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A Characterization of M_1-Spaces

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Abstract. In this paper, we prove the following theorem which gives a characterization of M_1-spaces by g-function.

Theorem. *A space (X, μ) is an M_1-space if and only if there exists a function $g : \omega \times X \to \mu$ such that:*
1. $\{x\} = \cap_n g(n, x)$.
2. $y \in g(n, x) \Rightarrow g(n, y) \subset g(n, x)$.
3. *There exists a collection $\mathcal{U} = \cup_n \mathcal{U}_n$ of open sets in (X, μ) such that:*
   (a). $\cup\{g(n, x) : x \in U_{n\alpha}\}$ *is regular open for each* $U_{n\alpha} \in \mathcal{U}$.
   (b). *If $H \subset X$ is closed with $y \notin H$, then $y \notin Cl_\mu(\cup\{g(n, x) : x \in U_{n\alpha}\})$ for some $U_{n\alpha} \in \mathcal{U}_n$ with $H \subset U_{n\alpha}$.*

1. Introduction

Ceder [2] defined M_i-spaces, $i = 1, 2, 3$ and proved $M_1 \Rightarrow M_2 \Rightarrow M_3$. It is an interesting problem that whether or not these implications can be reversed. Recall that a space X is an M_1-*space* if X has a σ-closure preserving base $\mathcal{B}$. Recall that a collection $\mathcal{B}$ is a *quasi-base* for X if for each open set U of X and a point $x \in U$, there is $B \in \mathcal{B}$ such that $x \in IntB \subset B \subset U$. A space X is an M_2-*space* if X has a σ-closure preserving *quasi-base* and an M_3-*space* if X has a σ-cushioned pair-base.

Borges [1] gave some important results on M_3-spaces and renamed M_3-spaces as stratifiable spaces. Gruenhage [4] and Junnila [5] independently proved that stratifiable spaces are M_2-spaces. This is an important progress to the problem since stratifiable spaces have been shown to have many useful properties and are preserved by countable products, closed images, arbitrary subspaces; M_1-spaces have a simple and natural definition. In 1990, Rudin [7] suggested several conjectures on general topology for the 21th century. One of the conjectures is called "$M_3 \Rightarrow M_1$ problem ". According to her guess, the problem should have a positive answer. Also the problem is called "Problem on General Metric Spaces" in Wang [8].

In 2007, Lin [6] gave a survey of the most important results about M_i-spaces. It included discussions of the characteristics of M_2-spaces. In order to consider

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the problem, in 2008, Chen [3] gave some new characterizations and properties of stratifiable spaces. The new characterizations of Chen showed a specific geometry construction of M_3-spaces.

But, so far, M_1-spaces have only a simple and natural definition. It will be helpful to consider the $M_3 \Rightarrow M_1$ problem if we have a characterization of M_1-spaces. To do it, we construct a g-function which shows a characterization of M_1-spaces.

In this paper, the letter N denotes the set of positive integers. m and n are used to denote members in N.

2. Section2

In this section, we prove the following theorem which gives a characterization of M_1-spaces by g-function.

Theorem 2.1. *A space (X,μ) is an M_1-space if and only if there exists a function $g : \omega \times X \to \mu$ such that:*

1. $\{x\} = \cap_n g(n,x)$.
2. $y \in g(n,x) \Rightarrow g(n,y) \subset g(n,x)$.
3. *There exists a collection $\mathcal{U} = \cup_n \mathcal{U}_n$ of open sets in (X,μ) such that:*
   (a). $\cup\{g(n,x) : x \in U_{n\alpha}\}$ *is regular open for each* $U_{n\alpha} \in \mathcal{U}$.
   (b). *If $H \subset X$ is closed with $y \notin H$, then $y \notin Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$ for some $U_{n\alpha} \in \mathcal{U}_n$ with $H \subset U_{n\alpha}$.*

Proof. $(\Longrightarrow)$ Let (X,μ) be an M_1-space. Then (X,μ) has a σ-closure preserving base $\mathcal{B} = \cup_n \mathcal{B}_n$. Let $g(n,x) = X - \cup\{Cl_\mu B_{n\alpha} : B_{n\alpha} \in \mathcal{B}_n, x \notin Cl_\mu B_{n\alpha}\}$.

