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Partial Fraction Expansions for Newton's and Halley's Iterations for Square Roots

Omran Kouba
Department of Mathematics, Higher Institute for Applied Sciences and Technology, P. O. Box 31983, Damascus, Syria
e-mail : `omran_kouba@hiast.edu.sy`

Abstract. When Newton's method, or Halley's method is used to approximate the pth root of $1-z$, a sequence of rational functions is obtained. In this paper, a beautiful formula for these rational functions is proved in the square root case, using an interesting link to Chebyshev's polynomials. It allows the determination of the sign of the coefficients of the power series expansion of these rational functions. This answers positively the square root case of a proposed conjecture by Guo(2010).

1. Introduction

Let p be an integer greater than 1, and z any complex number. If we apply Newton's method to solve the equation $x^p = 1 - z$ starting from the initial value 1, we obtain the sequence of rational functions $(F_k)_{k\geq 0}$ in the variable z defined by the following iteration

$$F_{k+1}(z) = \frac{1}{p}\left((p-1)F_k(z) + \frac{1-z}{F_k^{p-1}(z)}\right), \qquad F_0(z) \equiv 1.$$

Similarly, if we apply Halley's method to solve the equation $x^p = 1 - z$ starting from the same initial value 1, we get the sequence of rational functions $(G_k)_{k\geq 0}$ in the variable z defined by the iteration

$$G_{k+1}(z) = \frac{(p-1)G_k^p(z) + (p+1)(1-z)}{(p+1)G_k^p(z) + (p-1)(1-z)}G_k(z), \qquad G_0(z) \equiv 1.$$

It was shown in [3] and more explicitly stated in [4] that both F_k and G_k have power series expansions that are convergent in the neighbourhood of $z = 0$, and that these expansions start similar to the power series expansion of $z \mapsto \sqrt[p]{1-z}$. More precisely, the first 2^k coefficients of the power series expansion of F_k are identical to

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the corresponding coefficients in the power series expansion of $z \mapsto \sqrt[p]{1-z}$, and the same holds for the first 3^k coefficients of the power series expansion of G_k. It was conjectured [3, Conjecture 12], that the coefficients of the power series expansions of $(F_k)_{k\geq 2}$, $(G_k)_{k\geq 1}$ and $z \mapsto \sqrt[p]{1-z}$ have the same sign pattern.

In this article, we will consider only the case $p = 2$. In this case an unsuspected link to Chebyshev polynomials of the first and the second kind is discovered, it will allow us to find general formulæ for F_k and G_k as sums of partial fractions. This will allow us to prove Guo's conjecture in this particular case. Finally, we note that for $p \geq 3$ the conjecture remains open.

Before proceeding to our results, let us fix some notation and definitions. We will use freely the properties of Chebyshev polynomials $(T_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ of the first and the second kind. In particular, they can be defined for $x > 1$ by the folmulæ:

$$\forall\, \varphi \in \mathbb{R}, \quad T_n(\cos\varphi) = \cos(n\varphi), \quad \text{and} \quad U_n = \frac{1}{n+1}T_n'. \tag{1.1}$$

For these definitions and more on the properties of these polynomials we invite the reader to consult [5] or any general treatise on special functions for example [1, Chapter 22], [2, §13.3-4] or [6, Part six], and the references therein. Let $\Omega = \mathbb{C} \setminus [1, +\infty)$, (this is the complex plane cut along the real numbers greater or equal to 1.) For $z \in \Omega$, we denote by $\sqrt{1-z}$ the square root of $1 - z$ with positive real part. We know that $z \mapsto \sqrt{1-z}$ is holomorphic in Ω. Moreover,

$$\forall\, z \in \overline{D(0,1)}, \qquad \sqrt{1-z} = \sum_{m=0}^{\infty} \lambda_m z^m, \tag{1.2}$$

where

$$\forall\, m \geq 0, \qquad \lambda_m = \frac{1}{(1-2m)2^{2m}}\binom{2m}{m}. \tag{1.3}$$

The standard result, is that (1.2) is true for z in the open unit disk, but the fact that for $n \geq 1$ we have $\lambda_n = \mu_{n-1} - \mu_n < 0$, where $\mu_n = 2^{-2n}\binom{2n}{n} \sim 1/\sqrt{\pi n}$, proves the uniform convergence of the series $\sum \lambda_m z^m$ in the closed unit disk, and (1.2) follows by Abel's Theorem since $z \mapsto \sqrt{1-z}$ can be continuously extended to $\overline{D(0,1)}$.

