

**ON SOME EXACT RECIPROCITY FORMULAS
FOR TWISTED SECOND MOMENTS
OF DIRICHLET L -FUNCTIONS**

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ABSTRACT. We derive the exact reciprocity formula connecting the twisted second moments at 1 for families of odd Dirichlet characters corresponding to two prime moduli, which parallels such formula discovered earlier for the special point $1/2$.

1. Introduction

For a Dirichlet character χ modulo q , let

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1,$$

denote the corresponding Dirichlet L -function. Understanding the special values of this function is one of the central and most classical goals in number theory. We mention for example, the relation between the class number of a quadratic field and $L(1, \chi)$ for quadratic character χ and related problems leading to the investigation of so called Landau-Siegel zeros.

One of the main tools in the study of special values of L -functions is asymptotic calculation or bounding of various moments in the associated families. Asymptotic evaluation of twisted moments is the main building block in the processes of mollification (leading to nonvanishing results) or amplification (leading to subconvexity results).

Motivation for this note is the reciprocity phenomenon discovered by Conrey in [2], Theorem 10, between the twisted second moments at $1/2$:

$$\sum_{\chi \bmod p}^* |L(1/2, \chi)|^2 \chi(h) \quad \text{and} \quad \sum_{\chi \bmod h}^* |L(1/2, \chi)|^2 \chi(-p),$$

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where \sum^* denotes the summation over primitive characters only and $h < p$ are prime numbers. Reciprocity is given in the form of the following *asymptotic* formula:

$$\begin{aligned}
 (1.1) \quad & \sum_{\chi \bmod p}^* |L(1/2, \chi)|^2 \chi(h) \\
 &= \frac{\sqrt{p}}{\sqrt{h}} \sum_{\chi \bmod h}^* |L(1/2, \chi)|^2 \chi(-p) + \frac{p}{\sqrt{h}} \left(\gamma - \log 8\pi + \log \frac{p}{h} \right) \\
 & \quad + \zeta(1/2)^2 \sqrt{p} + O(hp^{-\frac{1}{2}+\varepsilon} + h^{-\frac{1}{2}+\varepsilon} p^{\frac{1}{2}}).
 \end{aligned}$$

This improved error term and hence asymptotic valid uniformly for $h < p^{1-\varepsilon}$, is obtained by M. Young in [5], with a different method. For $h = O(p^{1/2})$ the second term on the right in (1.1) is dominating, but for larger h the first term sometimes dominates (for example if $p \equiv -1 \pmod{h}$), illustrating the significance of “reciprocity” terms.

In this note we are interested in the corresponding twisted second moments at positive integral points and obtain the *exact* reciprocity formula between them in the special point $m = 1$.

Definitions of Bernoulli polynomials $B_m(x)$, functions $\bar{B}_m(x)$ and numbers B_m are recalled in Section 2. Let then

$$s_m(h, d) = \frac{1 + (-1)^m}{2} B_m^2 + \sum_{b=1}^{d-1} \bar{B}_m\left(\frac{b}{d}\right) \bar{B}_m\left(\frac{bh}{d}\right),$$

where the sum on the right is empty for $d = 1$.

Theorem 1.1. *For $m \in \mathbb{Z}_{>0}$, the family of Dirichlet characters χ to modulus $q \geq 3$ and of the same parity as m , and for a positive integer h , $(h, q) = 1$ we have:*

$$(1.2) \quad \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^m}} |L(m, \chi)|^2 \chi(h) = \frac{(2\pi)^{2m} \phi(q)}{4(m!)^2 q^{2m}} \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{2m-1} s_m(h, d).$$

If we specialize to the case of *prime* $q \geq 3$ and m *odd*, we get for $(h, q) = 1$:

$$(1.3) \quad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(m, \chi)|^2 \chi(h) = \frac{(2\pi)^{2m} \phi(q)}{4(m!)^2} \frac{1}{q} s_m(h, q).$$

Here we recall that

$$s_1(h, q) = \sum_{b=1}^{q-1} \bar{B}_1\left(\frac{b}{q}\right) \bar{B}_1\left(\frac{hb}{q}\right),$$

defined for positive integers h, q , $(h, q) = 1$, agrees with the classical Dedekind sum. For these sums holds Dedekind’s reciprocity formula: for positive integers

