 \neq f \in [[R^{S,\leq},\omega]]$, $\mathrm{supp}(f)$ has a minimal element, we denote it by $\pi(f)$, and for any $0 \neq \varphi \in [M^{S,\leq}]$, $\mathrm{supp}(\varphi)$ has a maximal element, we denote it by $\sigma(\varphi)$.

In the finial of this section, we explain some notations and facts involved. To any $r \in R$ and any $s \in S$ we associate the maps $\lambda_r^s \in [[R^{S,\leq},\omega]]$ defined by

$$\lambda_r^s(x) = \begin{cases} r, & \text{if } x = s,\\ 0, & \text{if } x \neq s.\end{cases}$$

In particular, denote $\lambda_r^0 = c_r$, and $\lambda_1^s = e_s$. For any $m \in M$ and any $s \in S$, we define $\phi_{s,m} \in [M^{S,\leq}]$ via

$$\phi_{s,m}(x) = \begin{cases} m, & \text{if } x = s,\\ 0, & \text{if } x \neq s.\end{cases}$$

For any $s \in S$, set $G_s = \{\phi_{s,m} \mid m \in M\}$. Then G_s is a right R-module by the right R-action $\phi_{s,m}r = \phi_{s,m}c_r$, and there exists an isomorphism of right R-modules $\alpha_s : M \longrightarrow G_s$ defined by $\alpha_s(m) = \phi_{s,m}$.

For any $\varphi \in [M^{S,\leq}]$, since $\mathrm{supp}(\varphi)$ is finite, φ can be written as

$$\varphi = \sum_{s \in S} \phi_{s,\varphi(s)}.$$

This fact will also be used freely in our next discussions.

2. Uniform dimension

Let us first recall the notion of uniform dimension, often abbreviated by "u.dim".

Definition 2.1. We say that M_R has uniform dimension n, if there is an essential submodule $V_R \leq M_R$ that is a direct sum of n uniform submodules. We write $\mathrm{u.dim}(M_R) = n$. If no such integer n exists, we write $\mathrm{u.dim}(M_R) = \infty$.

An intuitive description of uniform dimension is perhaps best reflected by the following result [5, Corollary 6.6].

Lemma 2.2. *For any nonzero module M_R,*

$$\mathrm{u.dim}(M_R) = \sup\{k \mid M_R \ \textit{contains a direct sum of } k \textit{ nonzero submodules}\}.$$

One checks easily that $\mathrm{u.dim}(M_R) = 0$ if and only if $M_R = 0$, and that $\mathrm{u.dim}(M_R) = 1$ if and only if M_R is uniform. Also, it is clearly possible for $\mathrm{u.dim}(M_R)$ to be infinite. In fact, we can characterize this situation as well [5, Proposition 6.4].

Lemma 2.3. *A module M_R has infinite uniform dimension if and only if M_R contains an infinite direct sum of nonzero submodules.*

The proof of the main result of this section relies on some elementary initial results.

Lemma 2.4. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism and N a submodule of M_R. Then N is a uniform submodule of M if and only if $[N^{S,\leq}]$ is a uniform submodule of $[M^{S,\leq}]$.*

Proof. $\Longrightarrow$) Let $0 \neq \varphi_1, \varphi_2 \in [N^{S,\leq}]$, and assume that $\sigma(\varphi_1) = s_1$, $\sigma(\varphi_2) = s_2$. Then $\varphi_1(s_1)R \cap \varphi_2(s_2)R \neq 0$ since N is a uniform submodule of M. Since $\omega_{s_1}, \omega_{s_2} \in \mathrm{Aut}(R)$, we may select $r_1, r_2 \in R$ so that $\varphi_1(s_1)\omega_{s_1}^{-1}(r_1) = \varphi_2(s_2)\omega_{s_2}^{-1}(r_2) \neq 0$. Then for any $s \in S$,

$$(\varphi_i \lambda_{r_i}^{s_i})(s) = \sum_{x \in S} \varphi_i(x+s)\omega_{x+s}^{-1}\big(\lambda_{r_i}^{s_i}(x)\big) = \begin{cases} \varphi_i(s_i)\omega_{s_i}^{-1}(r_i), & s = 0, \\ 0, & s \neq 0. \end{cases}$$

Thus $0 \neq \varphi_1\lambda_{r_1}^{s_1} = \varphi_2\lambda_{r_2}^{s_2} \in \varphi_1[[R^{S,\leq},\omega]] \cap \varphi_2[[R^{S,\leq},\omega]]$. Hence $[N^{S,\leq}]$ is a uniform submodule of $[M^{S,\leq}]$.

