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AMALGAMATED DUPLICATION OF SOME SPECIAL RINGS

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Abstract. Let R be a commutative Noetherian ring and let I be an ideal of R. In this paper we study the amalgamated duplication ring $R \bowtie I$ which is introduced by D'Anna and Fontana. It is shown that if R is generically Cohen-Macaulay (resp. generically Gorenstein) and I is generically maximal Cohen-Macaulay (resp. generically canonical module), then $R \bowtie I$ is generically Cohen-Macaulay (resp. generically Gorenstein). We also defined generically quasi-Gorenstein ring and we investigate when $R \bowtie I$ is generically quasi-Gorenstein. In addition, it is shown that $R \bowtie I$ is approximately Cohen-Macaulay if and only if R is approximately Cohen-Macaulay, provided some special conditions. Finally it is shown that if R is approximately Gorenstein, then $R \bowtie I$ is approximately Gorenstein.

1. Introduction

Throughout this paper all rings are considered commutative with identity element and all ring homomorphisms are unital. In [8], D'Anna and Fontana considered a different type of construction obtained involving a ring R and an ideal $I \subset R$ that is denoted by $R \bowtie I$, called amalgamated duplication, and it is defined as the following subring of $R \times R$:

$$R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\}\cdot$$

In [6] D'Anna showed that if R is a Noetherian local ring, then $R \bowtie I$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is maximal Cohen-Macaulay. In [1] it is shown that if R is a Noetherian local ring, then $R \bowtie I$ is Gorenstein if and only if R is Cohen-Macaulay and I is a canonical module for R, and then R/I is Cohen-Macaulay of dimension $\dim(R) - 1$. In this paper it is shown that if $R \bowtie I$ is a Gorenstein ring where I is a non-zero flat ideal of Noetherian zero dimensional ring R, then R is Gorenstein (see Proposition 2.2). Recently, the authors in [4] showed that if R is a Noetherian local ring and I is a proper ideal of R such that $\operatorname{Ann}_R(I) = 0$, then $R \bowtie I$ is a quasi-Gorenstein

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ring if and only if $\widehat{R}$ satisfies Serre's condition (S_2) and I is a canonical ideal of R.

Recall that a Noetherian ring R is called generically Cohen-Macaulay (resp. generically Gorenstein) if the ring $R_\mathfrak{p}$ is Cohen-Macaulay (resp. Gorenstein) for all $\mathfrak{p} \in \operatorname{Ass}(R)$. Every Cohen-Macaulay (resp. Gorenstein) ring is also generically Cohen-Macaulay (resp. generically Gorenstein) and every Artinian generically Cohen-Macaulay (resp. generically Gorenstein) ring is Cohen-Macaulay (resp. Gorenstein). In Section 2 we define a generically quasi-Gorenstein ring and we investigate when $R \bowtie I$ is a generically Cohen-Macaulay (resp. generically Gorenstein, generically quasi-Gorenstein) ring (see Theorem 2.8 and Proposition 2.9).

In [9] Goto defined approximately Cohen-Macaulay ring and in [13] the authors examined how this property transfers under flat maps and tensor product operations. In [10] Hochster defined approximately Gorenstein ring. In Section 3 we provide necessary and sufficient conditions which led $R \bowtie I$ be an approximately Cohen-Macaulay (resp. approximately Gorenstein) ring (see Proposition 3.2 and Theorem 3.4).

2. Generically Cohen-Macaulay, generically Gorenstein and generically quasi-Gorenstein rings

As general reference for terminology and well-known results, we refer the reader to [5]. This section deals with some general results about generically Cohen-Macaulay, generically Gorenstein and generically quasi-Gorenstein properties of a general construction, introduced in [8], called amalgamated duplication of a ring along an ideal.

Let R be a commutative ring with unit element 1 and let I be a proper ideal of R. Set

$$R \bowtie I = \{(r,s) \mid r,s \in R, s-r \in I\}\cdot$$

It is easy to check that $R \bowtie I$ is a subring, with unit element $(1,1)$, of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\}$. In the following we bring some main properties of the ring $R \bowtie I$ from [6].