Finally, we consider the sequences of rational functions $(V_n)_{n\geq 0}$, $(F_n)_{n\geq 0}$ and $(G_n)_{n\geq 0}$ defined by

$$V_{n+1}(z) = \frac{1 - z + V_n(z)}{1 + V_n(z)}, \qquad V_0(z) \equiv 1, \tag{1.4}$$

$$F_{n+1}(z) = \frac{1}{2}\left(F_n(z) + \frac{1-z}{F_n(z)}\right), \qquad F_0(z) \equiv 1, \tag{1.5}$$

$$G_{n+1}(z) = \frac{G_n^3(z) + 3(1-z)G_n(z)}{3G_n^2(z) + 1 - z}, \qquad G_0(z) \equiv 1, \tag{1.6}$$

where $(F_n)_{n\geq 0}$ and $(G_n)_{n\geq 0}$ are Newton's and Halley's iterations mentioned before, (in the case $p = 2$.) Since the sequence $(V_n)_{n\geq 0}$ is simpler than the other two sequences, we will prove our main result for the V_n's, then we will deduce the corresponding properties for F_n and G_n.

2. The main results

We start this section by proving a simple property that shows why it is sufficient to study the sequence of V_n's to deduce the properties of Newton's and Halley's iterations the F_n's and G_n's :

Lemma 2.1. *The sequences $(V_n)_n$, $(F_n)_n$ and $(G_n)_n$ of the rational functions defined inductively by* (1.4), (1.5) *and* (1.6), *satisfy the following properties :*

(a) *For $z \in \Omega$ and $n \geq 0$, we have :*

$$\frac{V_n(z) - \sqrt{1-z}}{V_n(z) + \sqrt{1-z}} = \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^{n+1},$$

(b) *For $n \geq 0$ we have $F_n = V_{2^n-1}$, and $G_n = V_{3^n-1}$.*

Proof. First, let us suppose that $z = x \in (0,1)$. In this case, we see by induction that all the terms of the sequence $(V_n(x))_{n\geq 0}$ are well defined and positive, and we have the following recurrence relation :

$$\begin{aligned}\frac{V_{n+1}(x) - \sqrt{1-x}}{V_{n+1}(x) + \sqrt{1-x}} &= \frac{1-x+V_n(x) - \sqrt{1-x}(1+V_n(x))}{1-x+V_n(x) + \sqrt{1-x}(1+V_n(x))} \\ &= \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \cdot \frac{V_n(x) - \sqrt{1-x}}{V_n(x) + \sqrt{1-x}}.\end{aligned}$$

Therefore, using simple induction, we obtain

$$\forall\, n \geq 0, \qquad \frac{V_n(x) - \sqrt{1-x}}{V_n(x) + \sqrt{1-x}} = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{n+1}, \tag{2.1}$$

Now, for a given $n \geq 0$, let us define $R(z)$ and $L(z)$ by the formulæ :

$$L(z) = \frac{V_n(z) - \sqrt{1-z}}{V_n(z) + \sqrt{1-z}} \quad \text{and} \quad R(z) = \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^{n+1}.$$

Clearly, R is analytic in Ω since $\Re(\sqrt{1-z}) > 0$ for $z \in \Omega$. On the other hand, if $V_n = A_n/B_n$ where A_n and B_n are two co-prime polynomials then

$$L(z) = \frac{(A_n(z) - \sqrt{1-z}B_n(z))^2}{A_n^2(z) - (1-z)B_n^2(z)}.$$

So L is meromorphic with, (at the most,) a finite number of poles in Ω. Using analyticity, we conclude from (2.1) that $L(z) = R(z)$ for $z \in \Omega \setminus \mathcal{P}$ for some finite set $\mathcal{P} \subset \Omega$, but this implies that the points in $\mathcal{P}$ are removable singularities, and that L is holomorphic and identical to R in Ω. This concludes the proof of (a).