$\Longleftarrow$) Let $0 \neq n_1, n_2 \in N$. Then $0 \neq \phi_{0,n_i} \in [N^{S,\leq}]$, $i = 1, 2$. Since $[N^{S,\leq}]$ is a uniform submodule of $[M^{S,\leq}]$, $\phi_{0,n_1}[[R^{S,\leq},\omega]] \cap \phi_{0,n_2}[[R^{S,\leq},\omega]] \neq 0$. Let $\phi_{0,n_1}f_1 = \phi_{0,n_2}f_2 \neq 0$. Then for any $s \in S$,

$$(\phi_{0,n_i}f_i)(s) = \sum_{x\in S} \phi_{0,n_i}(x+s)\omega_{x+s}^{-1}\big(f_i(x)\big) = \begin{cases} n_if_i(0), & s = 0, \\ 0, & s \neq 0. \end{cases}$$

Thus $n_1f_1(0) = n_2f_2(0) \neq 0$. So $n_1R \cap n_2R \neq 0$. Hence N is a uniform submodule of M. □

Lemma 2.5. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism and N a submodule of M_R. Then N is an essential submodule of M if and only if $[N^{S,\leq}]$ is an essential submodule of $[M^{S,\leq}]$.*

Proof. $\Longrightarrow$) Let $0 \neq \varphi \in [M^{S,\leq}]$, and assume that $\sigma(\varphi) = s$. Then there exists $r \in R$ such that $0 \neq \varphi(s)\omega_s^{-1}(r) \in N$ since N is an essential submodule of M and $\omega_s \in \mathrm{Aut}(R)$. Then for any $x \in S$,

$$(\varphi\lambda_r^s)(x) = \sum_{y\in S} \varphi(x+y)\omega_{x+y}^{-1}\big(\lambda_r^s(y)\big) = \begin{cases} \varphi(s)\omega_s^{-1}(r) \in N, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Thus $0 \neq \varphi\lambda_r^s \in [N^{S,\leq}]$. Hence $[N^{S,\leq}]$ is an essential submodule of $[M^{S,\leq}]$.

$\Longleftarrow$) Let $0 \neq m \in M$. Then $0 \neq \phi_{0,m} \in [M^{S,\leq}]$. Since $[N^{S,\leq}]$ is an essential submodule of $[M^{S,\leq}]$, there exists an $f \in [[R^{S,\leq},\omega]]$ such that $0 \neq \phi_{0,m}f \in [N^{S,\leq}]$. Then for any $x \in S$,

$$(\phi_{0,m}f)(x) = \sum_{y\in S} \phi_{0,m}(x+y)\omega_{x+y}^{-1}\big(f(y)\big) = \begin{cases} mf(0), & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Thus $0 \neq mf(0) \in N$. Hence N is an essential submodule of M. □

Now, we can prove the main result of this section.

Theorem 2.6. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. Then for any right R-module M, we have*

$$\mathrm{u.dim}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \mathrm{u.dim}\left(M_R\right).$$

Proof. Assume that $\mathrm{u.dim}(M_R) = n < \infty$. By definition, we can find uniform submodules $N_1, N_2, \ldots, N_n$ of M such that $N_1 \bigoplus N_2 \bigoplus \cdots \bigoplus N_n$ is an essential submodule of M. By Lemma 2.4 and Lemma 2.5, $[N_1^{S,\leq}], [N_2^{S,\leq}], \ldots, [N_n^{S,\leq}]$ are uniform submodules of $[M^{S,\leq}]$, and $[N_1^{S,\leq}] \bigoplus [N_2^{S,\leq}] \bigoplus \cdots \bigoplus [N_n^{S,\leq}]$ is an essential submodule of $[M^{S,\leq}]$. Thus $\mathrm{u.dim}\big([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\big) = n$.

If $\text{u.dim}(M_R) = \infty$. Then, by Lemma 2.3, there exist nonzero submodules $N_1, N_2, \dots$ of M such that $\bigoplus_{i=1}^{\infty} N_i \leq M$. Thus $0 \neq [N_i^{S,\leq}] \leq [M^{S,\leq}]$, and $\bigoplus_{i=1}^{\infty}[N_i^{S,\leq}] \leq [M^{S,\leq}]$. This means that $\text{u.dim}\big([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\big) = \infty$.

Therefore, $\text{u.dim}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \text{u.dim}\,(M_R)$. □

Corollary 2.7. *Let $\alpha \in \text{Aut}(R)$. Then for any right R-module M, we have*

$$\text{u.dim}\left(M[x^{-1}]_{R[[x;\alpha]]}\right) = \text{u.dim}\,(M_R)\,.$$

Any submodule of the additive monoid $\mathbb{N}\cup\{0\}$ is called a *numerical* monoid.

