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Lens Surgeries along the n-twisted Whitehead Link

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Dedicated to Professor Akio Kawauchi for his 60th birthday.

Abstract. We determine lens surgeries (i.e. Dehn surgery yielding a lens space) along the n-twisted Whitehead link. To do so, we first give necessary conditions to yield a lens space from the Alexander polynomial of the link as: (1) $n = 1$ (i.e. the Whitehead link), and (2) one of surgery coefficients is 1, 2 or 3. Our interests are not only lens surgery itself but also how to apply the Alexander polynomial for this kind of problems.

1. Introduction

For a μ-component link $L = K_1 \cup \ldots \cup K_\mu$ in an integral homology 3-sphere Σ, *Dehn surgery* is an operation to Σ by attaching solid tori to the boundaries of the exterior of L, where the way to attach a solid torus is parametrized by a rational number or $1/0 = \infty$ or $\emptyset$. The parameter is called a *surgery coefficient*. The result of $(r_1, \ldots, r_\mu)$-surgery along L is obtained by Dehn surgery along K_i with a surgery coefficient $r_i \in \mathbb{Q} \cup \{\infty, \emptyset\}$ for every $i = 1, \ldots, \mu$. We say that Dehn surgery is a *lens surgery* if the resulting space is a lens space. Let $W_n = K_1 \cup K_2$ ($n \in \mathbb{Z}$) be the n-twisted Whitehead link as in Figure 1, where a rectangle with an integer m

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implies a righthand m-full twists if $m \geq 0$, or a lefthand $|m|$-full twists if $m < 0$. In the present paper, we determine when Dehn surgery along W_n yields a lens space by using the Reidemeister torsion and Rolfsen moves.

In the present paper, we are mainly concerned with the restriction on the Alexander polynomial of a link to admit a lens surgery. Our interests are not only lens surgery itself but also how to apply the Alexander polynomial for this kind of problems. For examples: (i) The first author [8] gave necessary conditions on the Alexander polynomial of an algebraically split component-preservingly amphicheiral link. Consideration on the sign ε_n in Theorem 1.1. (relation with chirality of the links) motivates the work (see Remark 6.4.). (ii) The first author [9] determined lens surgeries along the Milnor links, and clarified that we cannot obtain the result by only the Alexander polynomial. Our method extends to algebraically same links with W_n (see Section 6).

L. Moser [17] determined Dehn surgery along every torus knot by the Seifert fibered structure of the exterior. Recently, the first author and the third author [10] determined Dehn surgery along every torus link by essentially the same method. R. Fintushel and R. J. Stern [3], and the second author [16] gave examples of hyperbolic knots yielding lens spaces. Moreover the second author [16] pointed out that a 2-bridge link $C(m, m)$ where m is odd in Conway's notation (cf. [13, Section 2]) can yield a lens space. Note that W_n is also a 2-bridge link $C(2, 2n, -2)$. J. Berge [1] showed that a doubly primitive knot yields a lens space. It is conjectured that a knot in S^3 yielding a lens space is a doubly primitive knot. Ordinarily, when we study lens surgeries along a knot or a link, we use a geometric structure of the complement of it [15], and apply Cyclic Surgery Theorem [2] or knot Floer homology [18] or more geometric cut and paste arguments [4].

Let $M = (W_n; p_1/q_1, p_2/q_2)$ denote the result of $(p_1/q_1, p_2/q_2)$-surgery along W_n. Since the linking number of W_n is zero, the first homology $H_1(M)$ is finite cyclic if and only if $\gcd(p_1, p_2) = 1$ and $p_1p_2 \neq 0$, and the order of $H_1(M)$ is $p = |p_1p_2|$. We note that W_0 is the 2-component trivial link, $W_{\pm 1}$ is the Whitehead link, and W_{-n} is the mirror image of W_n. Hence it is sufficient to consider the case $n > 0$. Thus we fix the following setting.

Setting (1) $W_n = K_1 \cup K_2$ is the 2-component link in S^3 of Figure 1, where $n > 0$.

(2) $M = (W_n; p_1/q_1, p_2/q_2)$ is the result of $(p_1/q_1, p_2/q_2)$-surgery along W_n, where $q_i \geq 1$ $(i = 1, 2)$, $\gcd(p_1, p_2) = 1$ and $p = |p_1p_2| \geq 2$.

Throughout this paper, ζ_d is a primitive d-th root of unity and $\mathbb{Q}(\zeta_d)$ is the d-th cyclotomic field, for an integer $d \geq 2$.

Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$, and T a generator of $H_1(M)$. Let $d \geq 2$ be a divisor of p, and $\psi : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_d)$ a ring homomorphism such that $\psi(T) = \zeta_d$. Then $\tau^{\psi}(M) \in \mathbb{Q}(\zeta_d)$, the *Reidemeister torsion* of M associated to ψ, is determined up to multiplications by $\pm\zeta_d^m$ $(m \in \mathbb{Z})$

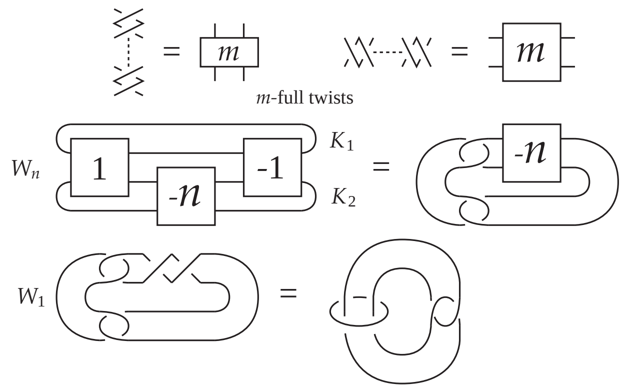

Figure 1: n-twisted Whitehead link W_n

(see [22, 23] for details on the Reidemeister torsion). For A and B in $\mathbb{Q}(\zeta_d)$, if there exists an integer m such that $A = \pm\zeta_d^m B$, then we denote by $A \doteq B$.

We first state a key theorem of the present paper.

Theorem 1.1. *Let $M = (W_n; p_1/q_1, p_2/q_2)$ be as in the setting above. Then we have the following:*

(1) *Let $d \geq 2$ be a divisor of p_2, and $\psi : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_d)$ a ring homomorphism defined by $\psi([m_1]) = 1$ and $\psi([m_2]) = \zeta_d$, where m_i is a meridian of K_i. Then we have*

$$\tau^{\psi}(M) \doteq \{nq_1(\zeta_d - 1)^2 + \varepsilon_n p_1 \zeta_d\}(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}_2} - 1)^{-1},$$

where $\varepsilon_n = 1$ or -1, and $q_2\bar{q}_2 \equiv 1 \pmod{p_2}$.