Proof of (1). By the definition of $g(n,x)$, we have $x \in g(n,x)$. So $x \in \cap_n g(n,x)$. On the other hand, if $y \in \cap_n g(n,x)$, then $y \in g(n,x)$ for each $n \in N$. Suppose $y \neq x$. Then there exist open sets V_1, V_2 such that $y \in V_1, x \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then there exists a $B_{m\beta} \in \mathcal{B}_m$ such that $y \in B_{m\beta} \subset V_1$ for some $m \in N$. So $x \notin Cl_\mu B_{m\beta}$. Then $y \notin g(m,x)$, a contradiction to $y \in \cap_n g(n,x)$. So $y = x$. Then $\{x\} = \cap_n g(n,x)$.

Proof of (2). Take a $y \in g(n,x) = X - \cup\{Cl_\mu B_{n\alpha} : B_{n\alpha} \in \mathcal{B}_n, x \notin Cl_\mu B_{n\alpha}\}$. Then $x \notin Cl_\mu B_{n\alpha}$ implies $y \notin Cl_\mu B_{n\alpha}$. So $\cup\{Cl_\mu B_{n\alpha} : B_{n\alpha} \in \mathcal{B}_n, x \notin Cl_\mu B_{n\alpha}\} \subset \cup\{Cl_\mu B_{n\alpha} : B_{n\alpha} \in \mathcal{B}_n, y \notin Cl_\mu B_{n\alpha}\}$. So $g(n,y) \subset g(n,x)$.

Proof of (3). We prove it by the following Claims.

Claim 2.2. $Cl_\mu Int_\mu(Cl_\mu B_{n\alpha}) = Cl_\mu B_{n\alpha}$ *for each $n \in N$ and each $B_{n\alpha} \in \mathcal{B}_n$.*

Proof. $B_{n\alpha} \subset Int_\mu Cl_\mu B_{n\alpha}$ implies $Cl_\mu B_{n\alpha} \subset Cl_\mu Int_\mu(Cl_\mu B_{n\alpha})$. On the other hand, if $x \notin Cl_\mu B_{n\alpha}$, then there exist open sets V_1, V_2 such that $x \in V_1, Cl_\mu B_{n\alpha} \subset V_2$ and $V_1 \cap V_2 = \emptyset$. So $Int_\mu Cl_\mu B_{n\alpha} \subset V_2$. Then $x \notin Cl_\mu Int_\mu(Cl_\mu B_{n\alpha})$. So $Cl_\mu Int_\mu(Cl_\mu B_{n\alpha}) \subset Cl_\mu B_{n\alpha}$. Then $Cl_\mu Int_\mu(Cl_\mu B_{n\alpha}) = Cl_\mu B_{n\alpha}$. □

Claim 2.3. $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} = X - Cl_\mu B_{n\alpha}$ *for each $n \in N$ and each*

$B_{n\alpha} \in \mathcal{B}_n$.

Proof. Pike an $x \in X - Cl_\mu B_{n\alpha}$. Then $g(n,x) \cap Cl_\mu B_{n\alpha} = \emptyset$. So $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} \supset X - Cl_\mu B_{n\alpha}$ since $x \in g(n,x)$. On the other hand, if $y \notin X - Cl_\mu B_{n\alpha}$, then $y \in Cl_\mu B_{n\alpha}$. Then $y \notin \cup\{g(n,x) : g(n,x) \cap Cl_\mu B_{n\alpha} = \emptyset\}$. So $y \notin \cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\}$. Then $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} \subset X - Cl_\mu B_{n\alpha}$. So $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} = X - Cl_\mu B_{n\alpha}$. □

Claim 2.4. *$X - Cl_\mu(X - Cl_\mu B_{n\alpha}) = Int_\mu Cl_\mu B_{n\alpha}$ for each $n \in N$ and each $B_{n\alpha} \in \mathcal{B}_n$.*

