KYUNGPOOK Math. J. 52(2012), 319-325
http://dx.doi.org/10.5666/KMJ.2012.52.3.319

On (ω)compactness and (ω)paracompactness

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Abstract. The notions of (ω)compactness and (ω)paracompactness are studied in product (ω)topology. The concept of countable (ω)paracompactness is introduced and some results on this notion are obtained.

1. Introduction

The definition of (ω)topological spaces is introduced in [1]. If X and Y are two (ω)topological spaces, we introduce in this paper, the product (ω)topology on $X \times Y$. We prove that the product of two (ω)compact spaces [1] is (ω)compact and the product of an (ω)paracompact space [2] with an (ω)compact space is (ω)paracompact. Also we show that the product of a countable (ω)paracompact space and an (ω)compact space is countably (ω)paracompact. Moreover, we prove two results: one analogous to a result of Dowker [3] and the other analogous to a result of Ishikawa [5].

2. Preliminaries

In the sequel for a topological space $(X, \mathcal{J})$, the closure of a set $A \subset X$ with respect to $\mathcal{J}$ is denoted by $(\mathcal{J})clA$. The set of natural numbers and the set of real numbers are denoted by N and R respectively. The members of N are denoted by i, j, k, l, m, n etc.

We require the following known definitions.

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* Corresponding Author.
Received May 23, 2011; accepted March 12, 2012.
2010 Mathematics Subject Classification: 54A10.
Key words and phrases: (ω)topology, product (ω)topology, (ω)compactness, (ω)paracompactness, countable (ω)paracompactness, condition(α).

Definition 2.1(Bose and Tiwari [1])**.** If $\{\mathcal{J}_n\}$ is a sequence of topologies on a set X with $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ for all $n \in N$, then the pair $(X, \{\mathcal{J}_n\})$ is called an (ω)*topological space.*

Definition 2.2(Bose and Tiwari [1])**.** Let $(X, \{\mathcal{J}_n\})$ be an (ω)topological space. A set $G \in \mathcal{J}_n$ for some n is called an (ω)*open set.* A set F is said to be (ω)*closed* if $X - F$ is (ω)open.

Definition 2.3(Bose and Tiwari [1])**.** An (ω)topological space X is said to be (ω)*compact* if every (ω)open cover of X has a finite subcover.

Definition 2.4(Michael [6])**.** A collection $\mathcal{A}$ of a subsets of a topological space $(X, \mathcal{J})$ is said to be $(\mathcal{J})$*locally finite* if every $x \in X$ has a $(\mathcal{J})$open neighbourhood intersecting finitely many sets $A \in \mathcal{A}$.

Definition 2.5(Bose and Tiwari [2])**.** An (ω)topological space $(X, \{\mathcal{J}_n\})$ is said to be (ω)*paracompact* if every (ω)open cover of X has, for some n, a $(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement.

Definition 2.6(Dugundji [4], p.152)**.** A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is said to be *point finite* if for every $x \in X$, there exist at most finitely many sets $A \in \mathcal{A}$ such that $x \in A$.

Theorem 2.7(Dugundji [4], p.152)**.** *A topological space $(X, \mathcal{J})$ is normal iff for every point finite open cover $\{U_\alpha | \alpha \in A\}$ of X, there exists a $(\mathcal{J})$open cover $\{V_\alpha | \alpha \in A\}$ of X such that $(\mathcal{J})clV_\alpha \subset U_\alpha$ for each $\alpha \in A$, and $V_\alpha \neq \emptyset$ whenever $U_\alpha \neq \emptyset$.*

3. New definitions and theorems

We introduce the following definition.

Definition 3.1. Let $(X, \{\mathcal{A}_n\})$ and $(Y, \{\mathcal{B}_n\})$ be two (ω)topological spaces and for each n, $\mathcal{P}_n$ be the product topology on $X \times Y$ of the topologies $\mathcal{A}_n$ and $\mathcal{B}_n$ on X and Y respectively. Then $(X \times Y, \{\mathcal{P}_n\})$ is called the *product* (ω)*topological space* of the spaces $(X, \{\mathcal{A}_n\})$ and $(Y, \{\mathcal{B}_n\})$.