(2) *Let $d \geq 2$ be a divisor of p_1, and $\psi : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_d)$ a ring homomorphism defined by $\psi([m_1]) = \zeta_d$ and $\psi([m_2]) = 1$, where m_i is a meridian of K_i. Then we have*

$$\tau^{\psi}(M) \doteq \{nq_2(\zeta_d - 1)^2 + \varepsilon_n p_2 \zeta_d\}(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}_1} - 1)^{-1},$$

where $\varepsilon_n = 1$ or -1, and $q_1\bar{q}_1 \equiv 1 \pmod{p_1}$.

(3) *In (1) and (2), we have $\varepsilon_1 = 1$.*

We have two remarks on the proof of Theorem 1.1.. (i) Since W_n is an interchangeable link (i.e. as an ordered link, $K_1 \cup K_2$ is ambient isotopic to an ordered link $K_2 \cup K_1$), it is sufficient to show Theorem 1.1. (1). We will often omit a half of the proofs by the same reason (ex. Theorem 1.2., Lemma 4.1. and Lemma 4.2.). (ii)

To show Theorem 1.1., we applied the surgery formula of the Reidemeister torsion due to V. G. Turaev [22, 23] (cf. Lemma 2.1.).

Let $L(p,q)$ be a lens space which is defined as the result of p/q-surgery along the trivial knot. By comparing the Reidemeister torsion of M as in Theorem 1.1. and that of $L(p,q)$ (in Example 2.2.), we have:

Theorem 1.2. *Let $M=(W_n;p_1/q_1,p_2/q_2)$ be as in the setting above. Then we have the following:*

(A) If M is a lens space, then we have $n=1$.

(B) The resulting space $M=(W_1;p_1/q_1,p_2/q_2)$ is a lens space if and only if one of the following (1), (2), (3), (4), (5) or (6) holds:

(1) $p_1/q_1=1$ *and* $|p_2-6q_2|=1$.

(2) $p_1/q_1=2$ *and* $|p_2-4q_2|=1$.

(3) $p_1/q_1=3$ *and* $|p_2-3q_2|=1$.

(4) $p_2/q_2=1$ *and* $|p_1-6q_1|=1$.

(5) $p_2/q_2=2$ *and* $|p_1-4q_1|=1$.

(6) $p_2/q_2=3$ *and* $|p_1-3q_1|=1$.

Moreover if (1), (2), (3), (4), (5) *or* (6) *holds, then* $M=L(p_2,4q_2)$, $L(2p_2,8q_2-p_2)$, $L(3p_2,3q_2-2p_2)$, $L(p_1,4q_1)$, $L(2p_1,8q_1-p_1)$ *or* $L(3p_1,\ 3q_1-2p_1)$, *respectively.*

We remark that six cases in Theorem 1.2. are not exclusive, for example $(p_1/q_1,p_2/q_2)=(2,3)$ in (2) and (6), and $(p_1/q_1,p_2/q_2)=(3,2)$ in (3) and (5).

B. Martelli and C. Petronio [15] completely determined exceptional Dehn fillings of the complement of the chain link with three components by using hyperbolic geometry. The complement of W_{-1} is a certain Dehn filling of the 3-component chain link. Though their result overlaps with Theorem 1.2, the overlap is only partial, our method is different from theirs, and our targets are extended (i.e. our results are 'not' properly included in theirs).

In Section 2, we provide basic tools of this paper such as Reidemeister torsion and Rolfsen moves. In Section 3, we prove Theorem 1.1. In Section 4, we prove "only if part" of Theorem 1.2. by using Theorem 1.1. In Section 5, we prove "if part" of Theorem 1.2. by using Rolfsen moves. In Section 6, we will apply our method for a 2-component link and its components with the same Alexander polynomials as W_n.

We refer to [5, 6, 7, 9, 11, 12] for studies on Dehn surgery by using the Reidemeister torsion.

2. Preliminaries

2.1. Reidemeister torsion

We rewrite a surgery formula due to Turaev to be suitable for the present paper. For details, see [22, 23], and see also [6, Section 2].

Let R be a commutative ring with nonzero identity element. Then we denote the classical ring of quotient by $Q(R)$. Let X be a finite CW complex. Then the *maximal abelian torsion* of X, denoted by $\tau(X)$, is an element of $Q(\mathbb{Z}[H_1(X)])$ that is determined up to multiplication by an element of $\pm H_1(X)$, which is defined from a chain complex $\mathbf{C}_*$ induced by the maximal abelian covering of X.

Let $L = K_1 \cup \cdots \cup K_\mu$ be an oriented μ-component link in an integral homology 3-sphere Σ, and $\Delta_L(t_1, \dots, t_\mu)$ the Alexander polynomial of L, where a variable t_i is represented by a meridian of K_i $(i = 1, \dots, \mu)$. We note that if the orientation of K_i is reversed, then the variable t_i is replaced with t_i^{-1}, and that the set of the μ-variable Alexander polynomials of μ-component links in S^3 coincides with the set of the μ-variable Alexander polynomials of μ-component links in any homology 3-sphere Σ.

Let $L = K_1 \cup K_2 \cup K_3$ be a 3-component link in an integral homology 3-sphere Σ, E_L the exterior of L, m_i and l_i a meridian and a longitude of K_i $(i = 1, 2, 3)$ on ∂E_L respectively. Let $M = (L; p_1/q_1, p_2/q_2, p_3/q_3)$ be the result of p_i/q_i-surgery along K_i, and set

$$M = E_L \cup V_1 \cup V_2 \cup V_3 \quad \text{and} \quad M_0 = E_L \cup V_1 \cup V_2,$$

where V_i is a solid torus glued in doing surgery along K_i. Let l'_i be the core of V_i. Note that the homology class of l'_i is uniquely determined in M_0 $(i = 1, 2)$ and in M $(i = 3)$. We assume that M is a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Let T be a generator of $H_1(M)$, $d \geq 2$ a divisor of p and $\psi : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_d)$ a ring homomorphism such that $\psi(T) = \zeta_d$. We define $\psi_0 : \mathbb{Z}[H_1(M_0)] \to \mathbb{Q}(\zeta_d)$ by $\psi_0 = \psi \circ \iota$ where $\iota : \mathbb{Z}[H_1(M_0)] \to \mathbb{Z}[H_1(M)]$ is a ring homomorphism induced from the natural inclusion $M_0 \hookrightarrow M$. Then we have the following *surgery formula* for the Reidemeister torsion.