Proof. Take an $x \in X - Cl_\mu(X - Cl_\mu B_{n\alpha})$. Then $x \notin Cl_\mu(X - Cl_\mu B_{n\alpha})$. Then there exists an open set V such that $x \in V$ and $V \cap (X - Cl_\mu B_{n\alpha}) = \emptyset$. So $V \subset Cl_\mu B_{n\alpha}$. Then $x \in Int_\mu Cl_\mu B_{n\alpha}$. So $X - Cl_\mu(X - Cl_\mu B_{n\alpha}) \subset Int_\mu Cl_\mu B_{n\alpha}$. On the other hand, if $x \notin X - Cl_\mu(X - Cl_\mu B_{n\alpha})$, then $x \in Cl_\mu(X - Cl_\mu B_{n\alpha})$. Then $U \cap (X - Cl_\mu B_{n\alpha}) \neq \emptyset$ if U is a neighborhood of x. So $x \notin Int_\mu Cl_\mu B_{n\alpha}$. Then $Int_\mu Cl_\mu B_{n\alpha} \subset X - Cl_\mu(X - Cl_\mu B_{n\alpha})$. So $X - Cl_\mu(X - Cl_\mu B_{n\alpha}) = Int_\mu Cl_\mu B_{n\alpha}$. □

Claim 2.5. *$\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\}$ is regular open for each $n \in N$ and each $B_{n\alpha} \in \mathcal{B}_n$.*

Proof. $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} \subset Cl_\mu(\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\})$ implies $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} \subset Int_\mu Cl_\mu(\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\})$.

On the other hand, take a $y \in Int_\mu Cl_\mu(\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\})$, then there exists an open set U such that $y \in U \subset Cl_\mu(\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\})$. Then $y \in U \subset X - Int_\mu Cl_\mu B_{n\alpha}$ by Claim 2.3 and Claim 2.4. So $y \notin Cl_\mu Int_\mu(Cl_\mu B_{n\alpha}) = Cl_\mu B_{n\alpha}$ by Claim 2.2. Then $y \in \cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\}$ by Claim 2.3. So $Int_\mu Cl_\mu(\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\}) \subset \cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\}$. Then $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\} = Int_\mu Cl_\mu(\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\})$. So $\cup\{g(n,x) : x \in X - Cl_\mu B_{n\alpha}\}$ is regular open. □

Proof of (3)(*continued*). Let $U_{n\alpha} = X - Cl_\mu B_{n\alpha}$ for $B_{n\alpha} \in \mathcal{B}_n$, $\mathcal{U}_n = \{U_{n\alpha} : \alpha \in \Lambda\}$, $\mathcal{U} = \cup_n \mathcal{U}_n$. Then $\cup\{g(n,x) : x \in U_{n\alpha}\}$ is regular open for each $U_{n\alpha} \in \mathcal{U}$ by Claim 2.5. If $y \notin H$, where H is closed, then there exist open sets V_1, V_2 such that $y \in V_1, H \subset V_2$ and $V_1 \cap V_2 = \emptyset$. So there exists a $B_{n\alpha} \in \mathcal{B}_n$ such that $y \in B_{n\alpha} \subset V_1$ for some $n \in N$. Then $Cl_\mu B_{n\alpha} \cap V_2 = \emptyset$. So $H \subset U_{n\alpha}$ and $B_{n\alpha} \cap (\cup\{g(n,x) : x \in U_{n\alpha}\}) = \emptyset$. Then $y \notin Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$.

($\Longleftarrow$) Now we prove that (X,μ) is an M_1-space if (X,μ) satisfies (1),(2) and (3). To do it, we prove the following claim at first.

Claim 2.6. $Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})) = X - \cup\{g(n,x) : x \in U_{n\alpha}\}$ if $\cup\{g(n,x) : x \in U_{n\alpha}\}$ is regular open for each $U_{n\alpha} \in \mathcal{U}$.

Proof. $\cup\{g(n,x) : x \in U_{n\alpha}\} \subset Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$ implies $X - \cup\{g(n,x) : x \in U_{n\alpha}\} \supset X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$. Then $X - \cup\{g(n,x) : x \in U_{n\alpha}\} \supset Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\}))$.

