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A New Approach to the Lebesgue-Radon-Nikodym Theorem with respect to Weighted p-adic Invariant Integral on $\mathbb{Z}_p$

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ABSTRACT. We will give a new proof of the Lebesgue-Radon-Nikodym theorem with respect to weighted p-adic q-measure on Z_p, using Mahler expansion of continuous functions, studied by the authors in 2012. In the special case, $q = 1$, we can derive the same result as in Kim, 2012, Kim et al, 2011.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols Z_p, Q_p, and C_p denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of Q_p, respectively. The p-adic norm $|.|_p$ is defined by $|x|_p = p^{-v_p(x)} = p^{-r}$ for $x = p^r \frac{s}{t}$ with $(p, s) = (p, t) = 1$ and $r \in Q$ (see [1–11]).

When one speaks of q-extension, q can be regard as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and we use the notations of q-numbers as follows:

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \quad \text{and} \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. The fermionic invariant measure on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\mu_{-1}(a + p^n Z_p) = (-1)^a, \tag{1.1}$$

where

$$a + p^n Z_p = \{x \in Z_p | x \equiv a \pmod{p^n}\},$$

and $a \in Z$ with $0 \le a < p^n$ (see [3,5,6,9]).

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From (1.1), the fermionic p-adic invariant integral on Z_p is defined by Kim as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1.2}$$

where $f \in C(Z_p)$ (see [3,5,6,8,9,10,11]).

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure Z_p satisfying

$$\left|\mu_{-1}(a+p^nZ_p) - \mu_{-1}(a+p^{n+1}Z_p)\right|_p \le \delta_n, \quad \text{(see [3, 10])}, \tag{1.3}$$

where $\delta_n \to 0$, a is a element of Z_p, and δ_n is independent of a (for strongly fermionic measure, δ_n is replaced by Cp^{-n}, where C is a positive constant).

Let $f(x)$ be a function defined on Z_p. The fermionic integral of f with respect to a weakly fermionic measure μ_{-1} is

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)\mu_{-1}(x+p^nZ_p),$$

if the limit exists.

If μ_{-1} is a weakly fermionic measure on Z_p, then we can define Radon-Nikodym derivative of μ_{-1} with respect to the Haar measure on Z_p as follows:

$$f_{\mu_{-1}}(x) = \lim_{n\to\infty} \mu_{-1}(x+p^nZ_p), \quad \text{(see [6, 13])}. \tag{1.4}$$

Note that $f_{\mu_{-1}}$ is only a continuous function on Z_p. Let $UD(Z_p)$ be the space of uniformly differentiable functions on Z_p. For $f \in UD(Z_p)$, let us define $\mu_{-1,f}$ as follows:

$$\mu_{-1,f}(x+p^nZ_p) = \int_{x+p^nZ_p} f(x)d\mu_{-1}(x), \quad \text{(see [6, 13])}, \tag{1.5}$$

where the integral is the fermionic p-adic invariant integral. From (1.5), we can easily note that $\mu_{-1,f}$ is a strongly fermionic measure on Z_p. Since

$$\begin{aligned}\left|\mu_{-1,f}(x+p^nZ_p) - \mu_{-1,f}(x+p^{n+1}Z_p)\right|_p &= \Big|\sum_{x=0}^{p^n-1} f(x)(-1)^x - \sum_{x=0}^{p^n} f(x)(-1)^x\Big|_p \\ &= \Big|\frac{f(p^n)}{p^n}\Big|\big|p^n\big| \le Cp^{-n},\end{aligned}$$

where C is positive constant.

The purpose of this paper is to give a new proof of a Lebesgue-Radon-Nikodym type theorem with respect to the fermionic weighted p-adic q-measure on Z_p (see

[13]), by using Mahler expansion of continuous functions in [12].

2. The Lebesgue-Radon-Nikodym theorem with respect to the weighted p-adic q-measure

For any positive integer a and n with $a < p^n$ and $f \in UD(Z_p)$, we define $\tilde{\mu}_{f,-q}$, weighted ferminonic measure on Z_p as follows:

$$\tilde{\mu}_{f,-q}(a + p^n Z_p) = \int_{a+p^n\mathbb{Z}_p} q^x f(x) d\mu_{-1}(x), \tag{2.1}$$

where the integral is the fermionic p-adic invariant integral on Z_p.