Lemma 2.1(surgery formula; Turaev [22, 23])**.**

(1) If $[l'_i]$ $(i = 1, 2)$ has infinite order in $H_1(M_0)$, then we have

$$\tau(M_0) \doteq \Delta_L([m_1], [m_2], [m_3])([l'_1] - 1)^{-1}([l'_2] - 1)^{-1} \quad \text{in } Q(\mathbb{Z}[H_1(M_0)]).$$

(2) If $\tau(M_0) \neq 0$ and $\psi([l'_3]) \neq 1$, then we have

$$\tau^\psi(M) \doteq \psi_0(\tau(M_0))(\psi([l'_3]) - 1)^{-1}.$$

Example 2.2(Reidemeister [19]). Let T be a generator of $H_1(L(p,q))$. Let $d \geq 2$ be a divisor of p, and $\psi : \mathbb{Z}[H_1(L(p,q))] \to \mathbb{Q}(\zeta_d)$ a ring homomorphism such that $\psi(T) = \zeta_d$. Then we have

$$\tau^{\psi}(L(p,q)) \doteq (\zeta_d^i - 1)^{-1}(\zeta_d^{i\bar{q}} - 1)^{-1}$$

for some i where $\gcd(i,d) = 1$ and $q\bar{q} \equiv 1 \pmod{d}$.

Lemma 2.3(Torres formula [21]). Let $L = K_1 \cup \cdots \cup K_\mu \cup K_{\mu+1}$ ($\mu \geq 1$) be an oriented $(\mu+1)$-component link in an integral homology 3-sphere Σ and $L' = K_1 \cup \cdots \cup K_\mu$ a μ-component sublink. Then we have

$$\Delta_L(t_1, \dots, t_\mu, 1) \doteq \begin{cases} \dfrac{t^{\ell} - 1}{t - 1}\Delta_K(t) & (\mu = 1), \\ (t_1^{\ell_1} \cdots t_\mu^{\ell_\mu} - 1)\Delta_{L'}(t_1, \dots, t_\mu) & (\mu \geq 2), \end{cases}$$

where $\ell_i = \mathrm{lk}\,(K_i, K_{\mu+1})$ $(i = 1, \dots, \mu)$ is the linking number of K_i and $K_{\mu+1}$, and we set $L = K_1 = K$, $t = t_1$ and $\ell = \ell_1$ if $\mu = 1$.

Lemma 2.4(duality; Turaev [22]). Let $L = K_1 \cup \cdots \cup K_\mu$ be an oriented μ-component link in an integral homology 3-sphere Σ. We set ℓ_{ij} is the linking number of K_i and K_j $(1 \leq i \neq j \leq \mu)$ if $\mu \geq 2$, and $L = K_1 = K$ and $t = t_1$ if $\mu = 1$. Then we have the following:

$$\begin{aligned} \Delta_K(t) &= t^a \Delta_K(t^{-1}) && (\mu = 1), \\ \Delta_L(t_1, t_2, \dots, t_\mu) &= (-1)^\mu t_1^{a_1} t_2^{a_2} \cdots t_\mu^{a_\mu} \Delta_L(t_1^{-1}, t_2^{-1}, \dots, t_\mu^{-1}) && (\mu \geq 2), \end{aligned}$$

where a is even and $a_i \equiv 1 + \sum_{j \neq i} \ell_{ij} \pmod{2}$.

Remark 2.5. Torres [21] has already shown a duality of the Alexander polynomials. Lemma 2.4. is a refinement of the duality.

The following lemma is used effectively to prove "only if part" of Theorem 1.2. in Section 4.

Lemma 2.6. Let $\ell \geq 5$ be a prime. Suppose that two Laurent polynomials $F(t)$ and $G(t) \in \mathbb{Z}[t, t^{-1}]$ are of the form:

$$\begin{aligned} F(t) &= a_0 + \sum_{i=1}^{\frac{\ell-3}{2}} a_i(t^i + t^{-i}) \\ G(t) &= b_0 + \sum_{i=1}^{\frac{\ell-3}{2}} b_i(t^i + t^{-i}) \end{aligned} \quad \left(a_i, b_i \in \mathbb{Z}\ ;\ i = 0, 1, \dots, \frac{\ell - 3}{2}\right)$$

and $F(\zeta_\ell) = G(\zeta_\ell)$ holds for any ℓ-th root of unity ζ_ℓ. Then we have $F(t) = G(t)$.

Proof. By the assumption, $F(t)-G(t)$ is divisible by $t^{\ell-1}+t^{\ell-2}+\cdots+t+1$. Since the degree of $F(t)-G(t)$ does not exceed $\ell-3$ by the form, we have $F(t)-G(t)\equiv 0$.□

2.2. Rolfsen moves

We recall Rolfsen moves on Dehn surgery. It is known that a pair of Dehn surgeries describes the same 3-manifold if and only if they are moved to each other by Rolfsen moves [20]. Rolfsen move consists of two moves; an (R1)-move and an (R2)-move. Let $L=K_1\cup\cdots\cup K_\mu$ be a μ-component link, and $M=(L;r_1,\ldots,r_\mu)$ the result of Dehn surgery along L.

(R1)-move: When the i-th component K_i is unknotted, we may operate u-full twists along K_i where "u-full twists" means righthand u-full twists if $u\geq 0$, and lefthand $|u|$-full twists if $u<0$. Then K_i, r_i, K_j $(j\neq i)$ and r_j change into K_i', r_i', K_j' $(j\neq i)$ and r_j', respectively, where

$$r_i'=\frac{1}{u+1/r_i}\quad\text{and}\quad r_j'=r_j+u(\operatorname{lk}(K_i,K_j))^2$$

$(1/0=\infty$ and $1/\infty=0)$, and $\operatorname{lk}(K_i,K_j)$ is the linking number of K_i and K_j. In Figure 2, an (R1)-move from the lefthand side to the righthand side is 1-full twist along K_i.

(R2)-move: Adding a new component $K_{\mu+1}$ to L with a framing ∞, and its inverse.

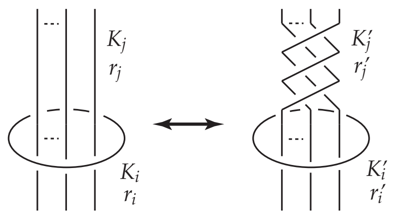

Figure 2: Rolfsen moves

Remark 2.7. R. Fintushel and R. J. Stern [3], and the second author [16] found families of knots yielding a lens space by using another method, called "Kirby moves" [14].

3. Proof of Theorem 1.1.

Proof of (1) *and* (2). We prove only the case (1). Let $d\geq 2$ be a divisor of p_2. Then $\gcd(d,p_1)=1$.