To prove (b), consider $z = x \in (0,1)$, as before, we see by induction that all the terms of the sequences $(F_n(x))_{n\geq 0}$ and $(G_n(x))_{n\geq 0}$ are well-defined and positive. Also, we check easily, using (1.5) and (1.6), that the following recurrence relations hold :

$$\frac{F_{n+1}(x)-\sqrt{1-x}}{F_{n+1}(x)+\sqrt{1-x}} = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2 \cdot \frac{F_n(x)-\sqrt{1-x}}{F_n(x)+\sqrt{1-x}},$$

and

$$\frac{G_{n+1}(x)-\sqrt{1-x}}{G_{n+1}(x)+\sqrt{1-x}} = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^3 \cdot \frac{G_n(x)-\sqrt{1-x}}{G_n(x)+\sqrt{1-x}}.$$

It follows that

$$\frac{F_n(x)-\sqrt{1-x}}{F_n(x)+\sqrt{1-x}} = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{2^n} = \frac{V_{2^n-1}(x)-\sqrt{1-x}}{V_{2^n-1}(x)+\sqrt{1-x}},$$

and

$$\frac{G_n(x)-\sqrt{1-x}}{G_n(x)+\sqrt{1-x}} = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{3^n} = \frac{V_{3^n-1}(x)-\sqrt{1-x}}{V_{3^n-1}(x)+\sqrt{1-x}}.$$

Hence $F_n(x) = V_{2^n-1}(x)$ and $G_n(x) = V_{3^n-1}(x)$ for every $x \in (0,1)$, and (b) follows by analyticity. □

Lemma 2.1 provides a simple proof of the following known result.

Corollary 2.2. *The sequences $(V_n)_n$, $(F_n)_n$ and $(G_n)_n$ of the rational functions defined inductively by* (1.4), (1.5) *and* (1.6)*, converge uniformly on compact subsets of Ω to the function $z \mapsto \sqrt{1-z}$.*

Proof. Indeed, this follows from Lemma 2.1 and the fact that for every non-empty compact set $K \subset \Omega$ we have :

$$\sup_{z\in K}\left|\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right| < 1,$$

which is easy to prove. □

One can also use Lemma 2.1 to study V_n in the neighbourhood of $z = 0$:

Corollary 2.3. *For every $n \geq 0$, and for z in the neighbourhood of* 0 *we have*

$$V_n(z) = \sqrt{1-z} + O(z^{n+1}), \tag{2.2}$$

and

$$V_n(z) = \sum_{m=0}^{n} \lambda_m z^m + O(z^{n+1}). \tag{2.3}$$

where λ_m is defined by (1.3)

Proof. Indeed, using Lemma 2.1 we have :

$$\begin{aligned}
V_n(z) &= \sqrt{1-z} \cdot \frac{(1+\sqrt{1-z})^{n+1} + (1-\sqrt{1-z})^{n+1}}{(1+\sqrt{1-z})^{n+1} - (1-\sqrt{1-z})^{n+1}}, \\
&= \sqrt{1-z} + \frac{2\sqrt{1-z}}{(1+\sqrt{1-z})^{n+1} - (1-\sqrt{1-z})^{n+1}} \cdot (1-\sqrt{1-z})^{n+1}, \\
&= \sqrt{1-z} + \frac{2\sqrt{1-z}}{(1+\sqrt{1-z})^{n+1} - (1-\sqrt{1-z})^{n+1}} \cdot \frac{z^{n+1}}{(1+\sqrt{1-z})^{n+1}}, \\
&= \sqrt{1-z} + \frac{2\sqrt{1-z}}{(1+\sqrt{1-z})^{2(n+1)} - z^{n+1}} \cdot z^{n+1},
\end{aligned}$$

In particular, for z in the neighbourhood of 0, we have $V_n(z) = \sqrt{1-z} + O(z^{n+1})$, which is (2.2).