Proposition 2.1. *Let R be a ring and let I be an ideal of R. Then the following statements hold.*

(1) *The map $f : R \oplus I \to R \bowtie I$ defined by $f((r,i)) = (r, r+i)$ is an R-isomorphism. Moreover, there is a split exact sequence of R-modules*

(a) $$0 \to R \xrightarrow{\varphi} R \bowtie I \xrightarrow{\psi} I \to 0,$$

where $\varphi(r) = (r,r)$ for all $r \in R$, and $\psi((r,s)) = s-r$ for all $(r,s) \in R \bowtie I$. We also have the short exact sequence of R-modules:

(b) $$0 \to I \xrightarrow{\psi'} R \bowtie I \xrightarrow{\varphi'} R \to 0,$$

where $\psi'(i) = (0, i)$ *and* $\varphi'((r, s)) = r$ *for every* $r \in R$ *and* $(r, s) \in R \bowtie I$. *Note that the exact sequence* (b) *is also a sequence of* $R \bowtie I$*-module, while the other one is not.*

(2) *Let* $\mathfrak{p}$ *be a prime ideal of* R *and set*:

$$\mathfrak{p}_0 = \{(p, p+i) \mid p \in \mathfrak{p}, i \in I \cap \mathfrak{p}\},$$
$$\mathfrak{p}_1 = \{(p, p+i) \mid p \in \mathfrak{p}, i \in I\}, \text{ and}$$
$$\mathfrak{p}_2 = \{(p+i, p) \mid p \in \mathfrak{p}, i \in I\}.$$

(a) *If* $I \subseteq \mathfrak{p}$, *then* $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2$ *is a prime ideal of* $R \bowtie I$ *and it is the unique prime ideal of* $R \bowtie I$ *lying over* $\mathfrak{p}$ *and* $(R \bowtie I)_{\mathfrak{p}_0} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$.

(b) *If* $I \nsubseteq \mathfrak{p}$, *then* $\mathfrak{p}_1 \neq \mathfrak{p}_2$ *and* $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_0$. *Moreover,* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *are the only prime ideals of* $R \bowtie I$ *lying over* $\mathfrak{p}$, *and* $(R \bowtie I)_{\mathfrak{p}_1} \cong R_{\mathfrak{p}} \cong (R \bowtie I)_{\mathfrak{p}_2}$.

(3) R *and* $R \bowtie I$ *have the same Krull dimension and if* R *is a local ring with maximal ideal* $\mathfrak{m}$, *then* $R \bowtie I$ *is local with maximal ideal* $\mathfrak{m}_0 = \{(r, r+i) \mid r \in \mathfrak{m}, i \in I\}$. *Also, if* R *is a Noetherian ring, then* $R \bowtie I$ *is a finitely generated* R*-module.*

In [6, Discussion 10], D'Anna showed that if R is a local ring of dimension d and I is a non-unit ideal of R, then the ring $R \bowtie I$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is a maximal Cohen-Macaulay R-module. Recently in [1, Theorem 1.8], it is shown that if R is a Noetherian local ring, then $R \bowtie I$ is Gorenstein if and only if R is Cohen-Macaulay and I is a canonical module for R, and then R/I is Cohen-Macaulay of dimension $\dim(R) - 1$. In the following proposition we suppose that $R \bowtie I$ is Gorenstein and we would like to know when R is Gorenstein.

Proposition 2.2. *Let* I *be a non-zero flat ideal of Noetherian zero dimensional ring* R. *If* $R \bowtie I$ *is a Gorenstein ring, then* R *is Gorenstein.*

Proof. By Proposition 2.1(3), $\dim(R \bowtie I) = \dim(R) = 0$ and so $R \bowtie I$ is self-injective. Hence by [14, Corollary 3.4], $\mathrm{id}_R(R \bowtie I) = \mathrm{fd}_R(R \bowtie I)$. Now by assumption I is a flat ideal of R, so $R \bowtie I$ is a flat R-module. Therefore $R \bowtie I$ is an injective R-module and hence for every R-module M and every integer $i \geq 1$, we have

$$\begin{aligned} 0 &= \mathrm{Ext}^i_R(M, R \bowtie I) \\ &\cong \mathrm{Ext}^i_R(M, R) \oplus \mathrm{Ext}^i_R(M, I). \end{aligned}$$

So for every R-module M and for all $i \geq 1$, we have $\mathrm{Ext}^i_R(M, R) = 0$. Hence R is self-injective and therefore R is Gorenstein, since $\dim(R) = 0$. □

We recall the notion of quasi-Gorenstein ring due to Platte and Storch in [12].