When we use the surgery formula of the Reidemeister torsion (cf. Lemma 2.1.), not to make the denominator and the numerator vanish, we add the third component

K_3 to W_n as in Figure 3. Then $H_i = K_i \cup K_3$ $(i = 1, 2)$ is the Hopf link. We set $\overline{W} = K_1 \cup K_2 \cup K_3$ and orient $\overline{W}$ so that $\mathrm{lk}\,(K_i, K_3) = 1$ $(i = 1, 2)$. We compute the Reidemeister torison of $M = (\overline{W}; p_1/q_1, p_2/q_2, \infty)$. Note that the value does not depend on K_3 because we close up K_3 by ∞-surgery.

By the Torres formula (Lemma 2.3.) and that

$$\Delta_{W_n}(t_1, t_2) \doteq n(t_1 - 1)(t_2 - 1),$$

we may set as follows:

$$\Delta_{\overline{W}}(t_1, t_2, t_3) = n(t_1 t_2 - 1)(t_1 - 1)(t_2 - 1) + (t_3 - 1)g_n(t_1, t_2, t_3) \tag{3.1}$$

for some $g_n(t_1, t_2, t_3) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$.

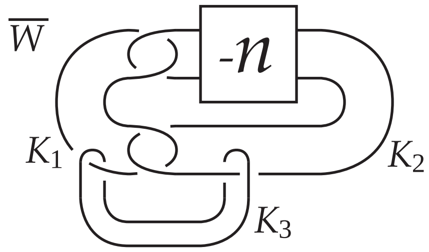

Figure 3: 3-component link $\overline{W}$

Then we have the following lemma:

Lemma 3.1.

$$g_n(t_1, 1, t_3) \doteq 1 \quad \text{and} \quad g_n(1, t_2, t_3) \doteq 1.$$

Proof. By (3.1), we have

$$\Delta_{\overline{W}}(t_1, 1, t_3) = (t_3 - 1)g_n(t_1, 1, t_3).$$

By the Torres formula (Lemma 2.3.),

$$\Delta_{\overline{W}}(t_1, 1, t_3) \doteq (t_3 - 1)\Delta_{H_1}(t_1, t_3) \doteq t_3 - 1.$$

Hence we have $g_n(t_1, 1, t_3) \doteq 1$. Similarly we have $g_n(1, t_2, t_3) \doteq 1$. □

We define an integer ε_n by $-\varepsilon_n = g_n(1, 1, 1)$. Then $\varepsilon_n = 1$ or -1.

Lemma 3.2.

$$g_n(t_1, 1, 1) = -\varepsilon_n t_1 \quad \text{and} \quad g_n(1, t_2, 1) = -\varepsilon_n t_2.$$

Proof. By the duality of the Alexander polynomial (Lemma 2.4.), there exists integers a, b and c such that

$$\Delta_{\overline{W}}(t_1, t_2, t_3) = -t_1^a t_2^b t_3^c \Delta_{\overline{W}}(t_1^{-1}, t_2^{-1}, t_3^{-1}) \tag{3.2}$$

By substituting $t_3 = 1$ to (3.2), we have

$$n(t_1t_2 - 1)(t_1 - 1)(t_2 - 1) = -nt_1^a t_2^b (t_1^{-1}t_2^{-1} - 1)(t_1^{-1} - 1)(t_2^{-1} - 1)$$

by (3.1). Then we have $a = b = 2$. By (3.1) and (3.2), we have

$$g_n(t_1, t_2, t_3) = t_1^2 t_2^2 t_3^{c-1} g_n(t_1^{-1}, t_2^{-1}, t_3^{-1}),$$

and hence $g_n(t_1, 1, 1) = t_1^2 g_n(t_1^{-1}, 1, 1)$. We then have the result by Lemma 3.1. □

Let $E_{\overline{W}}$ be the exterior of $\overline{W}$, m_i and l_i a meridian and a longitude of K_i ($i = 1, 2, 3$) on $\partial E_{\overline{W}}$ respectively, and set

$$M = E_{\overline{W}} \cup V_1 \cup V_2 \cup V_3 \quad \text{and} \quad M_0 = E_{\overline{W}} \cup V_1 \cup V_2,$$

where V_i is a solid torus glued in doing surgery along K_i. Let m_i' and l_i' be a meridian and a longitude of V_i respectively. We may assume that, in $H_1(E_{\overline{W}})$,

$$\begin{gathered}[m_i'] = [m_i]^{p_i}[l_i]^{q_i}, [l_i'] = [m_i]^{r_i}[l_i]^{s_i}, p_is_i - q_ir_i = -1 \ (i = 1, 2), \\ [m_3'] = [m_3], [l_3'] = [l_3], [l_1] = [l_2] = [m_3], [l_3] = [m_1][m_2].\end{gathered}$$

Here $[\ -\ ]$ denotes the homology class in $H_1(E_{\overline{W}})$. In the following, we also denote the homology class in $H_1(M_0)$ and $H_1(M)$ by the same symbol.

In $H_1(M_0)$, we have $[m_i'] = [m_i]^{p_i}[l_i]^{q_i} = 1$ ($i = 1, 2$). Hence we have

$$H_1(M_0) \cong \langle [m_1], [m_2], [m_3] \mid [m_i]^{p_i}[m_3]^{q_i} = 1 \ (i = 1, 2) \rangle \tag{3.3}$$

We set $T_i = [m_i]^{r_i}[m_3]^{s_i}$ ($i = 1, 2$). Then

$$\begin{aligned} [m_i] &= [m_i]^{-p_is_i + q_ir_i} \\ &= ([m_i]^{p_i}[m_3]^{q_i})^{-s_i}([m_i]^{r_i}[m_3]^{s_i})^{q_i} = T_i^{q_i} \ (i = 1, 2) \\ [m_3] &= [m_3]^{-p_1s_1 + q_1r_1} \\ &= ([m_1]^{p_1}[m_3]^{q_1})^{r_1}([m_1]^{r_1}[m_3]^{s_1})^{-p_1} = T_1^{-p_1} = T_2^{-p_2} \end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we have

$$H_1(M_0) \cong \langle T_1, T_2 \mid T_1^{p_1} = T_2^{p_2} \rangle \tag{3.5}$$

By the condition $\gcd(p_1, p_2) = 1$, there exists integers u_1, u_2 such that $u_2p_1 + u_1p_2 = 1$. We set $T = T_1^{u_1}T_2^{u_2}$. Then by (3.5), we have