In the following three theorems the (ω)topological spaces $(X, \{\mathcal{A}_n\})$ and $(Y, \{\mathcal{B}_n\})$ and their product space $(X \times Y, \{\mathcal{P}_n\})$ are denoted, simply, by X, Y and $X \times Y$ respectively.

Theorem 3.2. *If the spaces X and Y are (ω)compact, then the product space $X \times Y$ is (ω)compact.*

Proof. Let $\mathcal{U}$ be an (ω)open cover of $X \times Y$. For $x \in X$, $x \times Y$ is (ω)compact. Therefore a finite number of elements $U_1^x, U_2^x, ..., U_{k(x)}^x$ of $\mathcal{U}$ covers $x \times Y$. Suppose $U_i^x \in \mathcal{P}_{n_i^x}$, $i = 1, 2, ..., k(x)$ and $m_x = \max(n_1^x, n_2^x, ..., n_{k(x)}^x)$. Then $H_x = U_1^x \cup U_2^x \cup ... U_{k(x)}^x \in \mathcal{P}_{m_x}$ and $x \times Y \subset H_x$. We consider a cover $\mathcal{C}$ of $x \times Y$ by $(\mathcal{P}_{m_x})$open sets of the product space $X \times Y$ lying in H_x and suppose that $\mathcal{C}$ consists of the elements of the form $A_{m_x} \times B_{m_x}$ where $A_{m_x} \in \mathcal{A}_{m_x}$ and $B_{m_x} \in \mathcal{B}_{m_x}$. By the (ω)compactness of $x \times Y$, it follows that we can cover $x \times Y$ by finite number of elements:

$$A^1_{m_x} \times B^1_{m_x}, A^2_{m_x} \times B^2_{m_x}, ..., A^k_{m_x} \times B^k_{m_x}. \tag{3.1}$$

We assume that all the sets in (3.1), intersect $x \times Y$. We write

$$W_x = A^1_{m_x} \cap A^2_{m_x} \cap ... \cap A^k_{m_x} \in \mathcal{A}_{m_x}.$$

Let $(\alpha, \beta) \in W_x \times Y$. Then for some i, $(x, \beta) \in A^i_{m_x} \times B^i_{m_x}$. So $\beta \in B^i_{m_x}$. Hence $(\alpha, \beta) \in A^i_{m_x} \times B^i_{m_x}$. Thus it follows that the finite collection $A^1_{m_x} \times B^1_{m_x}, A^2_{m_x} \times B^2_{m_x}, ..., A^k_{m_x} \times B^k_{m_x}$ also covers $W_x \times Y$. Hence H_x and so the finite subclass $U^x_1, ..., U^x_{k(x)}$ of $\mathcal{U}$ covers $W_x \times Y$. Since X is (ω)compact, the (ω)open cover $\{W_x \mid x \in X\}$ of X has a finite subcover $\{W_1, W_2, ..., W_l\}$. So $\{W_1 \times Y, W_2 \times Y, ..., W_l \times Y\}$ covers $X \times Y$. Again each of $W_1 \times Y, W_2 \times Y, ..., W_l \times Y$ can be covered by finitely many elements of $\mathcal{U}$. Therefore $X \times Y$ can be covered by finitely many elements of $\mathcal{U}$. □

We now introduce the following definition which we require in the next two results.

Definition 3.3. An (ω)topological space $(X, \{\mathcal{J}_n\})$ is said to satisfy the *condition*(α) if for all $A \in \mathcal{J}_m$ and $B \in \mathcal{J}_n$ with $A \neq X$ and $n > m$, we have $A \cap B \in \mathcal{J}_m$.