Corollary 2.8. *Let $\alpha \in \text{Aut}(R)$, S a numerical monoid with the usual natural order of $\mathbb{N}\cup\{0\}$ and define $\omega : S \longrightarrow \text{Aut}(R)$ via $\omega_k = \alpha^k$ for every $k \in S$. Then for any right R-module M, we have*

$$\text{u.dim}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \text{u.dim}\,(M_R)\,.$$

Let α and β be ring automorphisms of R such that $\alpha\beta = \beta\alpha$. Let $S = (\mathbb{N}\cup\{0\}) \times (\mathbb{N}\cup\{0\})$ be endowed with the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N}\cup\{0\}$, and define $\omega : S \longrightarrow \text{Aut}(R)$ via $\omega(m,n) = \alpha^m\beta^n$ for any $m, n \in \mathbb{N}\cup\{0\}$. Then $[[R^{S,\leq},\omega]] = R[[x,y;\alpha,\beta]]$ and $[M^{S,\leq}] = M[x^{-1},y^{-1}]$, in which

$$\left(ax^iy^j\right)(bx^py^q) = a\alpha^i\beta^j(b)x^{i+p}y^{j+q},$$

where $i,j,p,q \in \mathbb{N}\cup\{0\}$ and $a,b \in R$, and

$$(mx^{-i}y^{-j})(rx^py^q) = \begin{cases} m\alpha^{-i}\beta^{-j}(r)x^{-i+p}y^{-j+q}, & p \leq i, q \leq j, \\ 0, & \text{otherwise}, \end{cases}$$

where $i,j,p,q \in \mathbb{N}\cup\{0\}$ and $r \in R, m \in M$.

Corollary 2.9. *For any right R-module M, we have*

$$\text{u.dim}\left(M[x^{-1},y^{-1}]_{R[[x,y;\alpha,\beta]]}\right) = \text{u.dim}\,(M_R)\,.$$

Corollary 2.10. *Let $(S,\leq)$ be a strictly totally ordered monoid which is also artinian. Then for any right R-module M, we have*

$$\text{u.dim}\left([M^{S,\leq}]_{[[R^{S,\leq}]]}\right) = \text{u.dim}\,(M_R)\,.$$

If S is the multiplicative monoid $(\mathbb{N},\cdot)$, endowed with the usual order $\leq$, then $[[R^{(\mathbb{N},\cdot),\leq}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution:

$$(fg)(n) = \sum_{d|n} f(d)g(n/d) \quad \text{for each } n \geq 1.$$

If M is a right R-module, then the right $[[R^{(\mathbb{N},\cdot),\leq}]]$-module $[M^{(\mathbb{N},\cdot),\leq}]$ is the set $\left\{\sum_{i=1}^{n} m_ix^{-i} \mid m_i \in M, i = 1,2,\dots,n, n \in \mathbb{N}\right\}$ with scalar multiplication as below:

$$\left(\sum_{j\geq 1} m_jx^{-j}\right)\left(\sum_{i\geq 1} r_ix^i\right) = \sum_{j\geq 1}\left(\sum_{i\geq 1} m_{i\cdot j}r_i\right)x^{-j},$$

where $\sum_{i\geq 1} r_i x^i \in [[R^{(\mathbb{N},\cdot),\leq}]]$ and $\sum_{j\geq 1} m_j x^{-j} \in [M^{(\mathbb{N},\cdot),\leq}]$.

Corollary 2.11. *For any right R-module M, we have*

$$\mathrm{u.dim}\left([M^{(\mathbb{N},\cdot),\leq}]_{[[R^{(\mathbb{N},\cdot),\leq}]]}\right) = \mathrm{u.dim}\,(M_R)\,.$$

3. Couniform dimension

As a dual of the uniform dimension of a module, Varadarajan introduced the couniform dimension of a module in his two 1979 papers [21] and [19], and obtained a number of results on couniform dimension. In [1, 2], Annin obtained result on the couniform dimension of the inverse polynomial module $M[x^{-1}]$ over skew polynomial rings. In this section, we study the couniform dimension of generalized inverse polynomial modules over skew generalized power series rings. At this point, we have everything we will need for our purposes firstly.

Definition 3.1. For any module M_R, we define

$$\mathrm{corank}(M_R) = \sup\{k \mid M_R \text{ surjects onto a direct sum of } k \text{ nonzero modules}\}\,.$$

In particular, $\mathrm{corank}(0) = 0$.

