

Isotropic Submanifolds of Real Space Forms

YOUNG HO KIM

*Department of Mathematics, Teachers' College, Kyungpook National University,
Daegu 702-701, Korea*
e-mail: yhkim@knu.ac.kr

ABSTRACT. We study some functions defined on the unit tangent space, which are formed with the second fundamental form of submanifolds of a real space form. These give an exact expression of isotropy of submanifolds in a real space form and a relationship between intrinsic invariants and extrinsic ones.

1. Introduction

The relationship between intrinsic and extrinsic invariants of submanifolds in Euclidean space or a space form is very interesting for geometers. One example for an obstruction to minimal isometric immersions into Euclidean space is that the Ricci curvature takes non-negative values. On the other hand, Chen ([6]) initiated the problems imposed on submanifolds of Euclidean space of arbitrary codimension related to Nash's embedding theorem. Many geometers have been working on related problems and applications since the late 1990. He began with the notion of delta-invariant $\delta(n_1, \dots, n_k)$ and sets up the maximal principle with it and the squared mean curvature to define the ideal immersion which is used to characterize 1-type immersions which characterize minimal submanifolds in Euclidean space or a space form. Furthermore, this contributes to giving obstruction theorems for minimal immersions in Euclidean space and Lagrangian immersions in complex Euclidean space.

Let M be a Riemannian n -manifold. For an orthonormal basis e_1, \dots, e_n of the tangent space $T_p M$ and the scalar curvature τ , the Riemannian invariant at p is defined by

$$\delta(n_1, \dots, n_k)(p) = 2\tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\}(p),$$

where infimum is taken for all possible choices of orthonormal subspaces L_1, \dots, L_k , satisfying $n_j = \dim L_j$, ($j = 1, \dots, k$).

Received April 27, 2011; accepted September 28, 2011.

2010 Mathematics Subject Classification: 53B25, 53C40.

Key words and phrases: Key words and phrases: unit tangent space, isotropic submanifold. Supported by Basic Science Research Program through National Research Foundation funded by the Ministry of Education, Science and Technology (2010-0007184).

Chen ([6]) introduced following:

For any n -dimensional submanifold M of a Riemannian space form $R^m(c)$ of constant sectional curvature c and for any k -tuple (n_1, \dots, n_k) , we have

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)c,$$

where

$$c(n_1, \dots, n_k) = \frac{n^2(n+k-1 - \sum n_j)}{2(n+k - \sum n_j)}$$

and

$$b(n_1, \dots, n_k) = \frac{1}{2}\{n(n-1) - \sum_{j=1}^k n_j(n_j-1)\}.$$

From this, we have immediately an obstruction theorem for minimal submanifold: Let M be a Riemannian n -manifold. If there exists a k -tuple (n_1, \dots, n_k) and a point p in M such that

$$\delta(n_1, \dots, n_k)(p) \geq \frac{1}{2}\{n(n-1) - \sum_j n_j(n_j-1)\}c,$$

then M admits no minimal isometric immersion into any Riemannian space form $R^m(c)$ with arbitrary codimension. In particular, if $\delta(n_1, \dots, n_k)(p) > 0$ at a point for some k -tuple (n_1, \dots, n_k) , then M admits no minimal isometric immersion into any Euclidean space with arbitrary codimension. He also get an inequality between an intrinsic invariant and mean curvature of a submanifold in a Riemannian space form with the notion of delta invariant:

$$\|H\|^2 \geq \frac{\tau}{n(n-1)} - c.$$

The unit tangent space U_pM at a point p in a Riemannian manifold M is regarded as a unit sphere in an ordinary Euclidean space in a natural manner. In the present paper, using the geometry of U_pM , we give a basic inequality relating some intrinsic and extrinsic invariants of submanifold of a real space form $R^m(c)$ and we can express the normal curvature of M at p in terms of the mean curvature and the scalar curvature if M is isotropic at p and we give an inequality on a submanifold M of a Riemannian space form $R^m(c)$, which characterizes that M is totally geodesic.