From (2.1), we note that

$$\begin{aligned} \tilde{\mu}_{f,-q}(a + p^n Z_p) &= \lim_{m\to\infty} \sum_{x=0}^{p^m-1} f(a + p^n x)(-1)^{a+p^n x} q^{a+p^n x} \\ &= (-1)^a q^a \lim_{m\to\infty} \sum_{x=0}^{p^{m-n}-1} f(a + p^n x)(-1)^x q^{p^n x} \\ &= (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x) q^{a+p^n x} d\mu_{-1}(x). \end{aligned} \tag{2.2}$$

By (2.2), we get

$$\tilde{\mu}_{f,-q}(a + p^n Z_p) = (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x) q^{a+p^n x} d\mu_{-1}(x). \tag{2.3}$$

Thus, by (2.3), we have

$$\tilde{\mu}_{\alpha f+\beta g,-q} = \alpha\tilde{\mu}_{f,-q} + \beta\tilde{\mu}_{g,-q}, \tag{2.4}$$

where $f, g \in UD(Z_p)$ and α, β are positive constants.

By (2.1), (2.2), (2.3) and (2.4), we get

$$\left|\tilde{\mu}_{f,-q}(a + p^n Z_p)\right|_p \leq \|f_q\|_\infty, \tag{2.5}$$

where $\|f_q\|_\infty = \sup_{x\in Z_p} |f(x)q^x|_p$.

Let $P(x) \in C_p[x]$ be an arbitrary polynomial. Now we show $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p-adic invariant measure on Z_p. Without a loss of generality, it is enough to prove the statement for $P(x) = x^k$.

For $a \in Z$ with $0 \leq a < p^n$, we have

$$\tilde{\mu}_{P,-q}(a + p^n Z_p) = \lim_{m\to\infty} (-q)^a \sum_{i=0}^{p^{m-n}-1} q^{p^n i}(a + ip^n)^k(-1)^i. \tag{2.6}$$

Note that

$$(a+ip^n)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} a^{k-\ell}(ip^n)^\ell \equiv a^k \pmod{p^n}. \tag{2.7}$$

By (2.6) and (2.7), we easily get

$$\begin{aligned} \tilde{\mu}_{P,-q}(a+p^nZ_p) &\equiv (-1)^a q^a a^k \frac{2}{[2]_{q^{p^n}}} \pmod{p^n} \\ &\equiv (-1)^a P(a) q^a \frac{2}{[2]_{q^{p^n}}} \pmod{p^n}. \end{aligned} \tag{2.8}$$

For $x \in Z_p$, let $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where x_n, $x_{n+1} \in Z$ with $0 \le x_n < p^n$ and $0 \le x_{n+1} < p^{n+1}$. Then we have

$$\left|\tilde{\mu}_{P,-q}(a+p^nZ_p) - \tilde{\mu}_{P,-q}(a+p^{n+1}Z_p)\right|_p \le Cp^{-n}, \tag{2.9}$$

where C is positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}_{P,-q}(a+p^nZ_p).$$

Then, by (2.8) and (2.9), we see that

$$f_{\tilde{\mu}_{P,-q}}(a) = (-1)^a q^a a^k = (-1)^a q^a P(a). \tag{2.10}$$

We can approach this procedure (2.10) following the idea of Kim, 2012 [4] and using the q-Euler numbers in [2].