A nonzero module M_R is called *hollow* if the sum of any two of its proper submodules is also a proper submodule. It is easy to see that $\mathrm{corank}(M_R) = 1$ if and only if M_R is hollow. We next record a few of the basic results from [21] and [19] (or see [1, 2]) on couniform dimension that will be needed below. We recall that a submodule K of M_R is called *superfluous* (or *small*) if, for every submodule $N \leq M_R$ with $N + K = M$, we have $N = M$. We will indicate that K is a superfluous submodule of M by the notation $K \ll M$.

Lemma 3.2. (1) *For any right R-module M,* $\mathrm{corank}(M_R) = k < \infty$ *if and only if there exists a surjection $\varphi : M \longrightarrow \bigoplus_{i=1}^k N_i$ with* $\mathrm{Ker}(\varphi) \ll M$ *and all N_i hollow.*

(2) *Let M be a right R-module and $N \leq M$. Then* $\mathrm{corank}(M/N)_R \leq \mathrm{corank}(M_R)$. *In particular, if $N \ll M$, then* $\mathrm{corank}(M_R) = \mathrm{corank}(M/N)_R$.

(3) *Given right R-modules $M_1, M_2, \ldots, M_n$, we have* $\mathrm{corank}(\bigoplus_{i=1}^n M_i) = \sum_{i=1}^n \mathrm{corank}(M_i)$.

For the proof of the main result of this section, there are some central lemmas that we must first establish.

Lemma 3.3. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism, M a right R-module and P a maximal R-submodule of $[M^{S,\leq}]$. For any $s \in S$, set $P_s = \{m \in M \mid \phi_{s,m} \in P\}$. Then for each $s \in S$, either $P_s = M$ or P_s is a maximal R-submodule of M_R. Moreover, there exists $s \in S$ for which the latter holds.*

Proof. Clearly, $\phi_{s,m_1+m_2} = \phi_{s,m_1} + \phi_{s,m_2}$ and $\phi_{s,m}\omega_s(r) = \phi_{s,mr}$ for any $r \in R$ and any $m, m_1, m_2 \in M$. Then it is easy to see that P_s is an R-submodule of M_R. Suppose that for some $s \in S$, $P_s \neq M$. To show that P_s is a maximal submodule of M_R, assume that $P_s \lneq N \leq M_R$. Let $n \in N - P_s$. Then $\phi_{s,n} \notin P$. Thus $[M^{S,\leq}]_R = P + \phi_{s,n}R$ since P is a maximal R-submodule of $[M^{S,\leq}]$. Then, for any $m \in M$, there exist $p \in P$ and $r \in R$ such that $\phi_{s,m} = p + \phi_{s,n}r$. Thus $p = \phi_{s,m} - \phi_{s,n}r \in P$. Note that $\phi_{s,m} - \phi_{s,n}r = \phi_{s,m-n\omega_s^{-1}(r)}$. Thus $m - n\omega_s^{-1}(r) \in P_s \lneq N$. Hence $m \in N$, and so $N = M$. This means that P_s is a maximal R-submodule of M_R. For the last assertion, assume to the contrary, meanly $P_s = M$ for all $s \in S$. Then for any $\varphi \in [M^{S,\leq}]$ and any $x \in S$, $\varphi(x) \in M = P_x$, and so $\phi_{x,\varphi(x)} \in P$. Hence $\varphi = \sum_{x \in S} \phi_{x,\varphi(x)} \in P$. Thus $P = [M^{S,\leq}]$, which contradicts to the hypothesis that P is a maximal R-submodule of $[M^{S,\leq}]$. □

Lemma 3.4. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism, M a right R-module and N a submodule of M. If R is a right perfect ring, then $N \ll M$ if and only if $[N^{S,\leq}] \ll [M^{S,\leq}]$.*

Proof. $\Longleftarrow$) If N is not superfluous in M, then we can find $L \lneq M$ such that $N + L = M$. Then $[M^{S,\leq}] = [N^{S,\leq}] + [L^{S,\leq}]$, and $[L^{S,\leq}] \lneq [M^{S,\leq}]$. This contradicts to the hypothesis that $[N^{S,\leq}] \ll [M^{S,\leq}]$.