2. Preliminaries

Let $R^m(c)$ be a Riemannian space form, i.e., a Riemannian manifold of constant curvature c that is simply connected and let M be a submanifold of $R^m(c)$. Let $\tilde{\nabla}$ be the Levi-Civita connection of $R^m(c)$ and ∇ the induced connection of M . Then, the Gauss formula is given by

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

where h is the second fundamental form of M and X, Y vector fields tangent to M . Let V be a vector field normal to M . Then, the Weingarten formula is given by

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

where A is the shape operator of M and ∇^\perp the normal connection of M . The shape operator A and the second fundamental form h are related by

$$\langle A_V X, Y \rangle = \langle h(X, Y), V \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the metric tensor of $R^m(c)$ as well as that of M . If R denotes the Riemann-Christoffel curvature tensor of M , then the Gauss, Codazzi and Ricci equations are obtained by

$$\begin{aligned} (2.2) \quad R(X, Y; Z, W) &= \langle R(X, Y)Z, W \rangle \\ &= c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ &\quad + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle), \end{aligned}$$

$$(2.3) \quad (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) = 0,$$

$$(2.4) \quad \langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle$$

for tangent vector fields X and Y and normal vector fields ξ and η , where $\bar{\nabla}$ denotes the operator of covariant differentiation on the direct sum of tangent bundle and normal bundle defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and R^\perp is the curvature tensor of the normal connection ∇^\perp .

Let p be a point of M . Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for the tangent space $T_p M$ of M at p . Then, the Ricci tensor S is defined by

$$\begin{aligned} (2.5) \quad S(X, Y) &= \text{tr}\{Z \rightarrow R(Z, X)Y\} = \sum_{i=1}^n \langle R(e_i, X)Y, e_i \rangle \\ &= c(n-1)\langle X, Y \rangle + n\langle H, h(X, Y) \rangle - \sum_{i=1}^n \langle h(X, e_i), h(Y, e_i) \rangle, \end{aligned}$$

where $H = \frac{1}{n} \text{tr} h$ denotes the mean curvature vector field of M . Let v be a unit tangent vector of M at p . Then, $S(v, v)$ is called the Ricci curvature of M in the direction of v .

A submanifold M of a Riemannian manifold \tilde{M} is called λ -isotropic at p if $\lambda = \|h(u, u)\|$ is independent of the choice of $u \in U_p M = \{x \in T_p M | \langle x, x \rangle = 1\}$, the unit tangent space of M at p .

According to O'Neill ([8]), if M is isotropic at p , then we have

$$(2.6) \quad \langle h(u, v), h(u, u) \rangle = 0,$$

$$(2.7) \quad \|h(u, u)\|^2 = \langle h(u, u), h(v, v) \rangle + 2\langle h(u, v), h(u, v) \rangle$$

for any orthonormal vectors u and v that are tangent to M at p .

A submanifold M of $R^m(c)$ is called minimal if the mean curvature vector field H is identically zero and totally geodesic if the second fundamental form vanishes identically.

3. Main results

Let M be an n -dimensional submanifold of an m -dimensional Riemannian space form $R^m(c)$ of constant curvature c . Let p be a point of M . Then, the tangent space T_pM of M at p admits a natural manifold structure which is isometric to Euclidean n -space E^n . The unit tangent space U_pM of M at p inherits a manifold structure isometric to the $(n-1)$ -unit sphere $S^{n-1}(1)$ in E^n . Let \tilde{D} be the Riemannian connection on T_pM and D that of U_pM induced from \tilde{D} . Let $\{e_1, e_2, \dots, e_{n-1}\}$ be a local orthonormal frame about a point $x \in U_pM$ such that $(D_{e_i}e_j)(x) = 0$. Then, $\{e_1, e_2, \dots, e_{n-1}, x\}$ forms an orthonormal basis for T_pM .

Let us define two functions f and F on U_pM as follows:

$$f(x) = \langle h(x, x), h(x, x) \rangle$$

and

$$F(x) = \sum_{i=1}^{n-1} \langle h(e_i, x), h(e_i, x) \rangle + f(x)$$

for $x \in U_pM$.

Lemma 3.1. *Let M be an n -dimensional submanifold of an m -dimensional real space form $R^m(c)$ whose Ricci curvature is not less than $c(n-1)$. Then, A_H is positive semi-definite.*

Proof. Suppose there exists a point p in M and $v \in U_pM$ such that $\langle A_H v, v \rangle < 0$. From (2.5), the Ricci curvature $S(v, v)$ in the direction of v is given by

$$(3.1) \quad S(v, v) = c(n-1) + n\langle A_H v, v \rangle - F(v),$$

which implies $S(v, v) < c(n-1)$, that is a contradiction. It completes the proof. \square

Let Δ_D be the operator of Laplacian on U_pM defined by

$$\Delta_D = \sum_{i=1}^{n-1} (\tilde{D}_{e_i} \tilde{D}_{e_i} - \tilde{D}_{D_{e_i} e_i})$$

where $\{e_1, \dots, e_{n-1}\}$ is a local orthonormal frame on U_pM defined as above.