Theorem 3.4. *If the space X is (ω)paracompact, Y is (ω)compact and both X and Y satisfy the condition(α), then the product space $X \times Y$ is (ω)paracompact.*

Proof. The space $X \times Y$ satisfies the condition(α), since the spaces X and Y do so. Let $\mathcal{U}$ be an (ω)open cover of $X \times Y$. We suppose that the elements U of $\mathcal{U}$ have the form $U = D \times H$, $D \in \mathcal{A}_n$ and $H \in \mathcal{B}_n$. Then $\{D\}$ and $\{H\}$ are (ω)open covers of X and Y respectively. Since X is (ω)paracompact, there exists, for some n, an $(\mathcal{A}_n)$open refinement $\{D'\}$ of $\{D\}$ and since Y is (ω)compact, $\{H\}$ has a finite subcover $H_1, H_2, ..., H_k$. Let $H_i \in \mathcal{B}_{n_i}$, $i = 1, 2, ..., k$ and $m = \max(n, n_1, n_2, ...n_k)$. Then $\mathcal{C} = \{D' \times H_i \mid D' \in \{D'\}$ and $i = 1, 2, ..., k\}$ is a $(\mathcal{P}_m)$open cover of $X \times Y$, which is not necessarily a refinement of $\mathcal{U}$. However, if the intersection of each member of $\mathcal{C}$ is taken with each member of $\mathcal{U}$, then since $X \times Y$ satisfies condition(α), we get a $(\mathcal{P}_m)$open refinement $\mathcal{K}$ of $\mathcal{U}$.

Let us consider a point $x \in X$. Then proceeding as in the previous theorem, we get a finite number of elements $K^x_1, K^x_2, ..., K^x_{p(x)}$ of $\mathcal{K}$ such that for some $(\mathcal{A}_m)$open neighbourhood V_x of x, we have $V_x \times Y \subset K_x = K^x_1 \cup K^x_2 \cup ... \cup K^x_{p(x)}$. Since X is (ω)paracompact, there exists, for some l, an $(\mathcal{A}_l)$locally finite $(\mathcal{A}_l)$open refinement $\mathcal{V}$ of the (ω)open cover $\{V_x \mid x \in X\}$ of X. If $r = \max(l, m)$, then the collection $\mathcal{W} = \{(V \times Y) \cap K^x_i \mid V \in \mathcal{V}, V \subset V_x, x \in X, i = 1, 2, ..., p(x)\}$ is a $(\mathcal{P}_r)$open refinement of $\mathcal{K}$ and hence of $\mathcal{U}$. Also for $(x, y) \in X \times Y$, there is a $(\mathcal{A}_l)$open neighbourhood M of x intersecting a finite number of $V \in \mathcal{V}$ and so the $(\mathcal{P}_l)$open neighbourhood $M \times Y$ of (x, y) intersects only a finite number of sets of $\mathcal{W}$. Thus $\mathcal{W}$ is $(\mathcal{P}_l)$locally finite and hence $(\mathcal{P}_r)$locally finite. □

We now provide two examples to show that the above theorem is not true if one of the two spaces X and Y does not satisfy the condition(α).

Example 3.5. If $X = N$ and if for all n, $\mathcal{A}_n$ is the discrete topology on X, then $(X, \{\mathcal{A}_n\})$ is an (ω)paracompact space satisfying the condition(α).

Next taking $Y = [0, 1]$, let us define an (ω)topological space $(Y, \{\mathcal{B}_n\})$ as follows:

Let $I_n = [0, 1 - \frac{1}{n+1})$ and $\mathcal{T}_n = \mathcal{U}/I_n$, where $\mathcal{U}$ denotes the usual topology on R. Now for any $n \in N$, define $\mathcal{B}_n$ to be the topology on Y generated by the base $\mathcal{T}_n \cup \mathcal{S}$, where $\mathcal{S}$ is the collection of all the left open subintervals of Y having 1 as the right end point. Then the space $(Y, \{\mathcal{B}_n\})$ is (ω)compact. Clearly $(Y, \{\mathcal{B}_n\})$ does not satisfy the condition(α) as $[0, \frac{2}{3}) \in \mathcal{B}_2$ and $(\frac{1}{2}, 1] \in \mathcal{B}_1$ but $(\frac{1}{2}, \frac{2}{3}) \notin \mathcal{B}_1$.