$\Longrightarrow$) Suppose that there exists $Q \lneq [M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$ with $Q + [N^{S,\leq}] = [M^{S,\leq}]$. Then $Q_R + [N^{S,\leq}]_R = [M^{S,\leq}]_R$. Since R is a right perfect ring, Q_R is contained in a maximal submodule $P_R \lneq [M^{S,\leq}]_R$. So, $P_R + [N^{S,\leq}]_R = [M^{S,\leq}]_R$, and it is clear that $[N^{S,\leq}]_R \nsubseteq P$. Thus there exists a $\varphi \in [N^{S,\leq}] \backslash P$. Note that $\varphi = \sum_{s \in S} \phi_{s,\varphi(s)}$, there are some $s \in S$ such that $\phi_{s,\varphi(s)} \notin P$. Let $P_s = \{m \in M \mid \phi_{s,m} \in P\}$. Then by Lemma 3.3, either $P_s = M$ or P_s is a maximal R-submodule of M_R. Since $\varphi(s) \in N \backslash P_s$, so the latter option holds. Thus, since $N \nsubseteq P_s$, we have $M_R = P_s + N$. The fact that $P_s \neq M$ now implies that N_R is not superfluous in M_R, completing the proof. □

Following [3], a monoid S is said to be *chain* if the ideals of S are totally ordered by set inclusion, i.e., for any $s, t \in S$, either $s + S \subseteq t + S$ or $t + S \subseteq s + S$.

Lemma 3.5. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism and M a right R-module. If R is a right perfect ring and S a chain monoid, then M is hollow if and only if $[M^{S,\leq}]$ is hollow.*

Proof. $\Longleftarrow$) If M_R is not hollow, we can find $K, L \lneq M$ with $M = K + L$. Then $[M^{S,\leq}] = [K^{S,\leq}] + [L^{S,\leq}]$ with $[K^{S,\leq}], [L^{S,\leq}] \lneq [M^{S,\leq}]$. This contradicts to the fact that $[M^{S,\leq}]$ is hollow.

$\Longrightarrow$) We complete by two steps.

Step 1: We show that if M is simple, then $[M^{S,\leq}]$ is hollow.

For this, firstly, we show that for any $0 \neq Q \lneq [M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$, there exists an $s \in S$ such that $\sigma(\varphi) \leq s$ for any $0 \neq \varphi \in Q$.

Assume the result is false, and let $0 \neq Q \lneq [M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$ be such that for any $s \in S$, there exists $0 \neq \varphi \in Q$ with $\sigma(\varphi) > s$. Let $s \in S$, set

$G_s = \{\phi_{s,m} \mid m \in M\}$. Then, G_s is a right R-module and $G_s \cong M_R$. Thus, G_s is a simple right R-module for any $s \in S$. We show that $G_s \subseteq Q$ by induction on $s \in S$.

For $s = 0$, by the hypothesis, choose a $0 \neq \varphi \in Q$ such that $\sigma(\varphi) > 0$. Assume that $\sigma(\varphi) = u$. Then $\varphi e_u = \phi_{0,\varphi(u)}$. Thus $0 \neq \phi_{0,\varphi(u)} \in Q \cap G_0 \subseteq G_0$. Since G_0 is simple, $G_0 = G_0 \cap Q$. Hence, $G_0 \subseteq Q$.

Now, let $0 < w \in S$. Assume that for any $s < w$, $G_s \subseteq Q$. We will show that $G_w \subseteq Q$. By the hypothesis, there exists a $0 \neq \varphi \in Q$ such that $\sigma(\varphi) > w$. Assume that $\sigma(\varphi) = u$. Since S is a chain monoid, there exists a $v \in S$ such that $u = w + v$. For any $x > w$, since $(S, \leq)$ is a strictly ordered monoid, $u = w + v < x + v$. Thus

$$(\varphi e_v)(x) = \sum_{y \in S} \varphi(x+y)\omega_{x+y}^{-1}(e_v(y)) = \varphi(x+v) = 0.$$

Hence $\sigma(\varphi e_v) \leq w$. Note that

$$(\varphi e_v)(w) = \sum_{y \in S} \varphi(w+y)\omega_{\omega+y}^{-1}(e_v(y)) = \varphi(w+v) = \varphi(u) \neq 0.$$

Thus $\sigma(\varphi e_v) = w$. Hence

$$\varphi e_v = \sum_{x \in S} \phi_{x,(\varphi e_v)(x)} = \phi_{w,\varphi(u)} + \sum_{x<w} \phi_{x,(\varphi e_v)(x)}.$$

By the hypothesis, $\sum_{x<w} \phi_{x,(\varphi e_v)(x)} \in Q$. Hence

$$0 \neq \phi_{w,\varphi(u)} = \varphi e_v - \sum_{x<w} \phi_{x,(\varphi e_v)(x)} \in Q \cap G_w \subseteq G_w.$$

Since G_w is a simple right R-module, $Q \cap G_w = G_w$, so $G_w \subseteq Q$.