In (2.6), we apply (2.7) to obtain

$$\begin{aligned} \tilde{\mu}_{p,-\epsilon}(a+p^nZ_p) &= (-q)^a \lim_{m\to\infty} \Big\{ a^k \sum_{i=0}^{p^{m-n}-1} q^{p^n i}(-1)^i + ka^{k-1} \sum_{i=0}^{p^{m-n}-1} (ip^n)(-1)^i + \\ &\qquad \cdots + \sum_{i=0}^{p^{m-n}-1} q^{p^n i}(ip^n)^k(-1)^i \Big\} \\ &= (-q)^a \Big\{ a^k \mathcal{E}_{0,q^{p^n}} + ka^{k-1}p^n \mathcal{E}_{1,q^{p^n}} + \cdots + p^{nk}\mathcal{E}_{n,q^{p^n}} \Big\}. \end{aligned}$$

Therefore

$$\begin{aligned} f_{\tilde{\mu}_{p,-q}}(a) &= (-q)^a \lim_{n\to\infty} \Big\{ a^k \mathcal{E}_{0,q^{p^n}} + ka^{k-1}p^n \mathcal{E}_{1,q^{p^n}} + \cdots + p^{nk}\mathcal{E}_{n,q^{p^n}} \Big\} \\ &= \lim_{n\to\infty} (-q)^a a^k \mathcal{E}_{0,q^{p^n}} \\ &= \lim_{n\to\infty} (-q)^a a^k \frac{2}{1+q^{p^n}} \\ &= (-q)^a a^k. \end{aligned}$$

Since $f_{\tilde{\mu}_{P,-q}}(x)$ is continuous function on Z_p. For $x \in Z_p$, we have

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x x^k, (k \in Z_+). \tag{2.11}$$

Let $g \in UD(Z_p)$. Then, by (2.9), (2.10) and (2.11), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) \tilde{\mu}_{P,-q}(x + p^n Z_p) \\ &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) q^x [2]_{q^{p^n}} x^k (-1)^x \\ &= \int_{\mathbb{Z}_p} g(x) q^x x^k d\mu_{-1}(x). \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.1. *Let $P(x) \in C_p[x]$ be an arbitrary polynomial. Then $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p-adic invariant measure on Z_p. That is,*

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x P(x) \quad \textit{for all} \quad x \in Z_p.$$

Furthermore, for any $g \in UD(Z_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) q^x d\mu_{-1}(x),$$

where the second integral is weighted fermionic p-adic invariant integral on Z_p.

Let $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ be the Mahler expansion of continuous function on Z_p. Then we note that $\lim_{n\to\infty} n|a_n|_p = 0$.

Let

$$f_m(x) = \sum_{i=0}^{m} a_n \binom{x}{i} \in C_p[x].$$

Then

$$\|f - f_m\|_\infty \leq \sup_{n \leq m} n|a_n|_p. \tag{2.13}$$

The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (2.4) and (2.13), we get

$$\begin{aligned} &\left|\tilde{\mu}_{f,-q}(a + p^n Z_p) - \tilde{\mu}_{f,-q}(a + p^{n+1} Z_p)\right| \\ &\quad \leq \max\{\left|\tilde{\mu}_{f_m,-q}(a + p^n Z_p) - \tilde{\mu}_{f_m,-q}(a + p^{n+1} Z_p)\right|, \\ &\qquad \left|\tilde{\mu}_{f-f_m,-q}(a + p^n Z_p) - \tilde{\mu}_{f-f_m,-q}(a + p^{n+1} Z_p)\right|\}. \end{aligned} \tag{2.14}$$

From Theorem 2.1, we note that

$$\big|\tilde{\mu}_{f-f_m,-q}(a+p^nZ_p)\big|_p \le C^*\|f-f_m\|_\infty \le C_1p^{-n}, \tag{2.15}$$

where C^* and C_1 are positive constants. For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$. So, we see that

$$\begin{aligned}\big|\tilde{\mu}_{f_m,-q}(a+p^nZ_p) - \tilde{\mu}_{f_m,-q}(a+p^{n+1}Z_p)\big|_p \\ = \Big|\frac{f_m(p^n)q^{p^n}}{p^n}\Big|_p\big|p^n\big|_p \le \|f_mq^x\|_\infty\big|p^n\big|_p \le C_2p^{-n},\end{aligned} \tag{2.16}$$

where C_2 is a positive constant.