The product space $(X \times Y, \{\mathcal{P}_n\})$ is not (ω)paracompact as seen below:

Let us consider the collection $\{\{1\} \times [0, 1]\} \cup \{\{2\} \times [0, \frac{1}{2}), \{2\} \times (\frac{1}{4}, 1]\} \cup \{\{3\} \times [0, \frac{2}{3}), \{3\} \times (\frac{1}{2}, 1], ..., \{2+k\} \times [0, \frac{k+1}{k+2}), \{2+k\} \times (\frac{k}{k+1}, 1], ...\}$. Clearly it is an (ω)open cover of $X \times Y$ which has no $(\mathcal{P}_n)$open refinement.

Example 3.6. Let $X = [0, 1]$ and $\mathcal{A}_n$ be the topology on X generated by the subbase $\mathcal{T}_n \cup \mathcal{D}$ where $\mathcal{T}_n$ is the collection defined in the above example and $\mathcal{D}$ is the collection of all subintervals of $[0, 1]$ having 1 as the right end point. Then $(X, \{\mathcal{A}_n\})$ is (ω)paracompact. The space $(X, \{\mathcal{A}_n\})$ and the (ω)compact space $(Y, \{\mathcal{B}_n\})$ of the above example do not satisfy the condition(α). Here also the product space $X \times Y$ is not (ω)paracompact.

We now introduce the following generalization of (ω)paracompactness.

Definition 3.7. An (ω)topological space $(X, \{\mathcal{J}_n\})$ is said to be *countably (ω)paracompact* if every countable (ω)open cover of X has for some $n \in N$, a $(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement.

Theorem 3.8. *If the space X is countably (ω)paracompact and Y is (ω)compact and both X and Y satisfy the condition(α), then the product space $X \times Y$ is countably (ω)paracompact.*

Proof. Let $\mathcal{U} = \{U_i \mid i \in N\}$ be a countable (ω)open cover of $X \times Y$. Replacing the (ω)open sets U_i by the basis elements having U_i as their union, we get a refinement $\mathcal{U}^*$ of $\mathcal{U}$. If for $U \in \mathcal{U}^*$, $U \subset U_i \in \mathcal{P}_n$, then we can write $U = D_U \times H_U$ with $D_U \in \mathcal{A}_n$ and $H_U \in \mathcal{B}_n$. Therefore $D_i = \cup\{D_U \mid U \subset U_i\} \in \mathcal{A}_n$ and so $\{D_i\}$ forms a countable (ω)open cover of X. Since X is countably (ω)paracompact, there is, for some $n_1 \in N$, a $(\mathcal{A}_{n_1})$open refinement $\mathcal{D}$ of $\{D_i\}$. Let $A_i = \cup\{D \in \mathcal{D} \mid D \subset D_i\}$. Then $A_i \in \mathcal{A}_{n_1}$ and $\{A_i\}$ covers X with $A_i \subset D_i$. Again $\{H_U \mid U \in \mathcal{U}^*\}$ is an (ω)open cover of Y. By (ω)compactness of Y, $\{H_U\}$ has a finite subcover $\mathcal{B}$. Since $\mathcal{B}$ is finite, the sets of $\mathcal{B}$ are $(\mathcal{B}_{n_2})$open for some $n_2 \in N$. Let $m = \max(n_1, n_2)$. Then $\mathcal{C} = \{A_i \times B \mid i \in N, B \in \mathcal{B}\}$ is a $(\mathcal{P}_m)$open cover of $X \times Y$. However, it may not be a refinement of $\mathcal{U}$. Now since X and Y both satisfy the condition(α), $X \times Y$ does so. Therefore as in the previous theorem we get a $(\mathcal{P}_m)$open refinement $\mathcal{K}$ of $\mathcal{U}$. Let $K_i = \cup\{K \in \mathcal{K} \mid K \subset U_i\}$. Then $\{K_i\}$ is a $(\mathcal{P}_m)$open cover of $X \times Y$ with $K_i \subset U_i$.