$(h, q) = 1$ we have

$$s_1(h, q) + s_1(q, h) = -\frac{1}{4} + \frac{1}{12} \left(\frac{q}{h} + \frac{h}{q} + \frac{1}{qh} \right).$$

This, together with (1.3) gives the following *exact* reciprocity formula for twisted second moments at the point 1:

Corollary 1.2. *For any two different prime numbers $q, h \geq 3$, we have:*

$$\begin{aligned} & \frac{q}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \chi(h) + \frac{h}{\phi(h)} \sum_{\substack{\chi \pmod{h} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \chi(q) \\ &= \frac{\pi^2}{12} \left(\frac{q}{h} + \frac{h}{q} + \frac{1}{qh} - 3 \right). \end{aligned}$$

We remark that the reciprocity in (1.1) in the proof from [5], comes from switching moduli in analysis of the corresponding divisor sums, while in this note, the source of this phenomenon is genuine reciprocity formula for Dedekind sums. On the other hand, one parallel in both cases is the restriction that both q and h are primes, necessary to obtain such simple reciprocity formulas.

More generally, the sums $s_m(h, d)$ appearing in (1.2) are closely related to the following generalized Dedekind-Rademacher sums:

$$(1.4) \quad S_{m,n}(a, b, c) = \sum_{\alpha \pmod{c}} \bar{B}_m \left(\frac{a\alpha}{c} \right) \bar{B}_n \left(\frac{b\alpha}{c} \right).$$

For example, for even m , we have that $s_m(h, q) = S_{m,m}(1, h, q)$.

There also exist reciprocity formulas connecting the sums (1.4), first proved in [1] and then generalized in many directions by various authors, cf. [3] for example. But in these reciprocity laws sums $S_{m,m}$ appear together with $S_{m,n}$, $m \neq n$, which when translated to twisted moments of Dirichlet L -functions, means that we need to consider the moments $\sum_{\chi} L(m, \chi)L(n, \bar{\chi})\chi(h)$ also. Calculation of these twisted moments for different m and n could be done analogously (using methods from [4] for example), but we refrain giving formulas connecting these “mixed” moments since they can be easily reconstructed from generalized reciprocity formulae for Dedekind-Rademacher sums.

For example, one such reciprocity formula for $S_{2,2}$ (hence for twisted second moment at 2) which could be found in [3] is the following:

$$\begin{aligned} \frac{6}{ab} S_{2,2}(a, b, c) &= \frac{4}{a^2} (S_{3,1}(a, b, c) + S_{3,1}(a, c, b)) + \frac{4}{b^2} (S_{3,1}(b, a, c) + S_{3,1}(b, c, a)) \\ &\quad - B_4 \left(\frac{a(b, c)^4}{b^3 c^3} + \frac{b(a, c)^4}{a^3 c^3} + \frac{3c(a, b)^4}{a^3 b^3} \right). \end{aligned}$$

Remark about notation: We will use $e(x)$ and $e_q(x)$ instead of $e^{2\pi i x}$ and $e^{2\pi i x/q}$, respectively, and the symbol \sum^* represents summation only over primitive residues or primitive characters. Moreover, the fractional part of $x \in \mathbb{R}$ is denoted by $\{x\} = x - [x]$.

2. Preliminaries

The classical Gauss sum, i.e., the discrete Fourier transform of a Dirichlet character χ modulo q is defined by:

$$\widehat{\chi}(a) = \frac{1}{\sqrt{q}} \sum_{x \bmod q} \chi(x) e_q(-ax).$$

Also we recall that Ramanujan's sums for $(n, q) = 1$ can be evaluated as

$$(2.1) \quad r_q(n) := \sum_{\alpha(q)}^* e_q(n\alpha) = \sum_{d|(n,q)} \mu\left(\frac{q}{d}\right) d.$$

Let $B_m(x)$ be the standard m -th Bernoulli polynomial, defined by the generating power series:

$$\frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} z^m, \quad |z| < 2\pi.$$

It satisfies the symmetry formula

$$(2.2) \quad B_m(1-x) = (-1)^m B_m(x)$$

and the following multiplication identity for $t \in \mathbb{Z}_{>0}$:

$$(2.3) \quad \sum_{\alpha=0}^{t-1} B_m\left(x + \frac{\alpha}{t}\right) = t^{1-m} B_m(tx).$$

Also let $\bar{B}_m(x)$ denotes the periodic extension into \mathbb{R} of the Bernoulli polynomial $B_m(x)$ on $[0, 1)$, that is $\bar{B}_m(x) := B(\{x\})$, with the exception that for $m = 1$ we set $\bar{B}_1(0) = 0$.