By (2.15), we get

$$\begin{aligned}&\big|(-1)^af(a)q^a - \tilde{\mu}_{f,-q}(a+p^nZ_p)\big|_p \\ &\le \max\{\big|q^af(a) - f_m(a)q^a\big|_p, \big|q^af_m(a) - \tilde{\mu}_{f_m,-q}(a+p^nZ_p)\big|_p, \big|\tilde{\mu}_{f-f_m,-q}(a+p^nZ_p)\big|_p\} \\ &\le \max\{\big|f(a) - f_m(a)\big|_p, \big|f_m(a) - \tilde{\mu}_{f_m,-q}(a+p^nZ_p)\big|_p, \|f-f_m\|_\infty\}.\end{aligned}$$

Let us assume that fix $\epsilon > 0$, and fix m such that $\|f - f_m\| < \epsilon$. Then we have

$$\big|(-q)^af(a) - \tilde{\mu}_{f,-q}(a+p^nZ_p)\big|_p \le \epsilon \quad \text{for} \quad n \gg 0. \tag{2.17}$$

Thus, by (2.17), we have

$$f_{\tilde{\mu}_{f,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}_{f,-q}(a+p^nZ_p) = (-1)^aq^af(a). \tag{2.18}$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \le p^{-n}$. Then we get

$$\begin{aligned}\tilde{\mu}_{f,-q}(a+p^nZ_p) &= \tilde{\mu}_{f_m,-q}(a+p^nZ_p) + \tilde{\mu}_{f-f_m,-q}(a+p^nZ_p) \\ &= (-1)^aq^af(a) \pmod{p^n}.\end{aligned}$$

For $g \in UD(Z_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)d\tilde{\mu}_{f,-q}(x) = \int_{\mathbb{Z}_p} f(x)g(x)q^xd\mu_{-1}(x).$$

Let f be the function from UD(Z_p) to Lip(Z_p). We easily see that $q^x\mu_{-1}(x + p^nZ_p)$ is a strongly weighted p-adic invariant measure on Z_p and

$$\big|(f_q)_{\mu_{-1}}(a) - q^a\mu_{-1}(a+p^nZ_p)\big|_p \le C_3p^{-n},$$

where $f_q(x) = f(x)q^x$ and C_3 is positive constant and $n \in Z_+$.

If $\tilde{\mu}_{1,-q}$ is associated with strongly weighted fermionic invariant measure on Z_p, then we have

$$\left|\tilde{\mu}_{1,-q}(a+p^nZ_p)-(f_q)_{\mu_{-1}}(a)\right|_p \leq C_4p^{-n},$$

where $n>0$ and C_4 is positive constant.

For $n \gg 0$, we have

$$\begin{aligned}\left|q^a\mu_{-1}(a+p^nZ_p)-\tilde{\mu}_{1,-q}(a+p^nZ_p)\right|_p &\leq \left|q^a\mu_{-1}(a+p^nZ_p)-(f_q)_{\tilde{\mu}_{-1}}(a)\right|_p \\ &\quad + \left|(f_q)_{\tilde{\mu}_{-1}}(a)-\tilde{\mu}_{1,-q}(a+p^nZ_p)\right|_p \\ &\leq K,\end{aligned} \tag{2.19}$$

where K is positive constant.

Hence, $q\mu_{-1}-\tilde{\mu}_{1,-q}$ is a weighted measure on Z_p. Therefore, we obtain the following theorem.

Theorem 2.2. *Let $q\mu_{-1}$ be a strongly weighted p-adic invariant measure on Z_p, and assume that the fermionic weighted Radon-Nikodym derivative $(f_q)_{\mu_{-1}}$ on Z_p is uniformly differentiable function. Suppose that $\tilde{\mu}_{1,-q}$ is the strongly weighted fermionic p-adic invariant measure associated with $(f_q)_{\mu_{-1}}$. Then there exists a weighted measure $\tilde{\mu}_{2,-q}$ on Z_p such that*

$$q^x\mu_{-1}(x+p^nZ_p)=\tilde{\mu}_{1,-q}(x+p^nZ_p)+\tilde{\mu}_{2,-q}(x+p^nZ_p).$$

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