On the other hand, using (1.2), we obtain

$$V_n(z) = \sum_{m=0}^{n} \lambda_m z^m + O(z^{n+1})$$

which is (2.3). □

Recall that we are interested in the sign pattern of the coefficients of the power series expansion of F_n and G_n, in the neighbourhood of $z = 0$. Lemma 2.1 reduces the problem to finding sign pattern of the coefficients of the power series expansion of V_n. But, V_n is rational function and a partial fraction decomposition would be helpful. The next theorem is our main result :

Theorem 2.4. *Let n be a positive integer, and let V_n be the rational function defined by the recursion* (1.4). *Then the partial fraction decomposition of V_n is as follows :*

$$\begin{aligned}
V_1(z) &= 1 - \frac{z}{2}, \\
V_n(z) &= 1 - \frac{z}{2} - \frac{z^2}{2(n+1)} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\sin^2(\frac{2\pi k}{n+1})}{1 - z\cos^2(\frac{\pi k}{n+1})}, \qquad \textit{for } n \geq 2.
\end{aligned}$$

Proof. Let us recall the fact that for $x > 1$ Chebyshev polynomials of the first and the second kind satisfy the following identities :

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}, \tag{2.4}$$

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x+\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}. \tag{2.5}$$

Thus, for $0 < x < 1$ we have

$$x^n T_n\left(\frac{1}{x}\right) = \frac{(1+\sqrt{1-x^2})^n + (1-\sqrt{1-x^2})^n}{2},$$
$$x^n U_n\left(\frac{1}{x}\right) = \frac{(1+\sqrt{1-x^2})^{n+1} - (1+\sqrt{1-x^2})^{n+1}}{2\sqrt{1-x^2}}.$$

So, using Lemma 2.1(a), we find that

$$V_n(x^2) = \frac{x^{n+1}T_{n+1}(1/x)}{x^n U_n(1/x)} = \frac{P_n(x)}{Q_n(x)}, \tag{2.6}$$

where $P_n(x) = x^{n+1}T_{n+1}(1/x)$ and $Q_n(x) = x^n U_n(1/x)$.

It is known that U_n has n simple zeros, namely $\{\cos(k\theta_n) : 1 \le k \le n\}$ where $\theta_n = \pi/(n+1)$. So, if we define

$$\Delta_{2m} = \{1, 2, \ldots, 2m\}, \quad \text{and} \quad \Delta_{2m-1} = \{1, 2, \ldots, 2m-1\} \setminus \{m\},$$

then the singular points of the rational function P_n/Q_n are $\{\lambda_k : k \in \Delta_n\}$ with $\lambda_k = \sec(k\theta_n)$. Moreover, since

$$\begin{aligned} P_n(\lambda_k) &= \lambda_k^{n+1} T_{n+1}(\cos(k\theta_n)) \\ &= \lambda_k^{n+1}\cos(\pi k) = (-1)^k \lambda_k^{n+1} \ne 0, \end{aligned}$$

we conclude that these singular points are, in fact, simple poles with residues given by

$$\operatorname{Res}\left(\frac{P_n}{Q_n}, \lambda_k\right) = -\lambda_k^3 \frac{T_{n+1}(1/\lambda_k)}{U_n'(1/\lambda_k)}.$$

But, from the identity $U_n(\cos\varphi) = \sin((n+1)\varphi)/\sin\varphi$ we conclude that

$$U_n'(\cos\varphi) = \frac{\cos\varphi \sin((n+1)\varphi)}{\sin^3\varphi} - (n+1)\frac{\cos((n+1)\varphi)}{\sin^2\varphi},$$

hence,

$$U_n'(\cos(k\theta_n)) = (n+1)\frac{(-1)^{k+1}}{\sin^2(k\theta_n)},$$

and finally,

$$\operatorname{Res}\left(\frac{P_n}{Q_n}, \lambda_k\right) = \frac{\lambda_k^3}{n+1}\sin^2(k\theta_n) = \frac{\lambda_k}{n+1}\tan^2(k\theta_n). \tag{2.7}$$