$$\begin{aligned} T_1 &= T_1^{u_2p_1 + u_1p_2} = (T_1^{u_1}T_2^{u_2})^{p_2}(T_1^{p_1}T_2^{-p_2})^{u_2} = T^{p_2} \\ T_2 &= T_2^{u_2p_1 + u_1p_2} = (T_1^{u_1}T_2^{u_2})^{p_1}(T_1^{p_1}T_2^{-p_2})^{-u_1} = T^{p_1}, \end{aligned} \tag{3.6}$$

and

$$H_1(M_0) \cong \langle T \mid - \rangle \cong \mathbb{Z}.$$

By (3.4) and (3.6), we have

$$\begin{array}{lrcl} & [m_1] & = & T_1^{q_1} = T^{p_2 q_1},\ [m_2] = T_2^{q_2} = T^{p_1 q_2},\ [m_3] = T_1^{-p_1} = T^{-p_1 p_2}, \\ (3.7) & [l_i'] & = & T_i = T^{p_i} \neq 1 \quad (i = 1, 2), \\ & [l_3] & = & [m_1][m_2] = T_1^{q_1} T_2^{q_2} = T^{p_2 q_1 + p_1 q_2} \neq 1 \end{array}$$

in $H_1(M_0)$.

Let ψ be as in the statement of Theorem 1.1. (1), and $\psi_0 = \psi \circ \iota$ where $\iota : \mathbb{Z}[H_1(M_0)] \to \mathbb{Z}[H_1(M)]$ is a ring homomorphism induced from the natural inclusion $M_0 \hookrightarrow M$. Then by Lemma 2.1. (1), (3.1) and (3.7), we have

$$\begin{aligned} \tau(M_0) &\doteq \Delta_{\overline{W}}(T^{p_2 q_1}, T^{p_1 q_2}, T^{-p_1 p_2})(T^{p_1} - 1)^{-1}(T^{p_2} - 1)^{-1} \\ &\doteq \frac{n(T^{p_2 q_1 + p_1 q_2} - 1)(T^{p_2 q_1} - 1)(T^{p_1 q_2} - 1)}{(T^{p_1} - 1)(T^{p_2} - 1)} \\ &\quad + \frac{(T^{-p_1 p_2} - 1)}{(T^{p_1} - 1)(T^{p_2} - 1)} g_n(T^{p_2 q_1}, T^{p_1 q_2}, T^{-p_1 p_2}) \\ &\doteq n(T^{p_2(q_1 - 1)} + T^{p_2(q_1 - 2)} + \cdots + T^{p_2} + 1) \\ &\quad \cdot \frac{(T^{p_2 q_1 + p_1 q_2} - 1)(T^{p_1 q_2} - 1)}{T^{p_1} - 1} \\ &\quad - T^{-p_1 p_2}(T^{p_2(p_1 - 1)} + T^{p_2(p_1 - 2)} + \cdots + T^{p_2} + 1) \\ &\quad \cdot \frac{g_n(T^{p_2 q_1}, T^{p_1 q_2}, T^{-p_1 p_2})}{T^{p_1} - 1}. \end{aligned}$$

Since $[m_2] = T^{p_1 q_2}$ and $\psi_0([m_2]) = \zeta_d$, we have $\psi_0(T) = \zeta_d^{\bar{p}_1 \bar{q}_2}$ where $p_1 \bar{p}_1 \equiv q_2 \bar{q}_2 \equiv 1 \pmod{d}$. Hence we have

$$\begin{aligned} \tau^{\psi}(M) &\doteq \left\{ \frac{n q_1 (\zeta_d - 1)^2}{\zeta_d^{\bar{q}_2} - 1} - \frac{p_1 g_n(1, \zeta_d, 1)}{\zeta_d^{\bar{q}_2} - 1} \right\} (\zeta_d - 1)^{-1} \\ &\doteq \{ n q_1 (\zeta_d - 1)^2 + \varepsilon_n p_1 \zeta_d \} (\zeta_d - 1)^{-1} (\zeta_d^{\bar{q}_2} - 1)^{-1} \end{aligned}$$

by Lemma 2.1. (2) and Lemma 3.2. □

Proof of (3). By computing the Alexander polynomial of $\overline{W}$ in Figure 3 for the case $n = 1$, we have

$$g_1(t_1, t_2, t_3) = -(2t_1 t_2 - t_1 - t_2 + 1),$$

$$g_1(1, \zeta_d, 1) = -\zeta_d,$$

and $\varepsilon_1 = 1$. □

Remark 3.3. We appreciate deeply that the referee computed

$$g_n(t_1, t_2, t_3) = -n(t_1 - 1)(t_2 - 1) - t_1 t_2,$$

and $\varepsilon_n = -g_n(1, 1, 1) = 1$. To tell the truth, we have already recognized that it is not so defficult to calculate $g_n(t_1, t_2, t_3)$ as the referee pointed out. But we do not calculate it, because we do not need the explicit expression. The arguments in this section and the next section can be applied for more extended situations after some modifications. In Section 6, we will discuss about it (for the meaning of ε_n, see Remark 6.4.).

4. Proof of "only if part" of Theorem 1.2.

We will prove two lemmas: In Lemma 4.1., we will study the case p_1 (or p_2) is divisible by a prime $\ell \geq 5$. In Lemma 4.2., we will study the case p_1 (or p_2) is divisible by 2 or 3. After that, we will prove "only if part" of Theorem 1.2. by the lemmas.

Lemma 4.1. Suppose that $M = (W_n; p_1/q_1, p_2/q_2)$ is a lens space. Then we have the following:

(1) If p_2 is divisible by a prime $\ell \geq 5$, then we have $n = 1$, $q_1 = 1$, and $p_1 = 1, 2$ or 3.

(2) If p_1 is divisible by a prime $\ell \geq 5$, then we have $n = 1$, $q_2 = 1$, and $p_2 = 1, 2$ or 3.

Proof. We prove only the case (1). Suppose that $M = (W_n; p_1/q_1, p_2/q_2)$ is a lens space.

By Theorem 1.1. and Example 2.2., there exists integers i, j and k with $\gcd(i, \ell) = \gcd(j, \ell) = \gcd(k, \ell) = 1$, $k \equiv \pm\bar{q}_2 \pmod{\ell}$,

$$1 \leq i,\ j \leq \frac{\ell - 1}{2}, \quad 1 \leq k \leq \ell - 1 \quad \text{and} \quad i + j \equiv k + 1 \pmod 2 \tag{4.1}$$

such that

$$\{nq_1(\zeta_\ell - 1)^2 + \varepsilon_n p_1 \zeta_\ell\}(\zeta_\ell^i - 1)(\zeta_\ell^j - 1) \doteq (\zeta_\ell - 1)(\zeta_\ell^k - 1). \tag{4.2}$$

Case 1 $i + j \equiv 1 \pmod 2$.