Therefore, by transfinite induction, we have shown that for any $s \in S$, $G_s \subseteq Q$. Thus, for any $\varphi \in [M^{S,\leq}]$, since $\varphi = \sum_{x \in S} \phi_{x,\varphi(x)}$, we have $\varphi \in Q$. Hence $Q = [M^{S,\leq}]$, which contradicts to the fact that Q is a proper submodule of $[M^{S,\leq}]$. Therefore, for any $0 \neq Q \lneq [M^{S,\leq}]$, there exists an $s \in S$ such that $\sigma(\varphi) \leq s$ for any $0 \neq \varphi \in Q$.

Now, we show that $[M^{S,\leq}]$ is a hollow module. Let $0 \neq P,\ Q \lneq [M^{S,\leq}]$. Then there exists $0 \neq u \in S$ such that $\sigma(\varphi) \leq u, \sigma(\psi) \leq u$ for any $0 \neq \varphi \in P$ and any $0 \neq \psi \in Q$. Thus, for any $0 \neq m \in M$, we have $\phi_{2u,m} \in [M^{S,\leq}] \backslash (P + Q)$. This implies that $[M^{S,\leq}]$ is a hollow right $[[R^{S,\leq}, \omega]]$-module.

Step 2: We show that if M is a hollow right R-module, then $[M^{S,\leq}]$ is a hollow right $[[R^{S,\leq}, \omega]]$-module.

Assume that there exist $Q, Q' \lneq [M^{S,\leq}]_{[[R^{S,\leq},\omega]]}$ such that $Q + Q' = [M^{S,\leq}]$. Since R is a right perfect ring, M_R has a maximal submodule, say L. Then $L \ll M$ since M_R is hollow. By Lemma 3.4, $[L^{S,\leq}] \ll [M^{S,\leq}]$. Hence, $[L^{S,\leq}] + Q$ and $[L^{S,\leq}] + Q'$ are both proper submodules of $[M^{S,\leq}]$. If $[L^{S,\leq}] + Q = [L^{S,\leq}]$, then $[L^{S,\leq}] + Q' = [L^{S,\leq}] + Q + Q' = [M^{S,\leq}]$, a contradiction. Thus $[L^{S,\leq}] \lneq$

$[L^{S,\leq}] + Q$. Similarly, $[L^{S,\leq}] \subsetneq [L^{S,\leq}] + Q'$. Hence, $([L^{S,\leq}] + Q)/[L^{S,\leq}]$, $([L^{S,\leq}] + Q')/[L^{S,\leq}]$ are all proper submodules of $[M^{S,\leq}]/[L^{S,\leq}]$, and

$$([L^{S,\leq}] + Q)/[L^{S,\leq}] + ([L^{S,\leq}] + Q')/[L^{S,\leq}] = [M^{S,\leq}]/[L^{S,\leq}].$$

On the other hand, since $[M^{S,\leq}]/[L^{S,\leq}] \cong [(M/L)^{S,\leq}]$, and M/L is a simple right R-module, by Step 1, $[(M/L)^{S,\leq}]$ is hollow. Hence $[M^{S,\leq}]/[L^{S,\leq}]$ is hollow, a contradiction. Now the result follows. □

Now, we prove our second main result.

Theorem 3.6. *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. If R is a right perfect ring and S is a chain monoid, then for any right R-module M, we have*

$$\mathrm{corank}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \mathrm{corank}\left(M_R\right).$$

Proof. Suppose first that $\mathrm{corank}(M_R) = n < \infty$. By Lemma 3.2(1), we have hollow R-modules $H_1, H_2, \ldots, H_n$, and a surjection $\alpha : M \longrightarrow \bigoplus_{i=1}^n H_i$ such that $\mathrm{Ker}\alpha \ll M_R$. Define $\beta : [M^{S,\leq}] \longrightarrow \left[(\bigoplus_{i=1}^n H_i)^{S,\leq}\right]$ via:

$$\beta(\varphi)(s) = \alpha(\varphi(s)), \quad \forall \varphi \in [M^{S,\leq}],\ \forall s \in S.$$

Since $\mathrm{supp}(\varphi)$ is finite, $\mathrm{supp}(\beta(\varphi))$ finite. Thus β is well-defined. For any $f \in [[R^{S,\leq}, \omega]]$, any $\varphi \in [M^{S,\leq}]$ and any $s \in S$,

$$\begin{aligned}
\beta(\varphi f)(s) &= \alpha\left((\varphi f)(s)\right) = \alpha\left(\sum_{x \in S} \varphi(x+s)\omega_{x+s}^{-1}\big(f(x)\big)\right) \\
&= \sum_{x \in S} \alpha\left(\varphi(x+s)\omega_{x+s}^{-1}\left(f(x)\right)\right) = \sum_{x \in S} \alpha\left(\varphi(x+s)\right)\omega_{x+s}^{-1}\left(f(x)\right) \\
&= \sum_{x \in S} \beta(\varphi)(x+s)\omega_{x+s}^{-1}\left(f(x)\right) = (\beta(\varphi) f)(s).
\end{aligned}$$

Thus $\beta(\varphi f) = \beta(\varphi) f$. Now, it is easy to see that β is a right $[[R^{S,\leq}, \omega]]$-homomorphism.