From this we conclude that the rational function R_n defined by

$$\begin{aligned} R_n(x) &= \frac{P_n(x)}{Q_n(x)} - \frac{1}{n+1}\sum_{k\in\Delta_n}\frac{\lambda_k\tan^2(k\theta_n)}{x-\lambda_k} \\ &= \frac{P_n(x)}{Q_n(x)} + \frac{1}{n+1}\sum_{k\in\Delta_n}\frac{\tan^2(k\theta_n)}{1-x\cos(k\theta_n)}, \end{aligned} \tag{2.8}$$

is, in fact, a polynomial, and $\deg R_n = \deg P_n - \deg Q_n \le (n+1)-(n-1) = 2$.

Noting that $k \mapsto n+1-k$ is a permutation of Δ_n we conclude that

$$\begin{aligned} \sum_{k\in\Delta_n}\frac{\tan^2(k\theta_n)}{1-x\cos(k\theta_n)} &= \sum_{k\in\Delta_n}\frac{\tan^2(k\theta_n)}{1+x\cos(k\theta_n)} \\ &= \frac{1}{2}\sum_{k\in\Delta_n}\left(\frac{\tan^2(k\theta_n)}{1+x\cos(k\theta_n)} + \frac{\tan^2(k\theta_n)}{1-x\cos(k\theta_n)}\right) \\ &= \sum_{k\in\Delta_n}\frac{\tan^2(k\theta_n)}{1-x^2\cos^2(k\theta_n)} \end{aligned}$$

and using the fact that

$$\frac{1}{1-z\cos^2\varphi} = 1 + z\cos^2\varphi + \frac{z^2\cos^4\varphi}{1-z\cos^2\varphi},$$

we find

$$\begin{aligned} \sum_{k\in\Delta_n}\frac{\tan^2(k\theta_n)}{1-x\cos(k\theta_n)} &= \sum_{k\in\Delta_n}\tan^2(k\theta_n) + x^2\sum_{k\in\Delta_n}\sin^2(k\theta_n) \\ &\quad + x^4\sum_{k=1}^{n}\frac{\cos^2(k\theta_n)\sin^2(k\theta_n)}{1-x^2\cos^2(k\theta_n)} \end{aligned}$$

where we added a "zero" term to the last sum for odd n.

Combining this conclusion with (2.8) we conclude that there exists a polynomial S_n with $\deg S_n \le 2$ such that

$$S_n(x) = \frac{P_n(x)}{Q_n(x)} + \frac{x^4}{4(n+1)}\sum_{k=1}^{n}\frac{\sin^2(2k\theta_n)}{1-x^2\cos^2(k\theta_n)}.$$

Moreover, S_n is even, since both P_n and Q_n are even. Thus, going back to (2.6) we conclude that there exists two constants α_n and β_n such that

$$V_n(z) = \alpha_n + \beta_n z + \frac{z^2}{4(n+1)}\sum_{k=1}^{n}\frac{\sin^2(2k\theta_n)}{1-z\cos^2(k\theta_n)} \tag{2.9}$$

But, from Corollary 2.3, we also have $V_n(z) = 1 - \frac{1}{2}z + O(z^2)$ for $n \geq 1$, thus $\alpha_n = 1$ and $\beta_n = -1/2$.

