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A New Hilbert-type Inequality with the Integral in Whole Plane

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Abstract. In this paper, by estimating the weight function, we give a new Hilbert-type inequality with the integral in whole plane. As its applications, we consider the equivalent and a particular result.

1. Introduction

If $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which has been extended by Hardy-Riesz:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ also is the best possible.

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Hilbet's inequality is important in analysis and its applications. It attracted some attention in the recent years. Actually, inequalities (1.1) and (1.2) have many generalizations and variants. (1.1) has been strengthened by Yang and others. (including double series inequalities) [2-18].

In 2008 Xie and Zitian Xie and Zeng Zheng gave a new Hilbert-type Inequality [2] as follows:

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x),g(x)\geq 0$ such that $0<\int_0^\infty x^{-1-p/2}f^p(x)dx<\infty$ and $0<\int_0^\infty x^{-1-q/2}g^q(x)dx<\infty$, then

$$\begin{aligned}(1.3)\qquad &\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+a^2y)(x+b^2y)(x+c^2y)}dxdy\\ &< K\left\{\int_0^\infty x^{-1-p/2}f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{-1-q/2}g^q(x)dx\right\}^{1/q},\end{aligned}$$

where the constant factor $K=\frac{\pi}{(a+b)(a+c)(b+c)}$ is the best possible.

In this paper, by obtaining the weight function, we give a new Hilbert-type inequality with the integral in whole plane.

In the following, we always suppose that $a=\cos\theta$, $\theta\in(0,\frac{\pi}{2})$; $p>1$, $1/p+1/q=1$; $m+n=2$, $m,\ n\in(0,2)$.

2. Some lemmas

Lemma 2.1. *If* $k_1:=\int_0^\infty\frac{dt}{(t^2+2at+1)t^{1-n}}$,$k_2:=\int_0^\infty\frac{dt}{(t^2-2at+1)t^{1-n}}$, *and* $k:=\int_{-\infty}^\infty\frac{dt}{(t^2+2at+1)|t|^{1-n}}$, *then*

$$(2.1)\qquad k_1=\begin{cases}\pi\csc\theta\csc(n\pi)\sin((1-n)\theta), & \text{if } n\neq 1,\\ \theta\csc\theta, & \text{if } n=1\end{cases}$$

$$k_2=\begin{cases}\pi\csc\theta\csc(n\pi)\sin((1-n)(\pi-\theta)), & \text{if } n\neq 1,\\ (\pi-\theta)\csc\theta, & \text{if } n=1\end{cases}$$

and

$$\begin{aligned}(2.2)\qquad k=k_1+k_2&=\begin{cases}2\pi\csc\theta\csc(n\pi)\sin\frac{(1-n)\pi}{2}\cos\frac{(1-n)(\pi-\theta)}{2}, & \text{if } n\neq 1,\\ \pi\csc\theta, & \text{if } n=1,\end{cases}\\ &=\begin{cases}\pi\csc\theta\sec\frac{n\pi}{2}\cos\frac{(1-n)(\pi-\theta)}{2}, & \text{if } n\neq 1,\\ \pi\csc\theta, & \text{if } n=1\end{cases}.\end{aligned}$$

Proof. Let $f(z)=\frac{1}{(1+2bz+z^2)z^\alpha}=\frac{1}{(z-z_1)(z-z_2)z^\alpha}$, $\alpha\in(-1,1)$, then
$k^*:=\int_0^\infty\frac{dt}{(t^2+2bt+1)t^\alpha}=\frac{2\pi i}{1-e^{-2\alpha\pi i}}[Res(f,z_1)+Res(f,z_2)]$;
if $b=\cos\vartheta$ $(0<\vartheta<\pi)$, and $z_1=-e^{i\vartheta}$, $z_2=-e^{-i\vartheta}$, then

$$k^*=\frac{2\pi i}{1-e^{-2\alpha\pi i}}\left[\frac{1}{(-2i\sin\vartheta)(-e^{i\vartheta})^\alpha}+\frac{1}{(2i\sin\vartheta)(-e^{-i\vartheta})^\alpha}\right]=\pi\csc\vartheta\csc(\alpha\pi)\sin(\alpha\vartheta),$$

setting $\alpha = 1-n$, $\vartheta = \theta$ and $\vartheta = \pi - \theta$, we have (2.1).