Bernoulli numbers are defined by $B_m = B_m(0)$ and we recall that for positive even m we have:

$$(2.4) \quad \zeta(m) = -\frac{(2\pi i)^m}{2(m!)} B_m.$$

For $m \in \mathbb{Z}_{>0}$ and a non-principal Dirichlet character χ modulo q of the same parity as m , $\chi(-1) = (-1)^m$, we have (Lemma 1 in [4]; the term $a = q$ is included since $\widehat{\chi}(q) = 0$ for nonprincipal characters):

$$(2.5) \quad L(m, \chi) = -\frac{1}{2\sqrt{q}} \frac{(2\pi i)^m}{m!} \sum_{a=1}^q \widehat{\chi}(a) B_m\left(\frac{a}{q}\right).$$

Jordan's totient function of order $m \geq 1$ is defined as

$$J_m(q) = \sum_{d|q} \mu\left(\frac{q}{d}\right) d^m = q^m \prod_{p|q} \left(1 - \frac{1}{p^m}\right),$$

and for the principal character χ_0 modulo q we have the following formula:

$$(2.6) \quad L(m, \chi_0) = \zeta(m) \frac{J_m(q)}{q^m}.$$

3. Twisted traces lemmas

Lemma 3.1. *For the family of even non-principal Dirichlet characters to modulus $q \geq 3$ and integers a, b and h such that $(h, q) = 1$, we have:*

$$\begin{aligned}
 (3.1) \quad & \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \widehat{\chi}(a)\widehat{\chi}(b)\chi(h) \\
 &= \frac{\phi(q)}{2q} \sum_{d|(a \pm bh, q)} \mu\left(\frac{q}{d}\right) d - \frac{1}{q} \sum_{d_1|(a, q)} \mu\left(\frac{q}{d_1}\right) d_1 \sum_{d_2|(b, q)} \mu\left(\frac{q}{d_2}\right) d_2.
 \end{aligned}$$

Proof. We first complete the sum to all even characters in order to exploit the orthogonality

$$\sum_{\substack{\chi \pmod q \\ \chi(-1)=1}} \chi(n) = \begin{cases} \frac{\phi(q)}{2}, & \text{if } n \equiv \pm 1 \pmod q \\ 0, & \text{otherwise,} \end{cases}$$

and then open the Gauss sums:

$$\begin{aligned}
 \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} &= \sum_{\substack{\chi \pmod q \\ \chi(-1)=1}} -\widehat{\chi_0}(a)\widehat{\chi_0}(b) \\
 &= \frac{1}{q} \sum_{\alpha(q)}^* \sum_{\beta(q)}^* e_q(-a\alpha - b\beta) \sum_{\substack{\chi \pmod q \\ \chi(-1)=1}} \chi(h\alpha)\overline{\chi}(\beta) \\
 &\quad - \frac{1}{q} \sum_{\alpha(q)}^* e_q(a\alpha) \sum_{\beta(q)}^* e_q(b\beta) \\
 &= \frac{\phi(q)}{2q} \sum_{\alpha(q)}^* (e_q(-a\alpha - bh\alpha) + e_q(-a\alpha + bh\alpha)) - \frac{1}{q} r_q(a)r_q(b) \\
 &= \frac{\phi(q)}{2q} (r_q(a + bh) + r_q(a - bh)) - \frac{1}{q} r_q(a)r_q(b).
 \end{aligned}$$

Here, substitution of (2.1) gives the required formula. □

Similarly, using the orthogonality of odd Dirichlet characters,

$$(3.2) \quad \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \chi(m)\overline{\chi}(n) = \begin{cases} \frac{\phi(q)}{2}, & \text{if } m \equiv n \pmod q, \\ -\frac{\phi(q)}{2}, & \text{if } m \equiv -n \pmod q, \\ 0, & \text{otherwise,} \end{cases}$$

which holds for integers $q \geq 3$, m and n such that $(mn, q) = 1$, one can obtain:

Lemma 3.2. *For the family of odd Dirichlet characters to modulus $q \geq 3$ and integers a, b and h such that $(h, q) = 1$, we have:*

$$\begin{aligned}
 (3.3) \quad \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} \widehat{\chi}(a)\widehat{\chi}(b)\chi(h) &= \frac{\phi(q)}{2q} (r_q(a + bh) - r_q(a - bh)) \\
 &= \frac{\phi(q)}{2q} \left(\sum_{d|(q, a+bh)} \mu\left(\frac{q}{d}\right) d - \sum_{d|(q, a-bh)} \mu\left(\frac{q}{d}\right) d \right).
 \end{aligned}$$

4. Proof of Theorem 1.1

For the family of *even* characters and an even integer m , $\chi(-1) = (-1)^m = 1$, we have by (2.5):

$$\begin{aligned}
 S(h) &= \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=1}} |L(m, \chi)|^2 \chi(h) \\
 &= L(m, \chi_0)^2 + \frac{(-1)^m (2\pi)^{2m}}{4q (m!)^2} \sum_{a=1}^q \sum_{b=1}^q B_m\left(\frac{a}{q}\right) B_m\left(\frac{b}{q}\right) \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \widehat{\chi}(a)\widehat{\chi}(b)\chi(h).
 \end{aligned}$$

Using (3.1), we have further

$$\begin{aligned}
 S(h) &= L(m, \chi_0)^2 + \frac{(-1)^m (2\pi)^{2m} \phi(q)}{8 (m!)^2 q^2} \sum_{d|q} \mu\left(\frac{q}{d}\right) d \sum_{\substack{a=1 \\ d|a \pm bh}}^q \sum_{b=1}^q B_m\left(\frac{a}{q}\right) B_m\left(\frac{b}{q}\right) \\
 &\quad - \frac{(-1)^m (2\pi)^{2m}}{4 (m!)^2 q^2} \sum_{d_1|q} \mu\left(\frac{q}{d_1}\right) d_1 \sum_{\substack{a=1 \\ d_1|a}}^q B_m\left(\frac{a}{q}\right) \sum_{d_2|q} \mu\left(\frac{q}{d_2}\right) d_2 \sum_{\substack{b=1 \\ d_2|b}}^q B_m\left(\frac{b}{q}\right).
 \end{aligned}$$

Here we have (using that $B_m(1) = B_m(0)$ for an even integer m , so that we can replace $\sum_{a=1}^q$ with $\sum_{a=0}^{q-1}$):

$$\begin{aligned}
 &\sum_{\substack{a=1 \\ d|a \pm bh}}^q \sum_{b=1}^q B_m\left(\frac{a}{q}\right) B_m\left(\frac{b}{q}\right) = \sum_{\substack{a=0 \\ d|a \pm bh}}^{q-1} \sum_{b=0}^{q-1} B_m\left(\frac{a}{q}\right) B_m\left(\frac{b}{q}\right) \\
 &= \sum_{b=0}^{d-1} \sum_{\beta=0}^{q/d-1} B_m\left(\frac{b+d\beta}{q}\right) \sum_{\alpha=0}^{q/d-1} \left(B_m\left(\frac{bh - [\frac{bh}{d}]d + d\alpha}{q}\right) + B_m\left(\frac{d - (bh - [\frac{bh}{d}]d) + d\alpha}{q}\right) \right),
 \end{aligned}$$

which is, using (2.3) further equal to

$$= \left(\frac{q}{d}\right)^{2-2m} \sum_{b=0}^{d-1} B_m\left(\frac{b}{d}\right) \left(B_m\left(\frac{bh - [\frac{bh}{d}]d}{d}\right) + B_m\left(\frac{d + [\frac{bh}{d}]d - bh}{d}\right) \right),$$

and by exploiting the symmetry (2.2),

$$= 2 \left(\frac{q}{d}\right)^{2-2m} \sum_{b=0}^{d-1} B_m \left(\frac{b}{d}\right) B_m \left(\left\{\frac{bh}{d}\right\}\right).$$

Therefore,

$$S(h) = \frac{(-1)^m (2\pi)^{2m} \phi(q)}{4 (m!)^2 q^{2m}} \sum_{d|q} \mu \left(\frac{q}{d}\right) d^{2m-1} \sum_{b=0}^{d-1} B_m \left(\frac{b}{d}\right) B_m \left(\left\{\frac{bh}{d}\right\}\right) + L(m, \chi_0)^2 - \frac{(-1)^m (2\pi)^{2m} B_m^2}{4 (m!)^2 q^{2m}} \left(\sum_{d|q} \mu \left(\frac{q}{d}\right) d^m\right)^2.$$

Formulas (2.4) and (2.6) show that the second line is 0, proving the theorem in even case.