Let $\psi \in \left[(\bigoplus_{i=1}^n H_i)^{S,\leq}\right]$. Then for any $s \in \mathrm{supp}(\psi)$, there exists an $m_s \in M$ such that $\alpha(m_s) = \psi(s)$. Define $\varphi \in [M^{S,\leq}]$ via:

$$\varphi(s) = \begin{cases} m_s, & s \in \mathrm{supp}(\psi), \\ 0, & s \notin \mathrm{supp}(\psi). \end{cases}$$

Then for any $s \in S$,

$$\beta(\varphi)(s) = \alpha(\varphi(s)) = \alpha(m_s) = \psi(s).$$

Thus $\beta(\varphi) = \psi$. Hence β is an epimorphism. Since

$$\begin{aligned}\mathrm{Ker}\beta &= \left\{\varphi \in [M^{S,\leq}] \mid \beta(\varphi) = 0\right\} \\ &= \left\{\varphi \in [M^{S,\leq}] \mid \alpha(\varphi(s)) = 0,\ \forall s \in S\right\} \\ &= \left\{\varphi \in [M^{S,\leq}] \mid \varphi(s) \in \mathrm{Ker}\alpha,\ \forall s \in S\right\} \\ &= \left[(\mathrm{Ker}\alpha)^{S,\leq}\right],\end{aligned}$$

so Lemma 3.4 implies that $\mathrm{Ker}\beta = \left[(\mathrm{Ker}\alpha)^{S,\leq}\right] \ll [M^{S,\leq}]$. Thus, by Lemma 3.2(2), we have

$$\mathrm{corank}([M^{S,\leq}]) = \mathrm{corank}\left([M^{S,\leq}]/\mathrm{Ker}\beta\right) = \mathrm{corank}\Big([(\bigoplus_{i=1}^{n} H_i)^{S,\leq}]\Big).$$

Note that $[(\bigoplus_{i=1}^{n} H_i)^{S,\leq}] \cong \bigoplus_{i=1}^{n}[H_i^{S,\leq}]$, we have

$$\mathrm{corank}([M^{S,\leq}]) = \mathrm{corank}\Big(\bigoplus_{i=1}^{n}[H_i^{S,\leq}]\Big).$$

Now, Lemma 3.2(3) implies that,

$$\mathrm{corank}([M^{S,\leq}]) = \mathrm{corank}\Big(\bigoplus_{i=1}^{n}[H_i^{S,\leq}]\Big) = \sum_{i=1}^{n}\mathrm{corank}\big([H_i^{S,\leq}]\big).$$

Since $H_1, H_2, \ldots, H_n$ are all hollow modules, by Lemma 3.5, $[H_1^{S,\leq}], [H_2^{S,\leq}], \ldots, [H_n^{S,\leq}]$ are hollow modules. Thus, $\mathrm{corank}([H_i^{S,\leq}]) = 1, i = 1, 2, \ldots, n$. Hence

$$\mathrm{corank}([M^{S,\leq}]) = \sum_{i=1}^{n}\mathrm{corank}\big([H_i^{S,\leq}]\big) = n.$$

Secondly, if $\mathrm{corank}(M_R) = \infty$, then for arbitrarily large k, we have a surjection $\alpha_k : M \longrightarrow \bigoplus_{i=1}^{k} N_i$ with $N_i \neq 0$. This induces a surjection $\beta_k : [M^{S,\leq}] \longrightarrow \bigoplus_{i=1}^{k}[N_i^{S,\leq}]$ for each such k, which shows that $\mathrm{corank}([M^{S,\leq}]) = \infty$.

Therefore, $\mathrm{corank}\left([M^{S,\leq}]_{[[R^{S,\leq}]]}\right) = \mathrm{corank}\left(M_R\right)$. □

Remark ([2, Example 2.7]). implied that the corank of $M[x^{-1}]$ may not equivalent to the corank of M_R without the assumption that R is a right perfect ring. Thus, the right perfect condition of the ring R is also essential in Theorem 3.6.

