

## On Generalizations of the Hadamard Inequality for $(\alpha, m)$ -Convex Functions

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ABSTRACT. In this paper we establish several Hadamard-type integral inequalities for  $(\alpha, m)$ -convex functions.

### 1. Introduction

One of the most important integral inequalities for convex functions is the Hadamard inequality (or the Hermite-Hadamard inequality). The following double inequality is well known as the Hadamard inequality in the literature.

**Theorem 1.** *If  $f$  is convex function on  $[a, b]$ , then*

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

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*Proof.* See [1]. □

If the function  $f$  is concave, (1.1) can be written as following:

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right).$$

For recent results related to the Hadamard inequality are given in [9], [10] and [17].

In the literature, the concepts of  $m$ -convexity and  $(\alpha, m)$ -convexity are well known. The concept of  $m$ -convexity was first introduced by G. Toader in [18] (see also [5], [6]) and it is defined as follows:

The function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be  $m$ -convex, where  $m \in [0, 1]$ , if for every  $x, y \in [0, b]$  and  $t \in [0, 1]$ , we have:

$$(1.2) \quad f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

The class of  $(\alpha, m)$ -convex functions was also first introduced in [8] and it is defined as follows:

The function  $f : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$ , is said to be  $(\alpha, m)$ -convex, where  $(\alpha, m) \in [0, 1]^2$ , if we have

$$(1.3) \quad f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ .

It can be easily seen that for  $(\alpha, m) \in \{(0, 0), (1, 1), (1, m)\}$  one obtains the following classes of functions: increasing, convex and  $m$ -convex functions respectively. The interested reader can find more about partial ordering of convexity in [15, P. 8,280]. For many papers connected with  $m$ -convex and  $(\alpha, m)$ -convex functions see ([2], [3], [6], [11], [12], [13], [14], [19]) and the references therein. There are similar inequalities for  $s$ -convex and  $h$ -convex functions in [7] and [16], respectively.

In [6], S. S. Dragomir and G. Toader proved the following Hadamard type inequality for  $m$ -convex functions.

**Theorem 2.** *Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $m$ -convex function with  $m \in (0, 1]$ . If  $0 \leq a < b < \infty$  and  $f \in L_1[a, b]$ , then the following inequality holds:*

$$(1.4) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}.$$

Some generalizations of this result can be found in [2], [3].

In [4] S. S. Dragomir established two new Hadamard-type inequalities for  $m$ -convex functions. They are given in the following Theorems.

**Theorem 3.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $m$ -convex function with  $m \in (0, 1]$ . If  $0 \leq a < b < \infty$  and  $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$ , then the following inequality holds:

$$(1.5) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx.$$

**Theorem 4.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $m$ -convex function with  $m \in (0, 1]$ . If  $f \in L_1[am, b]$  where  $0 \leq a < b$ , then the following inequality holds:

$$(1.6) \quad \frac{1}{m+1} \left[ \frac{1}{mb-a} \int_a^{mb} f(x) dx + \frac{1}{b-ma} \int_{ma}^b f(x) dx \right] \leq \frac{f(a) + f(b)}{2}.$$

The goal of this paper is to obtain new inequalities like those given in Theorems 1, 2, 3, 4, but now for the class of  $(\alpha, m)$ -convex functions.

## 2. Inequalities for $(\alpha, m)$ -convex functions

The following theorem is a generalization of the Hadamard inequality.

**Theorem 5.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $0 \leq a < b$  and  $(\alpha, m) \in [0, 1] \times (0, 1]$ . If  $f \in L_1[m^2a, (2-m)b] \cap L_1\left[ma, \frac{(2-m)b}{m}\right]$ , then the following inequalities hold:

$$(2.1) \quad \begin{aligned} & f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) \\ & \leq \frac{1}{2^\alpha} \frac{1}{b(2-m)-m^2a} \left\{ \int_{m^2a}^{(2-m)b} [f(x) + (2^\alpha - 1)m \right. \\ & \quad \left. \times f\left(\frac{(2-m)b}{m} \left(1 - \frac{x-m^2a}{2b-mb-m^2a}\right) + m \frac{x-m^2a}{2b-mb-m^2a} a\right)] dx \right\} \\ & \leq \frac{1}{2^\alpha(\alpha+1)} [f((2-m)b) \\ & \quad + (\alpha + (2^\alpha - 1))mf(am) + (2^\alpha - 1)\alpha m^2 f\left(\frac{(2-m)b}{m^2}\right)]. \end{aligned}$$