A similar way to prove Lemma 3.2 in [1], we can prove the following lemma.

Lemma 3.2 *Let M be an n -dimensional submanifold of a real space form $R^m(c)$. Let p be a point of M . Then we have*

$$(3.2) \quad \int_{U_pM} \langle A_Hx, x \rangle dx_p = \|H\|^2(p) \text{vol}(U_pM),$$

where dx_p is the canonical volume element of U_pM and $\text{vol}(U_pM)$ denotes the volume of U_pM .

Let $v \in U_pM$ and we choose an orthonormal frame $\{e_1, \dots, e_{n-1}\}$ for the tangent bundle of U_pM such that $(D_X e_i)(v) = 0$ for any $X \in T_v(U_pM)$. Then, the Laplacian of the Ricci curvature at the point v is obtained

$$\frac{1}{2}(\Delta_D S(x, x))(v) = -nS(v, v) + \tau(p),$$

where τ denotes the scalar curvature of M . Integrating this over U_pM and making use of the Hopf's lemma, we have

$$(3.3) \quad \int_{U_pM} S(x, x) dx_p = \frac{\tau(p)}{n} \text{vol}(U_pM).$$

On the other hand, if we integrate (3.1) over U_pM and using Lemma 3.1 and (3.3) we get

$$(3.4) \quad \int_{U_pM} F(x) dx_p = \{c(n-1) + n\|H\|^2 - \frac{\tau}{n}\} \text{vol}(U_pM).$$

Thus, we have

Lemma 3.3. *Let M be an n -dimensional submanifold of a real space form $R^m(c)$ and let p be a point of M . Then, we have*

$$\int_{U_pM} S(x, x) dx_p = \frac{\tau}{n} \text{vol}(U_pM),$$

$$\int_{U_pM} F(x) dx_p = \{c(n-1) + n\|H\|^2 - \frac{\tau}{n}\} \text{vol}(U_pM).$$

Proposition 3.4 *Let M be an n -dimensional submanifold of a real space form $R^m(c)$. Let p be a point of M . Then, we have*

$$\int_{U_pM} G(x) dx_p = \frac{1}{n-2} \{n(n-1)\|H\|^2 - \tau + n(n-1)c\} \text{vol}(U_pM)$$

where G is a function on U_pM defined by

$$G(x) = \sum_{i=1}^{n-1} \langle h(e_i, x), h(e_i, x) \rangle$$

where $\{e_1, \dots, e_{n-1}\}$ is a local orthonormal frame about each $x \in U_pM$.

Proof. Let $\{e_1, \dots, e_{n-1}\}$ is a local orthonormal frame of the tangent bundle of U_pM near $x \in U_pM$ such that $(D_X e_i)(x) = 0$ for any $X \in T_x(U_pM)$. Then, we have

$$\frac{1}{4}(\Delta_D f)(x) = n\langle H, h(x, x) \rangle - nf(x) + 2G(x).$$

It gives

$$\int_{U_pM} f(x) dx_p = \|H\|^2 \text{vol}(U_pM) + \frac{2}{n} \int_{U_pM} G(x) dx_p.$$

On the other hand, since $F(x) = G(x) + f(x)$, if we take account of Lemma 3.3, then we have

$$\int_{U_pM} G(x) dx_p = \frac{1}{n+2} \{n(n-1)c + n(n-1)\|H\|^2 - \tau\} \text{vol}(U_pM).$$

It completes the proof of the proposition. \square

From this proposition we have the following an inequality between intrinsic invariants and extrinsic invariants of submanifolds of real space forms, which was independently done by B.-Y. Chen.

Corollary 3.5([4]). *Let M be an n -dimensional submanifold of $R^m(c)$. Then, we have*

$$\|H\|^2 \geq \frac{\tau}{n(n-1)} - c$$

and the equality holds at p if and only if p is a umbilical point.