Let S_i be the set of points $x \in X$ such that $x \times Y \subset \cup_{j \leq i} K_j \in \mathcal{P}_m$. As in Theorem 3.2, for any $s \in S_i$, there exists an $N_s \in \mathcal{A}_m$ such that $s \in N_s$ and $N_s \times Y \subset \cup_{j \leq i} K_j$. Therefore $N_s \subset S_i$. Hence S_i is $(\mathcal{A}_m)$open. Also for every $x \in X$, $x \times Y$ is (ω)compact and hence $x \times Y$ is contained in some finite number of sets $\in \{K_i\}$. Thus $x \in S_i$ for some i. Therefore $\{S_i\}$ forms a countable (ω)open cover of X. From the countable (ω)paracompactness of X, we get for some n, an $(\mathcal{A}_n)$locally finite $(\mathcal{A}_n)$open refinement $\mathcal{S}$ of $\{S_i\}$. For each $S \in \mathcal{S}$, let $g(S)$ be the first S_i containing S and let $G_i = \cup\{S \in \mathcal{S} \mid g(S) = S_i\}$. Then $G_i \in \mathcal{A}_n$, $G_i \subset S_i$ and $\{G_i\}$ is an $(\mathcal{A}_n)$locally finite cover of X.

If $j \leq i$, let $G_{ij} = (G_i \times Y) \cap K_j$, then $G_{ij} \in \mathcal{P}_l$ where $l = \max(m, n)$. If $(x, y) \in X \times Y$ then for some i, $x \in G_i$ and hence $(x, y) \in x \times Y \subset \cup_{j \leq i} K_j$ and so $(x, y) \in K_j$ for some $j \leq i$. Hence $(x, y) \in G_{ij}$. Also $G_{ij} \subset K_j \subset U_j$. Thus $\{G_{ij}\}$ is a $(\mathcal{P}_l)$open refinement of $\{U_i\}$. If $(x, y) \in X \times Y$, then $x \in X$ and so there exists a set $M_x \in \mathcal{A}_n$ such that M_x intersects a finite number of sets of $\{G_i\}$. Then $M_x \times Y$ is a $(\mathcal{P}_l)$open neighbourhood of (x, y) intersecting a finite number of sets of $\{G_{ij}\}$. Therefore $X \times Y$ is countably (ω)paracompact. □

Since the product spaces $X \times Y$ considered in Example 3.5 and Example 3.6 are not even countably (ω)paracompact, the above theorem is not true if one of the spaces does not satisfy the condition(α).

Theorem 3.9. *Let $(X, \{\mathcal{J}_n\})$ be an (ω)topological space. Suppose for each $n \in N$, $(X, \mathcal{J}_n)$ is a normal topological space. Then the following statements are equivalent.*

1. *X is countably (ω)paracompact.*
2. *Every countable (ω)open cover of X has, for some n, a point finite $(\mathcal{J}_n)$open refinement.*
3. *Every countable (ω)open cover $\{U_i \mid i \in N\}$ has, for some n, a $(\mathcal{J}_n)$open refinement $\{V_i \mid i \in N\}$ with $(\mathcal{J}_n)clV_i \subset U_i$.*
4. *Every countable (ω)open cover $\{U_i \mid i \in N\}$ has, for some n, a $(\mathcal{J}_n)$open refinement. If for every i, $U_i \subset U_{i+1}$, then $\{U_i \mid i \in N\}$ has, for some n, a $(\mathcal{J}_n)$open refinement $\{V_i \mid i \in N\}$ with $V_i \subset U_i$ for each i and $\{V_i \mid i \in N\}$ has, for some m, a $(\mathcal{J}_m)$closed refinement $\{F_i \mid i \in N\}$ with $F_i \subset V_i$ for each i.*