*Proof.* Let  $U_1 = t(2-m)b + (1-t)m^2a$  and  $U_2 = (1-t)(2-m)b + tm^2a$ , where  $t \in [0, 1]$  is arbitrary. Then we get

$$f\left(\frac{U_1 + U_2}{2}\right) = f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right).$$

By the  $(\alpha, m)$ -convexity of  $f$  we can write the following inequality:

$$\begin{aligned} f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) &= f\left(\frac{U_1+U_2}{2}\right) \\ &\leq \frac{1}{2^\alpha}f(U_1) + \left(1 - \frac{1}{2^\alpha}\right)mf\left(\frac{U_2}{m}\right) \\ &= \frac{1}{2^\alpha}\left[f(U_1) + (2^\alpha - 1)mf\left(\frac{U_2}{m}\right)\right], \end{aligned}$$

or

$$\begin{aligned} f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) &\leq \frac{1}{2^\alpha}\left[f\left(t(2-m)b + (1-t)m^2a\right)\right. \\ &\quad \left.+ (2^\alpha - 1)mf\left(\frac{(1-t)(2-m)b}{m} + tma\right)\right]. \end{aligned}$$

Integrating over  $t \in [0, 1]$ , we get

$$\begin{aligned} &f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) \\ &\leq \frac{1}{2^\alpha} \int_0^1 \left[ f\left(t(2-m)b + (1-t)m^2a\right) \right. \\ (2.2) \quad &\left. + (2^\alpha - 1)mf\left(\frac{(1-t)(2-m)b}{m} + tma\right) \right] dt \\ &= \frac{1}{2^\alpha} \frac{1}{b(2-m) - m^2a} \left\{ \int_{m^2a}^{(2-m)b} [f(x) + (2^\alpha - 1)m \right. \\ &\quad \left. \times f\left(\frac{(2-m)b}{m} \left(1 - \frac{x-m^2a}{2b-mb-m^2a}\right) + m\frac{x-m^2a}{2b-mb-m^2a}a\right) \right] dx \right\}, \end{aligned}$$

where we used the change of the variable  $x = t(2-m)b + (1-t)m^2a$  or  $t = \frac{x-m^2a}{2b-mb-m^2a}$  and so

$$\int_0^1 f\left(t(2-m)b + (1-t)m^2a\right) dt = \frac{1}{(2-m)b - m^2a} \int_{m^2a}^{(2-m)b} f(x) dx$$

and

$$\begin{aligned} &\int_0^1 f\left(\frac{(1-t)(2-m)b}{m} + tma\right) dt \\ &= \frac{1}{(2-m)b - m^2a} \\ &\quad \times \int_{m^2a}^{(2-m)b} f\left(\frac{(2-m)b}{m} \left(1 - \frac{x-m^2a}{2b-mb-m^2a}\right) + m\frac{x-m^2a}{2b-mb-m^2a}a\right) dx. \end{aligned}$$

This completes the proof of the first inequality in (2.1).

Next, by the  $(\alpha, m)$ -convexity of  $f$ , we also have

$$\begin{aligned} f\left(t(2-m)b + (1-t)m^2a\right) &= f\left(t(2-m)b + m(1-t)ma\right) \\ &\leq t^\alpha f\left((2-m)b\right) + m(1-t^\alpha) f(ma) \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{(1-t)(2-m)b}{m} + tma\right) &= f\left(t(ma) + m(1-t)\left(\frac{(2-m)b}{m^2}\right)\right) \\ &\leq t^\alpha f(ma) + m(1-t^\alpha) f\left(\frac{(2-m)b}{m^2}\right). \end{aligned}$$

So

$$\begin{aligned} &\frac{1}{2^\alpha} \left[ f\left(t(2-m)b + (1-t)m^2a\right) + (2^\alpha - 1)mf\left(\frac{(1-t)(2-m)b}{m} + tma\right) \right] \\ (2.3) \quad &\leq \frac{1}{2^\alpha} \left\{ t^\alpha f((2-m)b) + m(1-t^\alpha) f(ma) \right. \\ &\quad \left. + (2^\alpha - 1)m \left[ t^\alpha f(ma) + m(1-t^\alpha) f\left(\frac{(2-m)b}{m^2}\right) \right] \right\}. \end{aligned}$$