Definition 2.3. A local ring R is said to be a quasi-Gorenstein ring if a canonical module of R exists and is a free R-module (of rank one). This is equivalent to saying that $H^d_{\mathfrak{m}}(R) \cong E_R(R/\mathfrak{m})$, where $d = \dim R$ and $\mathfrak{m}$ is the maximal ideal of R.

The ring R is Gorenstein if and only if it is quasi-Gorenstein and Cohen-Macaulay. In [4, Theorem 3.3], it is shown that if R is a Noetherian local ring and I is a proper ideal of R such that $\operatorname{Ann}_R(I) = 0$, then $R \bowtie I$ is a quasi-Gorenstein ring if and only if $\widehat{R}$ satisfies Serre's condition (S_2) and I is a canonical ideal of R.

Recall that a Noetherian ring R is called generically Cohen-Macaulay (resp. generically Gorenstein) if the ring $R_{\mathfrak{p}}$ is Cohen-Macaulay (resp. Gorenstein) for all $\mathfrak{p} \in \operatorname{Ass}(R)$. Every Cohen-Macaulay (resp. Gorenstein) ring is also generically Cohen-Macaulay (resp. generically Gorenstein) and every Artinian generically Cohen-Macaulay (resp. generically Gorenstein) ring is Cohen-Macaulay (resp. Gorenstein). We are ready now to introduce generically quasi-Gorenstein ring.

Definition 2.4. Let R be a Noetherian local ring. Then R is called generically quasi-Gorenstein if the ring $R_{\mathfrak{p}}$ is quasi-Gorenstein for all $\mathfrak{p} \in \operatorname{Ass}(R)$.

According to [2, Corollary 2.4], the localization of every quasi-Gorenstein ring is quasi-Gorenstein. Therefore every quasi-Gorenstein ring is generically quasi-Gorenstein. It is straightforward to see that if R is a zero dimensional local ring, then R is quasi-Gorenstein if and only if R is generically quasi-Gorenstein. It is routine to show that a Noetherian local ring R is generically Gorenstein if and only if R is generically quasi-Gorenstein and generically Cohen-Macaulay.

We are interested in understanding when $R \bowtie I$ is generically Cohen-Macaulay (resp. generically Gorenstein, generically quasi-Gorenstein). In the following lemma we investigate the associated prime ideals of the ring $R \bowtie I$.

Lemma 2.5. *Let R be a Noetherian ring and let I be a proper ideal of R. Consider the ring homomorphism $\varphi : R \to R \bowtie I$, where $\varphi(r) = (r, r)$. Then the following statements hold.*

(i) *If $\mathfrak{p} \in Ass(R \bowtie I)$, then $\varphi^{-1}(\mathfrak{p}) \in Ass(R)$.*
(ii) *If $\mathfrak{q} \in Ass(R)$, then there exists $\mathfrak{p} \in Ass(R \bowtie I)$ such that $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$.*

Proof. (i) The exact sequence $0 \to I \to R \bowtie I \to R \to 0$ of $R \bowtie I$-modules implies that

$$\begin{aligned}\operatorname{Ass}(R \bowtie I) &\subseteq \operatorname{Ass}_{R\bowtie I}(I) \cup \operatorname{Ass}_{R\bowtie I}(R)\\ &= \operatorname{Ass}_{R\bowtie I}(R).\end{aligned}$$

So by assumption $\mathfrak{p} \in \operatorname{Ass}_{R\bowtie I}(R)$. By [11, Exercise 6.7] we have $\varphi^{-1}(\mathfrak{p}) \in \operatorname{Ass}(R)$, since R is a finitely generated $R \bowtie I$-module.

(ii) From the R-monomorphism $\varphi : R \to R \bowtie I$, we have $\mathrm{Ass}_R(R) \subseteq \mathrm{Ass}_R(R \bowtie I)$. So by assumption $\mathfrak{q} \in \mathrm{Ass}_R(R \bowtie I)$ and by [11, Exercise 6.7] there exists $\mathfrak{p} \in \mathrm{Ass}_{R\bowtie I}(R \bowtie I)$ such that $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$. □

Definition 2.6. A finitely generated R-module M is called generically maximal Cohen-Macaulay (resp. generically canonical module) if the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is maximal Cohen-Macaulay (resp. canonical module) for all $\mathfrak{p} \in \mathrm{Ass}\,(R)$.