Lemma 3.6. *Let M be an n -dimensional submanifold of a real space form $R^m(c)$. If M is λ -isotropic at p , then λ is completely determined by the squared mean curvature and the scalar curvature in the following form :*

$$\lambda^2 = \frac{3n}{n+2} \|H\|^2 - \frac{2}{n(n+2)} \tau + \frac{2(n-1)}{n+2} c.$$

Proof. Suppose M is λ -isotropic at p . Then, $\lambda^2 = \langle h(x, x), h(x, x) \rangle$ is constant on U_pM . If we take an orthonormal frame $\{e_1, \dots, e_{n-1}\}$ of the tangent bundle of U_pM such that $(D_X e_i)(v) = 0$ for a point $v \in U_pM$ and $X \in T_v(U_pM)$, then we get

$$(3.5) \quad \langle h(e_i, x), h(x, x) \rangle = 0$$

for any $x \in U_p M$ near v . Let x be a vector field on $U_p M$. Taking the directional derivative of (3.4) in the direction of X and evaluating at v , we have

$$(X, e_i)\lambda^2 = \langle h(e_i, X), h(v, v) \rangle + 2\langle h(e_i, v), h(X, v) \rangle$$

where (\cdot, \cdot) denotes the natural inner product defined on $T_v(U_p M)$.

It follows that

$$\begin{aligned} n\lambda^2 &= n\langle A_H v, v \rangle + 2 \sum_{i=1}^{n-1} \langle h(e_i, v), h(e_i, v) \rangle \\ &= n\langle A_H v, v \rangle + 2(F(v) - \lambda^2). \end{aligned}$$

Since this holds for arbitrary point $v \in U_p M$, we have

$$(n + 2)\lambda^2 = n\langle A_H x, x \rangle + 2F(x)$$

for any $x \in U_p M$. Integrating the last expression over $U_p M$ and using Lemma 3.2 and (3.4), we obtain (3.5). \square

From Lemma 3.6, we immediately have

Theorem 3.7. *Let M be an n -dimensional submanifold of a real space form $R^m(c)$. If M is isotropic, then*

$$\|H\|^2 \geq \frac{2\tau}{3n^2} - \frac{2(n-1)}{3n}c$$

and the equality holds if and only if M is totally geodesic.

Corollary 3.8. *Let M be an n -dimensional Riemannian manifold. If there is a point p in M such that $\tau(p) > 0$, then there exists no minimal isotropic immersion of M into a Euclidean space.*

Corollary 3.9. *Let M be an n -dimensional flat Riemannian manifold. If M is immersed as a minimal isotropic submanifold in Euclidean space, it is totally geodesic.*

4. Applications

Application 4.1. Let M be a right helicoid parametrized by

$$x(u, v) = (v \cos u, v \sin u, au), \quad a \in \mathbb{R} - \{0\}.$$

It is easily seen that M is flat and minimal in E^3 . Using Corollary 3.9, we see that M cannot be isotropic.

Application 4.2. Let M be an n -dimensional pseudo-umbilical submanifold in Euclidean space E^m , i.e., the shape operator A_H in the direction of the mean curvature vector field H is proportional to the identity transformation. If H is

parallel in the normal bundle, M is minimally immersed in a hypersphere $S^{m-1}(c)$ of radius $\frac{1}{\sqrt{c}}$ for some $c > 0$ (see [7]). Then, the scalar curvature τ satisfies

$$\tau \leq n(n-1)c.$$

References

- [1] Ahn, S. S., Kim, D.-S. and Kim, Y. H., *Submanifolds of Euclidean space with non-negative Ricci curvature*, Kyungpook Math. J., **35**(1995), 387-392.
- [2] Chen, B.-Y., *Geometry of submanifolds*, Marcel Dekker, New York, 1973.
- [3] Chen, B.-Y., *Some pinching and classification theorems for minimal submanifolds*, Arch. der Math., **60**(1993), 568-578.
- [4] Chen, B.-Y., *Mean curvature and shape operator of isometric immersions in real space forms*, Glasgow Math. J., **38**(1996), 87-97.
- [5] Chen, B.-Y., *Some new obstructios to minimal and Lagragian isometric immersions*, Japanese J. Math., **26**(2000), 105-127.
- [6] Chen, B.-Y., *What can we do with Nashs embedding theorem?*, Soochow J. Math., **30**(2004), 303-338.
- [7] Chen, B.-Y. and K. Yano, *Pseudo-umbilical submanifolds in a Riemannian manifold of constant curvature*, Diff. Geom., in honor of K. Yano, Kinikuniya, Tokyo, 61-71, 1972.
- [8] O'Neill, B., *Isotropic and Kaehler immersions*, Canad. J. Math., **17**(1965), 907-915.