Integrating (2.3) over  $t$  on  $[0, 1]$ , we get

$$\begin{aligned} &\frac{1}{2^\alpha} \frac{1}{(2-m)b - m^2a} \left\{ \int_{m^2a}^{(2-m)b} [f(x) + (2^\alpha - 1)m \right. \\ &\quad \left. \times f\left(\frac{(2-m)b}{m} \left(1 - \frac{x - m^2a}{2b - mb - m^2a}\right) + m \frac{x - m^2a}{2b - mb - m^2a} a\right)] dx \right\} \\ &\leq \frac{1}{2^{\alpha(\alpha+1)}} \left\{ f((2-m)b) \right. \\ &\quad \left. + (\alpha + (2^\alpha - 1))mf(am) + (2^\alpha - 1)\alpha m^2 f\left(\frac{(2-m)b}{m^2}\right) \right\}. \end{aligned}$$

This completes the proof of the second inequality in (2.1). □

**Remark 1.** Choosing  $(\alpha, m) = (1, 1)$  in (2.1), from the first and the second inequalities of (2.1), respectively, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \frac{1}{b-a} \left[ \int_a^b [f(x) + f(a+b-x)] dx \right] \\ &= \frac{1}{2} \frac{1}{b-a} \left[ \int_a^b f(x) dx + \int_a^b f(x) dx \right] \\ &= \frac{1}{b-a} \int_a^b f(x) dx \end{aligned}$$

and

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \frac{1}{4} [f(b) + 2f(a) + f(b)] \\ &= \frac{f(a)+f(b)}{2}. \end{aligned}$$

Note that, we used

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

and so

$$\int_a^b [f(x) + f(a + b - x)] dx = 2 \int_a^b f(x) dx.$$

Clearly, we can drop the assumption  $f \in L_1 [m^2a, (2 - m)b] \cap L_1 [ma, \frac{(2-m)b}{m}] = L_1 [a, b]$ , and in this case (2.1) exactly becomes the Hermite-Hadamard inequalities for  $(\alpha, m) = (1, 1)$ .

**Theorem 6.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ . If  $0 \leq a < b < \infty$  and  $f \in L_1 [a, b]$ , then the following inequality holds:

$$(2.4) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha + 1}, \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha + 1} \right\}.$$

*Proof.* Since  $f$  is  $(\alpha, m)$ -convex, we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y)$$

for all  $x, y \geq 0$ , which gives:

$$f(ta + (1-t)b) \leq t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right)$$

and

$$f(tb + (1-t)a) \leq t^\alpha f(b) + m(1-t^\alpha) f\left(\frac{a}{m}\right)$$

for all  $t \in [0, 1]$ . Integrating on  $[0, 1]$ , we obtain

$$\int_0^1 f(ta + (1-t)b) dt \leq \frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha + 1}$$

and

$$\int_0^1 f(tb + (1-t)a) dt \leq \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha + 1}.$$

However,

$$\int_0^1 f(ta + (1-t)b) dt = \int_0^1 f(tb + (1-t)a) dt = \frac{1}{b-a} \int_a^b f(x) dx$$

and the inequality (2.4) is obtained.  $\square$

**Remark 2.** The inequality (2.4) yields inequality (1.4) for  $\alpha = 1$ .

**Theorem 7.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in$

$(0, 1]^2$ . If  $0 \leq a < b < \infty$  and  $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$ , then the following inequalities hold:

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha(b-a)} \int_a^b [f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)] dx \\
 &\leq \frac{1}{2^{\alpha+1}(\alpha+1)} [(f(a) + f(b)) \\
 (2.5) \quad &+ m(\alpha + 2^\alpha - 1)(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)) \\
 &+ \alpha m^2(2^\alpha - 1)(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right))].
 \end{aligned}$$

*Proof.* By the  $(\alpha, m)$ -convexity of  $f$ , we have

$$\begin{aligned}
 f\left(\frac{x+y}{2}\right) &= f\left(\frac{x}{2} + m\frac{y}{2m}\right) \\
 &\leq \frac{1}{2^\alpha} f(x) + m\left(1 - \frac{1}{2^\alpha}\right) f\left(\frac{y}{m}\right) \\
 &= \frac{1}{2^\alpha} [f(x) - mf\left(\frac{y}{m}\right)] + mf\left(\frac{y}{m}\right)
 \end{aligned}$$

for all  $x, y \in [0, \infty)$ .