Definition 2.7. The ring R is called generically (S_n) if $R_\mathfrak{p}$ satisfies Serre's condition (S_n) for all $\mathfrak{p} \in \mathrm{Ass}\,(R)$.

Theorem 2.8. *Let R be a Noetherian ring and let I be a proper ideal of R. Then the following statements hold.*

(i) *If $R \bowtie I$ is generically Cohen-Macaulay, then R is generically Cohen-Macaulay.*
(ii) *If R is generically Cohen-Macaulay* (*resp. generically Gorenstein*) *and I is generically maximal Cohen-Macaulay* (*resp. generically canonical module*), *then $R \bowtie I$ is generically Cohen-Macaulay* (*resp. generically Gorenstein*).
(iii) *If R is generically quasi-Gorenstein and I is a generically canonical ideal of R, then $R \bowtie I$ is generically quasi-Gorenstein.*
(iv) *If $Ann_R(I) = 0$, then R is generically (S_2) provided that $R \bowtie I$ is generically quasi-Gorenstein.*

Proof. We prove items (iii) and (iv). The proof of the others is similar.

(iii) Let $\mathfrak{p} \in \mathrm{Ass}\,(R \bowtie I)$. By Lemma 2.5, $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \in \mathrm{Ass}\,(R)$. According to Proposition 2.1(2), we have the following two cases:

Case (1). If $I \subseteq \mathfrak{q}$, then $(R \bowtie I)_\mathfrak{p} \cong R_\mathfrak{q} \bowtie I_\mathfrak{q}$. By assumption $I_\mathfrak{q}$ is a canonical ideal and $R_\mathfrak{q}$ is quasi-Gorenstein. Therefore $R_\mathfrak{q}$ satisfies Serre's condition (S_2) by [3, Remark 1.4]. Hence $\widehat{R_\mathfrak{q}}$ satisfies Serre's condition (S_2) by [3, Proposition 1.2]. Now according to [4, Theorem 3.3], $(R \bowtie I)_\mathfrak{p}$ is quasi-Gorenstein.

Case (2). If $I \nsubseteq \mathfrak{q}$, then $(R \bowtie I)_\mathfrak{p} \cong R_\mathfrak{q}$. So $(R \bowtie I)_\mathfrak{p}$ is quasi-Gorenstein.

(iv) Let $\mathfrak{q} \in \mathrm{Ass}\,(R)$. By Lemma 2.5, there exists $\mathfrak{p} \in \mathrm{Ass}\,(R \bowtie I)$ such that $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$ and, by Proposition 2.1(2), we have the following two cases:

Case (1). If $I \subseteq \mathfrak{q}$, then $(R \bowtie I)_\mathfrak{p} \cong R_\mathfrak{q} \bowtie I_\mathfrak{q}$. So by assumption $R_\mathfrak{q} \bowtie I_\mathfrak{q}$ is quasi-Gorenstein. Therefore by [4, Theorem 3.3], $\widehat{R_\mathfrak{q}}$ satisfies Serre's condition (S_2) and so $R_\mathfrak{q}$ satisfies Serre's condition (S_2) by [3, Proposition 1.2].

Case (2). If $I \nsubseteq \mathfrak{q}$, then $(R \bowtie I)_\mathfrak{p} \cong R_\mathfrak{q}$. So $R_\mathfrak{q}$ satisfies Serre's condition (S_2), by [3, Remark 1.4]. □

Proposition 2.9. *Let R be a Cohen-Macaulay ring and let I be a non-zero ideal of R such that $I_\mathfrak{q}$ is a flat $R_\mathfrak{q}$-module for all $\mathfrak{q} \in Ass\,(R)$. If $R \bowtie I$ is generically Gorenstein, then R is generically Gorenstein.*

Proof. Note that $\dim\,(R_\mathfrak{q}) = 0$ for all $\mathfrak{q} \in \mathrm{Ass}\,(R)$, since R is Cohen-Macaulay. The assertion follows by Propositions 2.2 and 2.1(3). □

3. Approximately Cohen-Macaulay and approximately Gorenstein rings

In this section we study when $R \bowtie I$ is approximately Cohen-Macaulay and when it is approximately Gorenstein. To state the first result of this section, we need the notion of approximately Cohen-Macaulay ring due to Goto in [9].