Then the one of i and j is odd, and the other is even. By (4.1), k is even, $3 \leq i + j \leq \ell - 2$ and $3 \leq k + 1 \leq \ell$. By (4.2), we have

$$\zeta_\ell^{-\frac{i+j-1}{2}} \cdot \{nq_1(\zeta_\ell - 1)^2 + \varepsilon_n p_1 \zeta_\ell\} \cdot \frac{(\zeta_\ell^i - 1)(\zeta_\ell^j - 1)}{(\zeta_\ell - 1)(\zeta_\ell^2 - 1)} = \eta \zeta_\ell^{-\frac{k-2}{2}} \cdot \frac{\zeta_\ell^k - 1}{\zeta_\ell^2 - 1} \in \mathbb{R}$$

where $\eta=\pm1$. By Lemma 2.6., we have

$$t^{-\frac{i+j-1}{2}}\cdot\{nq_1(t-1)^2+\varepsilon_n p_1 t\}\cdot\frac{(t^i-1)(t^j-1)}{(t-1)(t^2-1)}=\eta t^{-\frac{k-2}{2}}\cdot\frac{t^k-1}{t^2-1}\in\mathbb{Z}[t,t^{-1}]$$

Hence we have $n=1$ and $q_1=1$. Thus $(t-1)^2+p_1t$ is a divisor of t^k-1, and hence it is the third, fourth or sixth cyclotomic polynomial:

$$(t-1)^2+\varepsilon_1 p_1 t=t^2+t+1,\quad t^2+1\quad\text{or}\quad t^2-t+1.$$

Hence we have $p_1=\varepsilon_1, 2\varepsilon_1$ or $3\varepsilon_1$. Recall that $\varepsilon_1=1$ (Theorem 1.1. (3)). Therefore we have $p_1=1,2$ or 3.

Case 2 $i+j\equiv 0 \pmod 2$.

Then by (4.1), k is odd, $2\le i+j\le \ell-1$ and $2\le k+1\le\ell-1$. By (4.2), we have

$$\zeta_\ell^{-\frac{i+j}{2}}\cdot\{nq_1(\zeta_\ell-1)^2+\varepsilon_n p_1\zeta_\ell\}\cdot\frac{(\zeta_\ell^i-1)(\zeta_\ell^j-1)}{(\zeta_\ell-1)^2}=\eta\zeta_\ell^{-\frac{k-1}{2}}\cdot\frac{\zeta_\ell^k-1}{\zeta_\ell-1}\in\mathbb{R}$$

where $\eta=\pm1$. Suppose that $(i,j)\ne\left(\frac{\ell-1}{2},\frac{\ell-1}{2}\right)$. Then we have

$$t^{-\frac{i+j}{2}}\cdot\{nq_1(t-1)^2+\varepsilon_n p_1 t\}\cdot\frac{(t^i-1)(t^j-1)}{(t-1)^2}=\eta t^{-\frac{k-1}{2}}\cdot\frac{t^k-1}{t-1}\in\mathbb{Z}[t,t^{-1}]$$

by Lemma 2.6.. As in Case 1, we have the result.

Suppose that $i=j=\frac{\ell-1}{2}$. We set

$$\begin{aligned}A&=\zeta_\ell^{-\frac{\ell+1}{2}}\cdot\{nq_1(\zeta_\ell-1)^2+\varepsilon_n p_1\zeta_\ell\}\left(\zeta_\ell^{\frac{\ell-1}{2}}-1\right)^2,\\ B&=\zeta_\ell^{-\frac{k+1}{2}}\cdot(\zeta_\ell-1)(\zeta_\ell^k-1).\end{aligned}$$

Then $A=\eta B$ holds. By expanding A and B, we have

$$\begin{aligned}A&=-2(\varepsilon_n p_1-2nq_1)-2nq_1(\zeta_\ell+\zeta_\ell^{-1})+nq_1\left(\zeta_\ell^{\frac{\ell-3}{2}}+\zeta_\ell^{-\frac{\ell-3}{2}}\right)\\ &\quad+(\varepsilon_n p_1-nq_1)\left(\zeta_\ell^{\frac{\ell-1}{2}}+\zeta_\ell^{-\frac{\ell-1}{2}}\right),\\ B&=-\left(\zeta_\ell^{\frac{k-1}{2}}+\zeta_\ell^{-\frac{k-1}{2}}\right)+\left(\zeta_\ell^{\frac{k+1}{2}}+\zeta_\ell^{-\frac{k+1}{2}}\right).\end{aligned}$$

If $\ell\ge7$, then

$$\begin{aligned}A&=-(3\varepsilon_n p_1-5nq_1)-(\varepsilon_n p_1+nq_1)(\zeta_\ell+\zeta_\ell^{-1})\\ &\quad-(\varepsilon_n p_1-nq_1)\sum_{i=2}^{\frac{\ell-5}{2}}(\zeta_\ell^i+\zeta_\ell^{-i})-(\varepsilon_n p_1-2nq_1)\left(\zeta_\ell^{\frac{\ell-3}{2}}+\zeta_\ell^{-\frac{\ell-3}{2}}\right).\end{aligned}$$

By Lemma 2.6., we have the following:

(i) If $k=1$, then no n, p_1 and q_1 satisfy $A=\eta B$.

(ii) If $3 \leq k \leq \ell-4$, then no n, p_1 and q_1 satisfy $A=\eta B$.

(iii) If $k=\ell-2$, then we have

$$B=-1-\sum_{i=1}^{\frac{\ell-5}{2}}(\zeta_\ell^i+\zeta_\ell^{-i})-2\left(\zeta_\ell^{\frac{\ell-3}{2}}+\zeta_\ell^{-\frac{\ell-3}{2}}\right),$$

and hence no n, p_1 and q_1 satisfy $A=\eta B$.

If $\ell=5$, then we have

$$\begin{aligned} A &= -2(\varepsilon_n p_1-2nq_1)-nq_1(\zeta_\ell+\zeta_\ell^{-1})+(\varepsilon_n p_1-nq_1)(\zeta_\ell^2+\zeta_\ell^{-2}) \\ &= -(3\varepsilon_n p_1-5nq_1)-\varepsilon_n p_1(\zeta_\ell+\zeta_\ell^{-1}). \end{aligned}$$

(i) If $k=1$, then we have $B=-2+(\zeta_\ell+\zeta_\ell^{-1})$, and hence we have $n=p_1=q_1=1$.

(ii) If $k=3$, then we have $B=-1-2(\zeta_\ell+\zeta_\ell^{-1})$, and hence we have $n=1$, $p_1=2$ and $q_1=1$.