Now, if we choose  $x = ta + (1-t)b$  and  $y = (1-t)a + tb$ , we deduce

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha} \left[ f(ta + (1-t)b) - mf\left(\frac{(1-t)a + tb}{m}\right) \right] + mf\left(\frac{(1-t)a + tb}{m}\right) \\
 &= \frac{1}{2^\alpha} \left[ f(ta + (1-t)b) + m(2^\alpha - 1)f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) \right]
 \end{aligned}$$

for all  $t \in [0, 1]$ .

Integrating over  $t \in [0, 1]$ , we get

$$\begin{aligned}
 (2.6) \quad f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha} \left[ \int_0^1 f(ta + (1-t)b) dt \right. \\
 &\quad \left. + m(2^\alpha - 1) \int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) dt \right].
 \end{aligned}$$

Taking into account that

$$\int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(x) dx$$

and

$$\int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) dt = \frac{1}{b-a} \int_a^b f\left(\frac{x}{m}\right) dx,$$

we deduce from (2.6) the first inequality in (2.5).

Next, by the  $(\alpha, m)$ -convexity of  $f$ , we also have

$$(2.7) \quad \begin{aligned} & \frac{1}{2^\alpha} \left[ f(ta + (1-t)b) + m(2^\alpha - 1) f\left(t\frac{b}{m} + (1-t)\frac{a}{m}\right) \right] \\ & \leq \frac{1}{2^\alpha} \left[ t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right. \\ & \quad \left. + m(2^\alpha - 1) \left( t^\alpha f\left(\frac{b}{m}\right) + m(1-t^\alpha) f\left(\frac{a}{m^2}\right) \right) \right]. \end{aligned}$$

Integrating over  $t$  on  $[0, 1]$ , we get

$$(2.8) \quad \begin{aligned} & \frac{1}{2^\alpha(b-a)} \int_a^b \left( f(x) + m(2^\alpha - 1) f\left(\frac{x}{m}\right) \right) dx \\ & \leq \frac{1}{2^\alpha} \left[ f(a) \int_0^1 t^\alpha dt + mf\left(\frac{b}{m}\right) \int_0^1 (1-t^\alpha) dt + \right. \\ & \quad \left. + m(2^\alpha - 1) f\left(\frac{b}{m}\right) \int_0^1 t^\alpha dt \right. \\ & \quad \left. + m^2(2^\alpha - 1) f\left(\frac{a}{m^2}\right) \int_0^1 (1-t^\alpha) dt \right] \\ & = \frac{1}{2^\alpha(\alpha+1)} \left[ f(a) + m(\alpha+2^\alpha-1) f\left(\frac{b}{m}\right) + \alpha m^2(2^\alpha-1) f\left(\frac{a}{m^2}\right) \right]. \end{aligned}$$

Similarly, changing the roles of  $a$  and  $b$ , we get

$$(2.9) \quad \begin{aligned} & \frac{1}{2^\alpha(b-a)} \int_a^b \left( f(x) + m(2^\alpha - 1) f\left(\frac{x}{m}\right) \right) dx \\ & \leq \frac{1}{2^\alpha(\alpha+1)} \left[ f(b) + m(\alpha+2^\alpha-1) f\left(\frac{a}{m}\right) + \alpha m^2(2^\alpha-1) f\left(\frac{b}{m^2}\right) \right]. \end{aligned}$$

Now adding (2.8) and (2.9) with each other, we obtain the second inequality in (2.5).  $\square$

**Remark 3.** Choosing  $\alpha = 1$  in the first part of (2.5), we get (1.5).

**Remark 4.** The inequality (2.5) yields the Hadamard inequality (1.1) for  $\alpha = 1$  and  $m = 1$ .

**Theorem 8.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ . If  $0 \leq a < b < \infty$  and  $f \in L_1[a, b]$ , then the following inequality holds:

$$(2.10) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{2} \left[ \frac{f(a) + f(b) + \alpha m \left( f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right)}{\alpha + 1} \right].$$