On the other hand,

$$k = \int_0^{\infty} \frac{dt}{(t^2+2at+1)t^{1-n}} + \int_{-\infty}^{0} \frac{dt}{(t^2+2at+1)(-t)^{1-n}} = k_1 + k_2$$
$$= \begin{cases} 2\pi \csc\theta \csc(n\pi) \sin\frac{(1-n)\pi}{2} \cos\frac{(1-n)(\pi-\theta)}{2}, & \text{if } n \neq 1, \\ \pi \csc\theta, & \text{if } n = 1. \end{cases}$$

We obtain (2.2). □

Lemma 2.2. *Define the weight functions as follow:*

$$w(x) := \int_{-\infty}^{\infty} \frac{|x|^m dy}{(x^2+2axy+y^2)|y|^{1-n}}, \quad \widetilde{w}(y) := \int_{-\infty}^{\infty} \frac{|y|^n dx}{(x^2+2axy+y^2)|x|^{1-m}}.$$

Then

$$w(x) = \widetilde{w}(y) = k. \tag{2.3}$$

Proof. We only prove that $w(x) = k$ for $x \in (-\infty, 0)$.
$w(x) = \int_{-\infty}^{0} \frac{|x|^m dy}{(x^2+2axy+y^2)|y|^{1-n}} + \int_0^{\infty} \frac{|x|^m dy}{(x^2+2axy+y^2)|y|^{1-n}} := w_1 + w_2$,
setting $y = tx$,then $w_1 = \int_{-\infty}^{0} \frac{(-x)^{2-n} dy}{(x^2+2axy+y^2)(-y)^{1-n}} = \int_0^{\infty} \frac{dt}{(t^2+2at+1)t^{1-n}} = k_1$.

Similarly, setting $y = -tx$, $w_2 = \int_0^{\infty} \frac{(-x)^{2-n} dy}{(x^2+2axy+y^2)y^{1-n}} = \int_0^{\infty} \frac{dt}{(t^2-2at+1)t^{1-n}} = k_2$, and $w(x) = k$.

Easily $\widetilde{w}(x) = k$, using lemma 2.1, the lemma is proved. □

Lemma 2.3. *For* $\varepsilon > 0$, *and* $(n - \max\{\frac{2\varepsilon}{p}, \frac{2\varepsilon}{q}\}) \in (0, 2)$, *define both functions,* $\widetilde{f}$ *and* $\widetilde{g}$, *as follow:*

$$\widetilde{f}(x) = \begin{cases} x^{m-1-2\varepsilon/p}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{m-1-2\varepsilon/p}, & \text{if } x \in (-\infty,-1) \end{cases}; \tag{2.4}$$

$$\widetilde{g}(x) = \begin{cases} x^{n-1-2\varepsilon/q}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{n-1-2\varepsilon/q}, & \text{if } x \in (-\infty,-1) \end{cases}.$$

Then

$$I(\varepsilon) := \varepsilon \left\{ \int_{-\infty}^{\infty} |x|^{p(1-m)-1} \widetilde{f}^p(x) dx \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} |x|^{q(1-n)-1} \widetilde{g}^q(x) dx \right\}^{1/q} = 1; \tag{2.5}$$

$$\widetilde{I}(\varepsilon) := \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\widetilde{f}(x)\widetilde{g}(y)}{x^2+2axy+y^2} dxdy = k + o(1) \quad (\varepsilon \to 0^+). \tag{2.6}$$

Proof. Easily,

$$I(\varepsilon) = 2\varepsilon \left\{ \int_1^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} \left\{ \int_1^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/q} = 1.$$

Let $y = -Y$, using $\widetilde{f}(-x) = \widetilde{f}(x)$, $\widetilde{g}(-x) = \widetilde{g}(x)$, and

$$\widetilde{f}(-x) \int_{-\infty}^{\infty} \frac{\widetilde{g}(y)dy}{x^2 - 2axy + y^2} = \widetilde{f}(x) \int_{-\infty}^{\infty} \frac{\widetilde{g}(Y)dY}{x^2 + 2axY + Y^2}$$

we have that $\widetilde{f}(x) \int_{-\infty}^{\infty} \frac{\widetilde{g}(y)dy}{x^2+2axy+y^2}$ is an even function on x, then

$$\begin{aligned}
\widetilde{I}(\varepsilon) &= 2\varepsilon \int_0^{\infty} \widetilde{f}(x) \left(\int_{-\infty}^{\infty} \frac{\widetilde{g}(y)}{x^2 + 2axy + y^2} dy \right) dx \\
&= 2\varepsilon \left[\int_1^{\infty} x^{m-1-\frac{2\varepsilon}{p}} \left(\int_{-\infty}^{-1} \frac{(-y)^{n-1-\frac{2\varepsilon}{q}}}{x^2 + 2axy + y^2} dy \right) dx \right. \\
&\qquad \left. + \int_1^{\infty} x^{m-1-\frac{2\varepsilon}{p}} \left(\int_1^{\infty} \frac{y^{n-1-\frac{2\varepsilon}{q}}}{x^2 + 2axy + y^2} dy \right) dx \right] \\
&:= I_1 + I_2.
\end{aligned}$$