Therefore this completes the proof. □

Lemma 4.2. Suppose that $M=(W_n; p_1/q_1, p_2/q_2)$ is a lens space. Then we have:

(1) If $n=1$ and $p_1/q_1=1$, then we have $|p_2-6q_2|=1$.

(2) If p_1 is divisible by 2, then we have $|\varepsilon_n p_2-4nq_2|=1$.

(3) If p_1 is divisible by 3, then we have $|\varepsilon_n p_2-3nq_2|=1$.

(4) If p_1 is divisible by 4, then we have $|\varepsilon_n p_2-2nq_2|=1$.

(5) If $n=1$ and $p_2/q_2=1$, then we have $|p_1-6q_1|=1$.

(6) If p_2 is divisible by 2, then we have $|\varepsilon_n p_1-4nq_1|=1$.

(7) If p_2 is divisible by 3, then we have $|\varepsilon_n p_1-3nq_1|=1$.

(8) If p_2 is divisible by 4, then we have $|\varepsilon_n p_1-2nq_1|=1$.

Proof. (1) If $n=1$ and $p_1/q_1=1$, then M is the result of p_2/q_2-surgery along the $(2,3)$-torus knot (i.e. the righthand trefoil). Hence we have $|p_2-6q_2|=1$ by the result of L. Moser [17], and then $M=L(p_2,4q_2)$. The case (5) is similarly shown.

We prove only (6), (7) and (8).

(6) Suppose that p_2 is divisible by 2. Since $\zeta_2=-1$, and i, j and k are odd in (4.2), we have

$$nq_1(-1-1)^2+\varepsilon_n p_1(-1)=4nq_1-\varepsilon_n p_1=\pm 1$$

and $|\varepsilon_n p_1-4nq_1|=1$ by (4.2). The case (2) is similarly shown.

(7) Suppose that p_2 is divisible by 3. Since $|\zeta_3 - 1| = |\zeta_3^i - 1| = |\zeta_3^j - 1| = |\zeta_3^k - 1| \neq 0$, and

$$nq_1(\zeta_3 - 1)^2 + \varepsilon_n p_1 \zeta_3 = \zeta_3(\varepsilon_n p_1 - 3nq_1),$$

we have $|\varepsilon_n p_1 - 3nq_1| = 1$ by (4.2). The case (3) is similarly shown.
(8) Suppose that p_2 is divisible by 4. Since $|\zeta_4 - 1| = |\zeta_4^i - 1| = |\zeta_4^j - 1| = |\zeta_4^k - 1| \neq 0$, and

$$nq_1(\zeta_4 - 1)^2 + \varepsilon_n p_1 \zeta_4 = \zeta_4(\varepsilon_n p_1 - 2nq_1),$$

we have $|\varepsilon_n p_1 - 2nq_1| = 1$ by (4.2). The case (4) is similarly shown. This completes the proof. □

Proof of the "only if part" of Theorem 1.2. By Lemma 4.2., it is sufficient to prove that $n = 1$, and at least one of p_1/q_1 and p_2/q_2 is 1, 2 or 3.

Case 1 At least one of p_1 and p_2 has a prime divisor $\ell \geq 5$.

Suppose that p_2 has a prime divisor $\ell \geq 5$. By Lemma 4.1. (1), we have $n = 1$, and $p_1/q_1 = 1, 2$ or 3. The case that p_1 has a prime divisor $\ell \geq 5$ is similarly shown.

Case 2 Otherwise, i.e. both $|p_1|$ and $|p_2|$ are of type $2^a 3^b$ $(a, b \in \mathbb{Z}; a \geq 0, b \geq 0)$.

Recall that p_1 and p_2 are coprime.

Case 2-1 Either p_1 or p_2 is divisible by 6.

Suppose that p_2 is divisible by 6. Then we have $p_1 = \pm 1$ by coprimeness. This case does not occur by Lemma 4.2. (6) or (7). The case that p_1 is divisible by 6 is similarly shown.

Case 2-2 Either p_1 or p_2 is divisible by 4.

Suppose that p_2 is divisible by 4. By Lemma 4.2. (6) and (8), we have $n = 1$, $q_1 = 1$ and $\varepsilon_n p_1 = 3$. By Theorem 1.1. (3), we have $p_1/q_1 = 3$. The case that p_1 is divisible by 4 is similarly shown.

Case 2-3 $\{|p_1|, |p_2|\} = \{1, 3^b\}$ or $\{2, 3^b\}$.

Suppose that $|p_1| = 1$ or 2, and $|p_2| = 3^b$. By Lemma 4.2. (7), we have $n = 1$, $q_1 = 1$ and $\varepsilon_n p_1 = 2$. By Theorem 1.1. (3), we have $p_1/q_1 = 2$. The case that $|p_2| = 1$ or 2, and $|p_1| = 3^b$ is similarly shown.

Case 2-4 $\{|p_1|, |p_2|\} = \{1, 2\}$.

By Lemma 4.2. (6), these cases do not occur. □

5. Proof of "if part" of Theorem 1.2.

We need the following fact proved in [10].

Lemma 5.1. Let L be a $(2, 2s)$-torus link where $|s| \geq 2$, and $M = (L; \alpha_1/\beta_1, \alpha_2/\beta_2)$ the result of Dehn surgery along L where α_i and β_i $(i = 1, 2)$ are integers such that $|\alpha_i - s\beta_i| \neq 0$. Then M is a lens space if and only if $|\alpha_1 - s\beta_1| = 1$ or $|\alpha_2 - s\beta_2| = 1$. Moreover if $|\alpha_2 - s\beta_2| = 1$, then $M = L(p, (\alpha_1 - s\beta_1)\beta_2 + \varepsilon\beta_1)$ where $p = \alpha_1\alpha_2 - s^2\beta_1\beta_2$ and $\varepsilon = \alpha_2 - s\beta_2 (= \pm 1)$.

Proof of the "if part" of Theorem 1.2.
(a) The case $p_1/q_1 = 1$, or $p_2/q_2 = 1$.

We have already shown the lens surgery in the proof of Lemma 4.2. (1).

(b) The case $p_1/q_1 = 2$, or $p_2/q_2 = 2$.

We prove only the case $p_1/q_1 = 2$. We have a framed link presentation of M as in Figure 4 which is Dehn surgery along a $(2, 4)$-torus link where we set $r = p_2/q_2$. Since this case is $s = 2$, $\alpha_1 = -2$, $\beta_1 = 1$, $\alpha_2 = p_2 - 2q_2$ and $\beta_2 = q_2$ in Lemma 5.1., M is a lens space if and only if $|(p_2 - 2q_2) - 2q_2| = |p_2 - 4q_2| = 1$, and then $M = L(2p_2, 8q_2 - p_2)$.