*Proof.* By the  $(\alpha, m)$ -convexity of  $f$ , we can write

$$f (ta + (1 - t) b) \leq t^\alpha f (a) + m (1 - t^\alpha) f \left( \frac{b}{m} \right)$$

and

$$f (tb + (1 - t) a) \leq t^\alpha f (b) + m (1 - t^\alpha) f \left( \frac{a}{m} \right)$$

for all  $t \in [0, 1]$ .

Adding the above inequalities, we get

$$\begin{aligned} & f (ta + (1 - t) b) + f (tb + (1 - t) a) \\ & \leq t^\alpha f (a) + m (1 - t^\alpha) f \left( \frac{b}{m} \right) + t^\alpha f (b) + m (1 - t^\alpha) f \left( \frac{a}{m} \right). \end{aligned}$$

Integrating over  $t \in [0, 1]$ , we obtain

$$\begin{aligned} (2.11) \quad & \int_0^1 f (ta + (1 - t) b) dt + \int_0^1 f (tb + (1 - t) a) dt \\ & \leq \int_0^1 t^\alpha (f (a) + f (b)) dt + \int_0^1 m (1 - t^\alpha) \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) dt \\ & = \frac{f (a) + f (b)}{\alpha + 1} + \frac{m\alpha}{\alpha + 1} \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) \\ & = \frac{f (a) + f (b) + m\alpha \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right)}{\alpha + 1}. \end{aligned}$$

As it is easy to see that

$$\int_0^1 f (ta + (1 - t) b) dt = \int_0^1 f (tb + (1 - t) a) dt = \frac{1}{b - a} \int_a^b f (x) dx,$$

from (2.11) we deduce the desired result, namely, the inequality (2.10). □

**Remark 5.** The inequality (2.10) yields the right side of the Hadamard inequality (1.1) for  $\alpha = 1$  and  $m = 1$ .

**Theorem 9.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ . If  $f \in L_1 [am, b]$  where  $0 \leq a < b$ , then the following inequality holds:

$$\begin{aligned} (2.12) \quad & \frac{1}{mb - a} \int_a^{mb} f (x) dx + \frac{1}{b - ma} \int_{ma}^b f (x) dx \\ & \leq \frac{1}{(\alpha + 1)} [(f (a) + f (b)) (1 + m\alpha)]. \end{aligned}$$

*Proof.* By  $(\alpha, m)$ -convexity of  $f$ , for all  $t \in [0, 1]$ , we can write:

$$\begin{aligned} f(ta + m(1-t)b) &\leq t^\alpha f(a) + m(1-t^\alpha)f(b), \\ f(tb + m(1-t)a) &\leq t^\alpha f(b) + m(1-t^\alpha)f(a), \\ f((1-t)a + mtb) &\leq (1-t)^\alpha f(a) + m(1-(1-t)^\alpha)f(b), \\ f((1-t)b + mta) &\leq (1-t)^\alpha f(b) + m(1-(1-t)^\alpha)f(a). \end{aligned}$$

Adding the above inequalities with each other, we get:

$$\begin{aligned} &f(ta + m(1-t)b) + f(tb + m(1-t)a) \\ &+ f((1-t)a + mtb) + f((1-t)b + mta) \\ &\leq [t^\alpha + m(1-t^\alpha) + (1-t)^\alpha + m(1-(1-t)^\alpha)](f(a) + f(b)). \end{aligned}$$

Now integrating over  $t \in [0, 1]$  and taking into account that:

$$\int_0^1 f(ta + m(1-t)b) dt = \int_0^1 f((1-t)a + mtb) dt = \frac{1}{mb-a} \int_a^{mb} f(x) dx$$

and

$$\int_0^1 f(tb + m(1-t)a) dt = \int_0^1 f((1-t)b + mta) dt = \frac{1}{b-ma} \int_{ma}^b f(x) dx,$$

we obtain the inequality (2.12).  $\square$

**Remark 7.** Choosing  $\alpha = 1$  in (2.12), we obtain (1.6).

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