For the family of *odd* characters and an odd integer m , $\chi(-1) = (-1)^m = -1$, we have similarly using (3.3), (2.2) and (2.3):

$$S(h) = \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(m, \chi)|^2 \chi(h) = \frac{(2\pi i)^{2m} \phi(q)}{8(m!)^2 q^2} \sum_{d|q} \mu \left(\frac{q}{d}\right) d \left(\sum_{\substack{a=1 \\ d|a+bh}}^q \sum_{b=1}^q B_m \left(\frac{a}{q}\right) B_m \left(\frac{b}{q}\right) - \sum_{\substack{a=1 \\ d|a-bh}}^q \sum_{b=1}^q B_m \left(\frac{a}{q}\right) B_m \left(\frac{b}{q}\right) \right).$$

Expression in the bracket is equal to

$$\begin{aligned} & \sum_{\substack{a=0 \\ d|a+bh}}^{q-1} \sum_{b=0}^{q-1} B_m \left(\frac{a}{q}\right) B_m \left(\frac{b}{q}\right) - \sum_{\substack{a=0 \\ d|a-bh}}^{q-1} \sum_{b=0}^{q-1} B_m \left(\frac{a}{q}\right) B_m \left(\frac{b}{q}\right) \\ &= \sum_{b=0}^{d-1} \sum_{\beta=0}^{q/d-1} B_m \left(\frac{b+d\beta}{q}\right) \left(\sum_{\substack{a=0 \\ a \equiv -bh \pmod{d}}}^{q-1} B_m \left(\frac{a}{q}\right) - \sum_{\substack{a=0 \\ a \equiv bh \pmod{d}}}^{q-1} B_m \left(\frac{a}{q}\right) \right) \\ &= \sum_{b=0}^{d-1} \left(\frac{q}{d}\right)^{1-m} B_m \left(\frac{b}{d}\right) \left(\sum_{\alpha=0}^{q/d-1} B_m \left(\frac{(d-b)h - \lfloor \frac{(d-b)h}{d} \rfloor d + \alpha d}{q}\right) \right. \\ & \quad \left. - \sum_{\alpha=0}^{q/d-1} B_m \left(\frac{bh - \lfloor \frac{bh}{d} \rfloor d + \alpha d}{q}\right) \right) \\ &= \left(\frac{q}{d}\right)^{2-2m} \sum_{b=0}^{d-1} B_m \left(\frac{b}{d}\right) \left(B_m \left(\frac{(d-b)h - \lfloor \frac{(d-b)h}{d} \rfloor d}{d}\right) - B_m \left(\frac{bh - \lfloor \frac{bh}{d} \rfloor d}{d}\right) \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{q}{d}\right)^{2-2m} \sum_{b=1}^{d-1} B_m\left(\frac{b}{d}\right) \left(B_m\left(1 - \left\{\frac{bh}{d}\right\}\right) - B_m\left(\left\{\frac{bh}{d}\right\}\right) \right) \\
&= -2 \left(\frac{q}{d}\right)^{2-2m} \sum_{b=1}^{d-1} B_m\left(\frac{b}{d}\right) B_m\left(\left\{\frac{bh}{d}\right\}\right),
\end{aligned}$$

where for $d = 1$ the sum is understood to be 0.

Substituting obtained back, we get

$$S(h) = -\frac{(2\pi i)^{2m} \phi(q)}{4(m!)^2 q^{2m}} \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{2m-1} \sum_{b=1}^{d-1} B_m\left(\frac{b}{d}\right) B_m\left(\left\{\frac{bh}{d}\right\}\right),$$

so that in both even and odd cases we obtain the required formula:

$$\sum_{\substack{\chi \bmod q \\ \chi(-1) = (-1)^m}} |L(m, \chi)|^2 \chi(h) = (-1)^m \frac{(2\pi i)^{2m} \phi(q)}{4(m!)^2 q^{2m}} \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{2m-1} s_m(h, d).$$

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