Finally, noting that the terms corresponding to k and $n+1-k$ in the sum (2.9) are identical, we arrive to the desired formula. This concludes the proof of Theorem 2.4. □

Theorem 2.4 allows us to obtain a precise information about the power series expansion of V_n in the neighbourhood of $z = 0$:

Corollary 2.5. *Let n be a positive integer greater than 1, and let V_n be the rational function defined by* (1.4)*. Then the radius of convergence of power series expansion $\sum_{m=0}^{\infty} A_m^{(n)} z^m$ of V_n in the neighbourhood of $z = 0$ is $\sec^2(\frac{\pi}{n+1}) > 1$, and the coefficients $(A_m^{(n)})_{m\geq 0}$ satisfy the following properties :*

(a) *For $0 \leq m \leq n$ we have $A_m^{(n)} = \lambda_m$, where λ_m is defined by* (1.3)*.*

(b) *For $m > n$ we have $A_m^{(n)} < 0$ and*

$$\sum_{m=n+1}^{\infty} \left(-A_m^{(n)}\right) = \frac{1}{2^{2n}}\binom{2n}{n} - \frac{1}{n+1}.$$

Moreover, for every $n \geq 0$ and every z in the closed unit disk $\overline{D(0,1)}$ we have

$$\left|V_n(z) - \sqrt{1-z}\right| \leq \frac{2}{\sqrt{\pi n}} |z|^{n+1}.$$

In particular, $(V_n)_{n\geq 0}$ converges uniformly on $\overline{D(0,1)}$ to the function $z \mapsto \sqrt{1-z}$.

Proof. Let us denote $\pi/(n+1)$ by θ_n. By Theorem 2.4 the poles of V_n, for $n > 1$, are $\{\sec^2(k\theta_n) : 1 \leq k \leq \lfloor n/2 \rfloor\}$ and the nearest one to 0 is $\sec^2(\theta_n)$. This proves the statement about the radius of convergence.

Also, we have seen in Corollary 2.3, that in the neighbourhood of $z = 0$ we have

$$V_n(z) = \sum_{m=0}^{n} \lambda_m z^m + O(z^{n+1}),$$

and this proves that for $0 \leq m \leq n$ we have $A_m^{(n)} = \lambda_m$ which is (a).

To prove (b), we note that for $1 \leq k \leq \lfloor n/2 \rfloor$, and $z \in D(0, \sec^2(\theta_n))$, we have

$$\frac{1}{1 - z\cos^2(k\theta_n)} = \sum_{m=0}^{\infty} \cos^{2m}(k\theta_n) z^m,$$

hence

$$V_n(z) = 1 - \frac{z}{2} - \sum_{m=0}^{\infty} \left(\frac{1}{2(n+1)} \sum_{k=1}^{\lfloor n/2 \rfloor} \cos^{2m}(k\theta_n)\sin^2(2k\theta_n) \right) z^{m+2}.$$

This gives the following alternative formulæ for $A_m^{(n)}$ when $m \geq 2$:

$$A_m^{(n)} = -\frac{2}{n+1}\sum_{k=1}^{\lfloor n/2\rfloor} \cos^{2(m-1)}(k\theta_n)\sin^2(k\theta_n),$$

$$(2.10) \qquad = -\frac{1}{n+1}\sum_{k=1}^{n} \cos^{2(m-1)}(k\theta_n)\sin^2(k\theta_n),$$

where (2.10) is also valid for $m = 1$. This proves in particular that $A_m^{(n)} < 0$ for $m > n$.

Since the radius of convergence of $\sum_{m=0}^{\infty} A_m^{(n)} z^m$ is greater than 1 we conclude that

$$(2.11) \qquad \sum_{m=0}^{\infty} A_m^{(n)} = V_n(1),$$

but, we can prove by induction from (1.4) that $V_n(1) = 1/(n+1)$. Therefore, (2.11) implies that, for $n \geq 1$ we have

$$\begin{aligned}\sum_{m=n+1}^{\infty} (-A_m^{(n)}) &= -\frac{1}{n+1} + \sum_{m=0}^{n} A_m^{(n)} = -\frac{1}{n+1} + 1 - \sum_{m=1}^{n} \lambda_m,\\ &= -\frac{1}{n+1} + 1 - \sum_{m=1}^{n} (\mu_{m-1} - \mu_m) = \mu_n - \frac{1}{n+1},\end{aligned}$$

where $\mu_m = 2^{-2m}\binom{2m}{m}$, which is (b).