Setting $y = tx$ then

$$\begin{aligned}
I_1 &= 2\varepsilon \left[\int_1^{\infty} x^{m-1-\frac{2\varepsilon}{p}} \left(\int_1^{\infty} \frac{y^{n-1-\frac{2\varepsilon}{q}}}{x^2 - 2axy + y^2} dy \right) dx \right] \\
&= 2\varepsilon \left[\int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{\frac{1}{x}}^{\infty} \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} dt \right) dx \right] \\
&= 2\varepsilon \left[\int_1^{\infty} x^{-1-2\varepsilon} \left(\int_1^{\infty} \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} dt \right) dx \right. \\
&\qquad \left. + \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{\frac{1}{x}}^{1} \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} dt \right) dx \right] \\
&= \int_1^{\infty} \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} dt + 2\varepsilon \int_0^1 \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} \left(\int_{\frac{1}{t}}^{\infty} x^{-1-2\varepsilon} dx \right) dt \\
&= \int_1^{\infty} \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} dt + \int_0^1 \frac{t^{n-1+\frac{2\varepsilon}{p}}}{t^2 - 2at + 1} dt \\
&= \int_0^{\infty} \frac{t^{n-1-\frac{2\varepsilon}{q}}}{t^2 - 2at + 1} dt + \int_0^1 (t^{\frac{2\varepsilon}{p}} - t^{-\frac{2\varepsilon}{q}}) \frac{t^{n-1}}{t^2 - 2at + 1} dt \\
&= \pi \csc\theta \csc((n - \frac{2\varepsilon}{q})\pi) \sin[(1 - (n - \frac{2\varepsilon}{q}))(\pi - \theta)] + \eta(\varepsilon).
\end{aligned}$$

there $\lim_{\varepsilon\to 0^+}\eta(\varepsilon)=0$, and we have $I_1 \to k_2 \quad (\varepsilon \to 0^+)$.

Similarly $I_2 \to k_1 \quad (\varepsilon \to 0^+)$. The lemma is proved. □

Lemma 2.4. *We have*

$$J := \int_{-\infty}^{\infty} |y|^{np-1} \left(\int_{-\infty}^{\infty} \frac{f(x)}{x^2+2axy+y^2} dx \right)^p dy \le k^p \int_{-\infty}^{\infty} |x|^{p(1-m)-1} f^p(x) dx \tag{2.7}$$

Proof. By lemma 2.2, we find

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} \frac{f(x)}{x^2+2axy+y^2} dx \right)^p \\
&= \left[\int_{-\infty}^{\infty} \frac{1}{x^2+2axy+y^2} \left(\frac{|x|^{(1-m)/q}}{|y|^{(1-n)/p}} f(x) \right) \left(\frac{|y|^{(1-n)/p}}{|x|^{(1-m)/q}} \right) dx \right]^p \\
&\le \int_{-\infty}^{\infty} \frac{1}{x^2+2axy+y^2} \frac{|x|^{(1-m)(p-1)}}{|y|^{1-n}} f^p(x) dx \\
&\qquad \left(\int_{-\infty}^{\infty} \frac{1}{x^2+2axy+y^2} \frac{|y|^{(1-n)(q-1)}}{|x|^{1-m}} dx \right)^{p-1} \\
&= k^{p-1} |y|^{-np+1} \int_{-\infty}^{\infty} \frac{1}{x^2+2axy+y^2} \frac{|x|^{(1-m)(p-1)}}{|y|^{1-n}} f^p(x) dx,
\end{aligned}$$

$$\begin{aligned}
J &\le k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{1}{x^2+2axy+y^2} \frac{|x|^{(1-m)(p-1)}}{|y|^{1-n}} f^p(x) dx \right] dy \\
&= k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{1}{x^2+2axy+y^2} \frac{|x|^{(1-m)(p-1)}}{|y|^{1-n}} dy \right] f^p(x) dx \\
&= k^p \int_{-\infty}^{\infty} |x|^{p(1-m)-1} f^p(x) dx.
\end{aligned}$$