Proof. $1 \Rightarrow 2$: Obvious.
$2 \Rightarrow 3$: Let $\mathcal{W}$ be a point finite $(\mathcal{J}_n)$open refinement of $\{U_i \mid i \in N\}$. For each open set $W \in \mathcal{W}$, let $g(W)$ be the first U_i containing W, and let $G_i = \cup\{W \mid g(W) = U_i\}$. Then clearly $\{G_i\}$ is a countable point finite $(\mathcal{J}_n)$open cover of X with $G_i \subset U_i$. By Theorem 2.7, $\{G_i\}$ has a $(\mathcal{J}_n)$open refinement $\{V_i\}$ such that $(\mathcal{J}_n)clV_i \subset G_i \subset U_i$.
$3 \Rightarrow 4$: Obvious.
$4 \Rightarrow 1$: Firstly let $\{U_i\}$ be a countable (ω)open cover of X, with $U_i \subset U_{i+1}$.

Then by (4), there exists, for some n, a $(\mathcal{J}_n)$open cover $\{V_i \mid i \in N\}$ of X with $V_i \subset U_i$ for each i. Also $\{V_i \mid i \in N\}$ has, for some m, a $(\mathcal{J}_m)$closed refinement $\{F_i \mid i \in N\}$ with $F_i \subset V_i$ for each i. Let $r = \max(m, n)$. Since $(X, \mathcal{J}_r)$ is normal, there exists, for each i, a $(\mathcal{J}_r)$open set W_i such that $F_i \subset W_i \subset (\mathcal{J}_r)clW_i \subset V_i$. Let $G_i = V_i - \cup_{j<i}(\mathcal{J}_r)clW_i$. Then $\{G_i\}$ is a $(\mathcal{J}_r)$locally finite $(\mathcal{J}_r)$open refinement of $\{V_i\}$ and hence of $\{U_i\}$.

Now let the countable (ω)open cover $\{U_i \mid i \in N\}$ does not satisfy the condition $U_i \subset U_{i+1}$. Then $\{U_i \mid i \in N\}$ has, for some $l \in N$, a $(\mathcal{J}_l)$open refinement $\{U_i^{'} \mid i \in N\}$. Therefore $H_i = \cup_{j \leq i} U_j^{'} \in \mathcal{J}_l$ for each i. So $\{H_i \mid i \in N\}$ is a countable $(\mathcal{J}_l)$open cover of X with $H_i \subset H_{i+1}$. Then proceeding as above, we get, for some $r \in N$, a $(\mathcal{J}_r)$locally finite $(\mathcal{J}_r)$open refinement $\{B_i \mid i \in N\}$ of $\{H_i \mid i \in N\}$. We write $D_i = (\cup_{j=i}^{\infty} B_i) \cap U_i^{'}$. Then $\{D_i \mid i \in N\}$ is a $(\mathcal{J}_p)$locally finite $(\mathcal{J}_p)$open refinement of $\{U_i^{'} \mid i \in N\}$ and hence of $\{U_i \mid i \in N\}$ where $p = \max(l, r)$. Thus X is countably (ω)paracompact. □

Theorem 3.10. *If an (ω)topological space $(X, \{\mathcal{J}_n\})$ is countably (ω)paracompact, then for any decreasing sequence $\{F_i \mid i \in N\}$ of (ω)closed sets with $\cap_{i=1}^{\infty} F_i = \emptyset$, there exists, for some n, a decreasing sequence $\{G_i \mid i \in N\}$ of $(\mathcal{J}_n)$open sets satisfying $G_i \supset F_i$ for each i and $\cap_{i=1}^{\infty}(\mathcal{J}_n)clG_i = \emptyset$. If every countable (ω)open cover of X has, for some n, a $(\mathcal{J}_n)$open refinement, then the converse is also true.*