On the other hand, $\cup\{g(n,x) : x \in U_{n\alpha}\}$ is a regular open set. Then $\cup\{g(n,x) :$

$x \in U_{n\alpha}\} = Int_\mu Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$. Pick an $x \in X - \cup\{g(n,x) : x \in U_{n\alpha}\}$. If $x \in X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$, then $x \in Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\}))$. If $x \in Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\}) - \cup\{g(n,x) : x \in U_{n\alpha}\}$, then $x \notin \cup\{g(n,x) : x \in U_{n\alpha}\} = Int_\mu Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$. Then $U \cap (X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})) \neq \emptyset$ if U is a neighborhood of x. Then $x \in Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\}))$. This implies $X - \cup\{g(n,x) : x \in U_{n\alpha}\} \subset Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\}))$. So $Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})) = X - \cup\{g(n,x) : x \in U_{n\alpha}\}$. □

Next we continue to prove the sufficiency of Theorem 2.1.

Let $B_{n\alpha} = X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$ for each $n \in N$ and each $U_{n\alpha} \in \mathcal{U}_n$. Let $\mathcal{B}_n = \{B_{n\alpha} : \alpha \in \Lambda\}$ and $\mathcal{B} = \cup_n \mathcal{B}_n$. Let $\Lambda_1 \subset \Lambda$. Then, by Claim 2.6,

$$\begin{aligned} &\cup\{Cl_\mu B_{n\alpha} : \alpha \in \Lambda_1\} \\ =&\cup\{Cl_\mu(X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})) : \alpha \in \Lambda_1\} \\ =&\cup\{X - \cup\{g(n,x) : x \in U_{n\alpha}\} : \alpha \in \Lambda_1\} \\ =&X - \cap\{\cup\{g(n,x) : x \in U_{n\alpha}\} : \alpha \in \Lambda_1\}. \end{aligned}$$

Take a $y \in \cap\{\cup\{g(n,x) : x \in U_{n\alpha}\} : \alpha \in \Lambda_1\}$. Then $y \in \cup\{g(n,x) : x \in U_{n\alpha}\}$ for each $\alpha \in \Lambda_1$. Then there exists an $x_{n\alpha} \in U_{n\alpha}$ such that $y \in g(n, x_{n\alpha})$ for each $\alpha \in \Lambda_1$. Then $g(n,y) \subset g(n, x_{n\alpha})$ by (2). So $g(n,y) \subset \cap\{\cup\{g(n,x) : x \in U_{n\alpha}\} : \alpha \in \Lambda_1\}$. Then $\cap\{\cup\{g(n,x) : x \in U_{n\alpha}\} : \alpha \in \Lambda_1\}$ is an open set. So $\cup\{Cl_\mu B_{n\alpha} : \alpha \in \Lambda_1\}$ is closed. Then $\mathcal{B}_n$ is a closure preserving collection.

Now we prove that $\mathcal{B} = \cup_n \mathcal{B}_n$ is a base.

To do it let O be an open set with $y \in O$. Then $y \notin H = X - O$ and H is closed. Then there exists an $U_{n\alpha} \in \mathcal{U}_n$ such that $H \subset U_{n\alpha}$ and $y \notin Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\})$ for some $n \in N$ by (3). So $y \in X - Cl_\mu(\cup\{g(n,x) : x \in U_{n\alpha}\}) \subset X - H = O$. Then $y \in B_{n\alpha} \subset O$. So $\mathcal{B} = \cup_n \mathcal{B}_n$ is a σ-closure preserving base. Then (X, μ) is an M_1-space. □

References

[1] C. R. Borges, *On stratifiable spaces*, Comm. Pacific Jour. Math., **17**(1966), 1-16.

[2] J. Ceder, *So generalizations of metric spaces*, Comm. Houston Pacific Jour. Math., **11**(1961), 105-125.

[3] H. Chen, *Characterizations and properties of stratifiable spaces*, Comm. Tsukuba J. Math., **2**(2008), 253-276.

[4] G. Gruenhage, *Stratifiable spaces are M_2*, Comm. Topology Proc., **1**(1976), 221-226.

[5] T. Junnila, *Neighbornets*, Comm. Pacific J. Math., **76**(1978), 83-108.

[6] S. Lin, Generalized Metric Spaces and maps (Chinese), Comm. Science Press, 2007, Beijing. 144-150.

[7] M. E. Rudin, Some conjectures, Open Problems in Topology, Comm. J. van Mill and G. M. Reed, editors, 183-193.

[8] G. Wang, Problem on Generalized Metric Spaces, 10000 Selected Problems in Sciences(Mathematics chapter, Chinese), Comm. Science Press, 2009, Beijing. 211.