Corollary 3.7. *Let R be a right perfect ring and $\alpha \in \mathrm{Aut}(R)$. Then for any right R-module M, we have*

$$\mathrm{corank}\left(M[x^{-1}]_{R[[x;\alpha]]}\right) = \mathrm{corank}\left(M_R\right).$$

Corollary 3.8. *Let R be a right perfect ring, $\alpha \in \mathrm{Aut}(R)$ and S a numerical monoid with the usual natural order of $\mathbb{N} \cup \{0\}$ and define $\omega : S \longrightarrow \mathrm{Aut}(R)$ via $\omega_k = \alpha^k$ for every $k \in S$. Then for any right R-module M, we have*

$$\mathrm{corank}\left([M^{S,\leq}]_{[[R^{S,\leq},\omega]]}\right) = \mathrm{corank}\left(M_R\right).$$

Corollary 3.9. *If R is a right perfect ring, then for any right R-module M, we have*

$$\operatorname{corank}\left(M[x^{-1},y^{-1}]_{R[[x,y;\alpha,\beta]]}\right)=\operatorname{corank}\left(M_R\right).$$

Acknowledgments. The author wishes to thank the referee for his/her valuable suggestions. This research is supported by the National Natural Science Foundation of China (10961021) and NWNU-LKQN-11-6.

References

[1] S. Annin, *Associated and attached primes over noncommutative rings*, Ph. D. Diss., University of California at Berkeley, 2002.
[2] ———, *Couniform dimension over skew polynomial rings*, Comm. Algebra **33** (2005), no. 4, 1195–1204.
[3] M. Ferrero, R. Mazurek, and A. Sant'Ana, *On right chain semigroups*, J. Algebra **292** (2005), no. 2, 574–584.
[4] P. Grzeszczuk, *Goldie dimension of differential operator rings*, Comm. Algebra **16** (1988), no. 4, 689–701.
[5] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics volume **189**, Springer-Verlag, Berlin-Heidelberg-New York, 1999.
[6] Z. K. Liu, *Endomorphism rings of modules of generalized inverse polynomials*, Comm. Algebra **28** (2000), no. 2, 803–814.
[7] ———, *Injectivity of modules of generalized inverse polynomials*, Comm. Algebra **29** (2001), no. 2, 583–592.
[8] ———, *Injective precover and modules of generalized inverse polynomials*, Chin. Ann. Math. Ser. B **25** (2004), no. 1, 129–138.
[9] ———, *Triangular matrix representations of rings of generalized power series*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 4, 989–998.
[10] Z. K. Liu and H. Cheng, *Quasi-duality for the rings of generalized power series*, Comm. Algebra **28** (2000), no. 3, 1175–1188.
[11] Z. K. Liu and Y. Fan, *Co-Hopfian modules of generalized inverse polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), no. 3, 431–436.
[12] J. Matczuk, *Goldie rank of Ore extensions*, Comm. Algebra **23** (1995), no. 4, 1455–1471.
[13] R. Mazurek and M. Ziembowski, *Uniserial rings of skew generalized power series*, J. Algebra **318** (2007), no. 2, 737–764.
[14] A. S. McKerrow, *On the injective dimension of modules of power series*, Quart. J. Math. Oxford Ser. (2) **25** (1974), 359–368.
[15] D. G. Northcott, *Injective envelopes and inverse polynomials*, London Math. Soc. **8** (1974), 290–296.
[16] S. Park, *The Macaulay-Northcott functor*, Arch. Math. **63** (1994), no. 3, 225–230.
[17] ———, *Inverse polynomials and injective covers*, Comm. Algebra **21** (1993), no. 12, 4599–4613.
[18] P. Ribenboim, *Semisimple rings and von Neumann regular rings of generalized power series*, J. Algebra **198** (1997), no. 2, 327–338.
[19] B. Sarath and K. Varadarajan, *Dual Goldie dimension II*, Comm. Algebra **7** (1979), no. 17, 1885–1899.
[20] R. C. Shock, *Polynomial rings over finite dimensional rings*, Pacific J. Math. **42** (1972), 251–257.
[21] K. Varadarajan, *Dual Goldie dimension*, Comm. Algebra **7** (1979), no. 6, 565–610.
[22] ———, *On a theorem of Shock*, Comm. Algebra **10** (1982), no. 20, 2205–2222.
[23] ———, *Dual Goldie dimension of certain extension rings*, Comm. Algebra **10** (1982), no. 20, 2223–2231.

[24] R. Y. Zhao and Z. K. Liu, *Artinness of generalized Macaulay-Northcott modules*, Comm. Algebra **37** (2009), no. 2, 525–531.

College of Economics and Management
Northwest Normal University
Lanzhou 730070, P. R. China
E-mail address: renyuzhao026@gmail.com