For $n \geq 1$ and $z \in \overline{D(0,1)}$ we have

$$\begin{aligned}\left|V_n(z) - \sum_{m=0}^{n} \lambda_m z^m\right| &\leq \left|\sum_{m=n+1}^{\infty} A_m^{(n)} z^m\right| \leq |z|^{n+1} \cdot \sum_{m=n+1}^{\infty} (-A_m^{(n)})\\ (2.12) \qquad &\leq \left(\frac{1}{2^{2n}}\binom{2n}{n} - \frac{1}{n+1}\right)|z|^{n+1}.\end{aligned}$$

On the other hand, using (1.2) we see that for $n \geq 1$ and $z \in \overline{D(0,1)}$ we have

$$\begin{aligned}\left|\sqrt{1-z} - \sum_{m=0}^{n} \lambda_m z^m\right| &\leq \left|\sum_{m=n+1}^{\infty} \lambda_m z^m\right| \leq |z|^{n+1} \cdot \sum_{m=n+1}^{\infty} (\mu_{m-1} - \mu_m)\\ (2.13) \qquad &\leq \frac{1}{2^{2n}}\binom{2n}{n}|z|^{n+1}.\end{aligned}$$

Combining (2.12) and (2.13), and noting that $2^{-2n}\binom{2n}{n} \leq 1/\sqrt{\pi n}$ we obtain

$$\left|V_n(z) - \sqrt{1-z}\right| \leq \frac{2}{\sqrt{\pi n}}|z|^{n+1},$$

which is the desired conclusion. □

The following corollary is an immediate consequence of Lemma 2.1 and Corollary 2.5. It proves that Conjecture 12 in [3] is correct in the case of square roots.

Corollary 2.6. *The following properties of* $(F_n(z))_{n\geq 0}$ *and* $(G_n(z))_{n\geq 0}$*, the Newton's and Halley's approximants of* $\sqrt{1-z}$*, defined by the recurrences* (1.5) *and* (1.6) *hold :*

(a) *For* $n > 1$*, the rational function* $F_n(z)$ *has a power series expansion* $1 - \sum_{m=1}^{\infty} B_m^{(n)} z^m$ *with* $\sec^2(2^{-n}\pi)$ *as radius of convergence, and* $B_m^{(n)} > 0$ *for every* $m \geq 1$*. Moreover,*

$$\forall\, z \in \overline{D(0,1)}, \qquad \left|F_n(z) - \sqrt{1-z}\right| \leq \frac{2}{\sqrt{\pi}} \cdot \frac{|z|^{2^n}}{\sqrt{2^n - 1}}.$$

(b) *For* $n \geq 1$*, the rational function* $G_n(z)$ *has a power series expansion* $1 - \sum_{m=1}^{\infty} C_m^{(n)} z^m$ *with* $\sec^2(3^{-n}\pi)$ *as radius of convergence, and* $C_m^{(n)} > 0$ *for every* $m \geq 1$*. Moreover,*

$$\forall\, z \in \overline{D(0,1)}, \qquad \left|G_n(z) - \sqrt{1-z}\right| \leq \frac{2}{\sqrt{\pi}} \cdot \frac{|z|^{3^n}}{\sqrt{3^n - 1}}.$$

Remark. Guo's conjecture [3, Conjecture 12] in the case of square roots is just the fact that $B_m^{(n)} > 0$ for $n > 1$, $m \geq 1$, and that $C_m^{(n)} > 0$ for $n,\ m \geq 1$. The case of pth roots for $p \geq 3$ remains open.

4. Conclusion

In this paper, we presented a link between the rational functions approximating $z \mapsto \sqrt{1-z}$ obtained from the application of Newton's and Halley's method, and Chebyshev Polynomials. This was used to find the partial fraction decomposition of this rational functions, and the sign pattern of the coefficients of their power series expansions was obtained. Finally, this was used to prove the square root case of Conjecture 12 in [3], and to give estimates of the approximation error for z in the closed unit disk.

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