Definition 3.1. The local ring $(R, \mathfrak{m})$ is called an approximately Cohen-Macaulay ring if either $\dim(R) = 0$ or there exists an element a of $\mathfrak{m}$ such that $R/a^n R$ is a Cohen-Macaulay ring of dimension $\dim(R) - 1$ for every integer $n > 0$.

It is straightforward to see that a Cohen-Macaulay local ring R is approximately Cohen-Macaulay and the converse is true when $\dim(R) = 0$. Also Goto in [9, Corollary 2.8], showed that if $(R, \mathfrak{m})$ is an approximately Cohen-Macaulay local ring such that $\dim(R) \geq 2$ and that $H^i_{\mathfrak{m}}(R)$ is finitely generated R-module for all $i \neq \dim(R)$, then R is Cohen-Macaulay.

The next result shows that $R \bowtie I$ is approximately Cohen-Macaulay if and only if R is approximately Cohen-Macaulay provided some special conditions.

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be a non-zero flat ideal of R. Assume that R is not a Cohen-Macaulay ring such that R is a homomorphic image of a Cohen-Macaulay local ring. Then $R \bowtie I$ is approximately Cohen-Macaulay if and only if R is approximately Cohen-Macaulay.*

Proof. Note that $\varphi : R \to R \bowtie I$ is a flat ring homomorphism. By [7, Proposition 5.1], we have $R \bowtie I/\mathfrak{m}_0 \cong R/\mathfrak{m}$, where $\mathfrak{m}_0 = \{(r, r+i) \mid r \in \mathfrak{m}, i \in I\}$ is the maximal ideal of $R \bowtie I$. So $R \bowtie I/\mathfrak{m}_0$ is a Cohen-Macaulay ring. Now the assertion follows from [13, Theorem 6]. □

Before stating our main results of this section, we recall the definition of approximately Gorenstein ring due to Hochster in [10].

Definition 3.3. A Notherian local ring $(R, \mathfrak{m})$ is called approximately Gorenstein, if for every integer $n > 0$ there is an ideal $I \subseteq \mathfrak{m}^n$ such that R/I is Gorenstein.

It is routine to see that every Gorenstein ring is approximately Gorenstein, and a zero dimensional ring is approximately Gorenstein if and only if it is Gorenstein. While approximately Gorenstein rings must have positive depth, they need not to be Cohen-Macaulay. In fact, every complete Noetherian domain is approximately Gorenstein [10, Theorem 1.6].

The next result shows that $R \bowtie I$ is approximately Gorenstein provided some special conditions.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a Notherian local ring and let I be a proper ideal of R. Then the following statements hold.*

(i) *If R is approximately Gorenstein, then $R \bowtie I$ is approximately Gorenstein.*

(ii) *If $R \bowtie I$ is Gorenstein and R is generically Gorenstein, then R is approximately Gorenstein.*

Proof. (i) According to Proposition 2.1(3), $(R \bowtie I, \mathfrak{m}_0)$ is a Notherian local ring. Let $n > 0$ be an integer. By assumption there exists an ideal $J \subseteq \mathfrak{m}^n$ such that R/J is Gorenstein. By [7, Proposition 5.1], $J \bowtie I$ is an ideal of $R \bowtie I$ and

$$\frac{R \bowtie I}{J \bowtie I} \cong \frac{R}{J}.$$

It is straightforward to see that $J \bowtie I \subseteq \mathfrak{m}^n \bowtie I = \mathfrak{m}_0^n$ and so $(R \bowtie I)/(J \bowtie I)$ is Gorenstein, therefore the assertion is proved.

(ii) By [1, Theorem 1.8], R is Cohen-Macaulay and I is a canonical ideal of R. The assertion follows from [10, Remarks (4.8b)]. □

Corollary 3.5. *Let R be a generically Gorenstein local ring and let I be a proper ideal of R. Assume that R is Cohen-Macaulay with canonical module. Then $R \bowtie I$ is approximately Gorenstein.*

Proof. According to [10, Remarks (4.8b)], R is approximately Gorenstein, so $R \bowtie I$ is approximately Gorenstein by Theorem 3.4(i). □

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