Proof. Firstly, suppose X is countably (ω)paracompact. Let $\{F_i \mid i \in N\}$ be a decreasing sequence of (ω)closed sets with $\cap_{i=1}^{\infty} F_i = \emptyset$. Then $\{X - F_i \mid i \in N\}$ forms a countable (ω)open cover of X. Therefore for some n, there exists a $(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement $\mathcal{V}$ of $\{X - F_i \mid i \in N\}$. If

$$G_i = \cup\{V \in \mathcal{V} \mid V \not\subset X - F_j, \text{ for } j = 1, 2, ...i\}, i \in N.$$

Then $G_i \in \mathcal{J}_n$, $F_i \subset G_i$ and $G_i \supset G_{i+1}$ for each i. Now let x be any point of X. Then there is a $(\mathcal{J}_n)$open neighbourhood N_x of x intersecting only a finitely many sets $V_1, V_2, ..., V_k$ of $\mathcal{V}$. Suppose, $V_j \subset X - F_{i_j}$, $j = 1, 2, ...k$, and $i_0 = \max\{i_1, i_2, ..., i_k\}$. Then for any $V (\in \mathcal{V})$ with $V \not\subset X - F_{i_0}$, we have $V \neq V_i$ for $i = 1, 2, ...k$ and so $N_x \cap V = \emptyset$. Thus $N_x \cap G_{i_0} = \emptyset$. So $N_x \cap (\mathcal{J}_n)clG_{i_0} = \emptyset$. Therefore $x \notin (\mathcal{J}_n)clG_{i_0}$. Hence $\cap_{i=1}^{\infty}(\mathcal{J}_n)clG_i = \emptyset$.

Conversely, suppose the conditions hold. Let $\{U_i \mid i \in N\}$ be a countable (ω)open cover of X. Then for some n, there exists a $(\mathcal{J}_n)$open refinement $\mathcal{W}$ of $\{U_i \mid i \in N\}$ and so proceeding as in the proof of (2) $\Rightarrow$ (3) (Theorem 3.9), we get a $(\mathcal{J}_n)$open refinement $\{V_i \mid i \in N\}$ of $\{U_i \mid i \in N\}$, with $V_i \subset U_i$ for each i. If $F_i = X - \cup_{j<i} V_j$, then $\{F_i \mid i \in N\}$ is a decreasing sequence of $(\mathcal{J}_n)$closed and hence (ω)closed sets with $\cap_{i=1}^{\infty} F_i = \emptyset$. So there is, for some m, a decreasing sequence $\{G_i \mid i = 1, 2, ...\}$ of $(\mathcal{J}_m)$open sets such that $G_i \supset F_i$ for each i and $\cap_{i=1}^{\infty}(\mathcal{J}_m)clG_i = \emptyset$. If $W_i = X - (\mathcal{J}_m)clG_i$, then $\{W_i \mid i \in N\}$ is a $(\mathcal{J}_m)$open cover of X with $(\mathcal{J}_m)clW_i \subset X - F_i = \cup_{j<i} V_j$. If $D_i = V_i - \cup_{j<i}(\mathcal{J}_m)clW_j$, then $\{D_i \mid i \in N\}$ is a $(\mathcal{J}_m)$locally finite $(\mathcal{J}_m)$open refinement of $\{V_i \mid i \in N\}$ and hence of $\{U_i \mid i \in N\}$. Therefore X is countably (ω)paracompact. □

Theorem 3.9 and Theorem 3.10 are analogous to Theorem 2(Dowker [3]) and the Theorem(Ishikawa [5]) respectively.

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