3. Main results

Theorem 3.1. *If both functions, $f(x)$ and $g(x)$ are nonnegative measurable functions, and satisfy*
$0 < \int_{-\infty}^{\infty} |x|^{p(1-m)-1} f^p(x) dx < \infty$ *and* $0 < \int_{-\infty}^{\infty} |x|^{q(1-n)-1} g^q(x) dx < \infty$, *then*

$$\begin{aligned}
I^* &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{x^2+2axy+y^2} dx dy \\
&< k \left(\int_{-\infty}^{\infty} |x|^{p(1-m)-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q(1-n)-1} g^q(x) dx \right)^{1/q},
\end{aligned} \tag{3.1}$$

and

$$J = \int_{-\infty}^{\infty} |y|^{p(1-m)-1} (\int_{-\infty}^{\infty} \frac{f(x)}{x^2+2axy+y^2} dx)^p dy < k^p \int_{-\infty}^{\infty} |x|^{p(1-n)-1} f^p(x) dx. \tag{3.2}$$

Inequalities (3.1) *and* (3.2) *are equivalent, and where the constant factors* k *and* k^p *are the best possible.*

Proof. If (2.7) takes the form of equality for some $y \in (-\infty, 0) \cup (0, \infty)$, then there exists constants M and N, such that they are not all zero, and

$$M \frac{|x|^{(1-m)(p-1)}}{|y|^{1-n}} f^p(x) = N \frac{|y|^{(1-n)(q-1)}}{|x|^{1-m}} \quad \text{a.e. in} \quad (-\infty, \infty),$$

Hence, there exists a constant C, such that

$$M|x|^{(1-m)p} f^p(x) = N|y|^{(1-n)q} = C \qquad \text{a.e. in} \quad (-\infty, \infty).$$

We claim that $M = 0$. In fact, if $M \neq 0$, then $|x|^{p(1-m)-1} f^p(x) = \frac{C}{M|x|}$ a.e. in $(-\infty, \infty)$ which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p(1-m)-1} f^p(x) dx < \infty$. In the same way, we claim that $N = 0$. This is too a contradiction and hence by (2.7), we have (3.2).

By Hölder's inequality with weight and (3.2), we have,

$$I^* = \int_{-\infty}^{\infty} \left[|y|^{n-1+\frac{1}{q}} \int_{-\infty}^{\infty} \frac{f(x)}{x^2+2axy+y^2} dx \right] \left[|y|^{1-n-\frac{1}{q}} g(y) \right] dy \leq (J)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q(1-n)-1} g^q(y) dy \right)^{1/q}. \tag{3.3}$$

Using (3.2), we have (3.1).

Setting $g(y) = |y|^{pn-1} (\int_{-\infty}^{\infty} \frac{f(x)}{x^2+2axy+y^2} dx)^{p-1}$, then $J = \int_{-\infty}^{\infty} |y|^{q(1-n)-1} g^q(y) dy$, by (2.7) we have $J < \infty$. If $J = 0$ then (3.2)is proved; if $0 < J < \infty$, by (3.1), we obtain

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-n)-1} g^q(y) dy = J = I^* < k \left(\int_{-\infty}^{\infty} |x|^{p(1-m)-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q(1-n)-1} g^q(x) dx \right)^{1/q},$$

$$\left(\int_{-\infty}^{\infty} |x|^{q(1-m)-1} g^q(x) dx \right)^{1/p} = J^{1/p} < k \left(\int_{-\infty}^{\infty} |x|^{p(1-n)-1} f^p(x) dx \right)^{1/p}.$$

Inequalities (3.1) and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with $h < k$), such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{x^2+2axy+y^2}dxdy < h\left(\int_{-\infty}^{\infty}|x|^{p(1-m)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(1-n)-1}g^q(x)dx\right)^{1/q}. \tag{3.4}$$

For $\varepsilon > 0$, by (3.4), using lemma 2.3, we have

$$k+o(1) < \varepsilon h\left(\int_{-\infty}^{\infty}|x|^{-1}\widetilde{f}^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}\widetilde{g}^q(x)dx\right)^{1/q} = h.$$

Hence we find, $k+o(1) < h$. For $\varepsilon \to 0^+$, it follows that $k \le h$, which contradicts the fact that $h < k$. Hence the constant k in (3.1) is the best possible.

Thus we complete the prove of the theorem. □

Remark. For $a = \frac{1}{2}$, $(\theta = \frac{\pi}{3}), n \neq 1$,in (3.1), we have the following particular result:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{x^2+xy+y^2}dxdy$$

$$< \frac{2}{\sqrt{3}}\pi\sec\frac{n\pi}{2}\cos\frac{(1-n)\pi}{3}\left(\int_{-\infty}^{\infty}|x|^{p(1-m)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|x|^{q(1-n)-1}g^q(x)dx\right)^{\frac{1}{q}}.$$

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