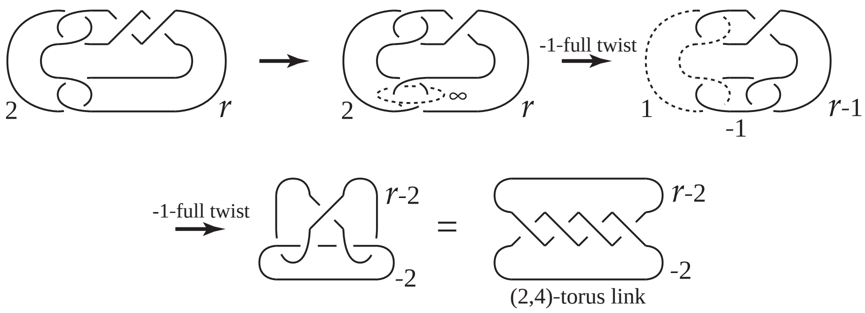

Figure 4: $(2, r)$-surgery along W_1

(c) The case $p_1/q_1 = 3$, or $p_2/q_2 = 3$.

We prove only the case $p_1/q_1 = 3$. We have a framed link presentation of M as in Figure 5 which is Dehn surgery along a $(2, -6)$-torus link where we set $r = p_2/q_2$. Since this case is $s = -3$, $\alpha_1 = -3$, $\beta_1 = 2$, $\alpha_2 = p_2 - 6q_2$ and $\beta_2 = q_2$ in Lemma 5.1., M is a lens space if and only if $|(p_2 - 6q_2) + 3q_2| = |p_2 - 3q_2| = 1$, and then $M = L(3p_2, 3q_2 - 2p_2)$. The case $p_2/q_2 = 3$ is similarly shown.

Therefore this completes the proof. □

6. Generalization of Theorem 1.2.

Our method extends to algebraically same links with W_n. Let $L = K_1 \cup K_2$ be a 2-component link in an integral homology 3-sphere Σ with its Alexander polynomials

(6.1) $\qquad \Delta_L(t_1, t_2) = n(t_1 - 1)(t_2 - 1)\ (n \geq 0), \Delta_{K_1}(t) \doteq 1$ and $\Delta_{K_2}(t) \doteq 1.$

Since we can take a 3-ball B in Σ such that $B \cap K_i \neq \emptyset$ $(i = 1, 2)$ and $(B, B \cap L)$ is a trivial 2-string tangle, we can add the third component K_3 in B such that

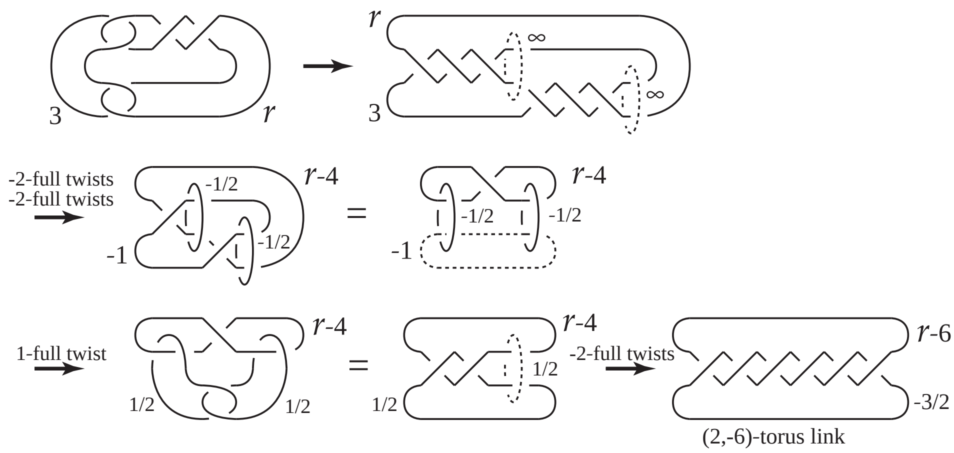

Figure 5: $(3, r)$-surgery along W_1

$H_i = K_i \cup K_3$ $(i = 1, 2)$ is the connected sum of K_i and the Hopf link, and $\mathrm{lk}\,(K_i, K_3) = 1$ $(i = 1, 2)$ by suitable orientations. We set $\overline{L} = L \cup K_3$. Then by the surgery formula (Lemma 2.1.) and (6.1), we have

$$\varDelta_{H_i}(t_1, t_2) \doteq \varDelta_{K_i}(t_i) \doteq 1 \ (i = 1, 2),$$

and by the Torres formula (Lemma 2.3.) and (6.1), we may set as follows:

$$(6.2) \qquad \varDelta_{\overline{L}}(t_1, t_2, t_3) = n(t_1t_2 - 1)(t_1 - 1)(t_2 - 1) + (t_3 - 1)g_n(t_1, t_2, t_3)$$

for some $g_n(t_1, t_2, t_3) \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}]$, which is just the same form as (3.1). We define an integer ε_n by $-\varepsilon_n = g_n(1, 1, 1)$. Then for the case $n > 0$, the same arguments as in Section 3 and Section 4 work by replacing W_n and $\overline{W}$ with L and $\overline{L}$, respectively, except the parts corresponding to Lemma 4.2. (1) and (5). In particular, Lemma 3.2. also holds for the case $n > 0$ in the present setting.

Lemma 6.1. In the situation above, if $n > 0$, then $\varepsilon_n = 1$ or -1 is uniquely determined (i.e. ε_n is well-defined), and $|\varepsilon_0| = 1$.

Proof. Since Lemma 3.1. also holds by replacing $\overline{W}$ with $\overline{L}$, we have $|\varepsilon_n| = 1$ for every n including the case $n = 0$. We show uniqueness of ε_n for the case $n > 0$. Let $M = (\overline{L}; \emptyset, 1, \infty)$, $M_0 = (\overline{L}; \emptyset, 1, \emptyset)$, $E_{\overline{L}}$ the exterior of $\overline{L}$, m_i and l_i a meridian and a longitude of K_i $(i = 1, 2, 3)$ on $\partial E_{\overline{L}}$ respectively, and set

$$M = E_{\overline{L}} \cup V_2 \cup V_3 \quad \text{and} \quad M_0 = E_{\overline{L}} \cup V_2,$$

where V_i is a solid torus glued in doing surgery along K_i. Let m_i' and l_i' be a meridian and a longitude of V_i respectively. We may assume that, in $H_1(E_{\overline{L}})$,

$$[m_2'] = [m_2][l_2], [l_2'] = [m_2], [m_3'] = [m_3], [l_3'] = [l_3],$$
$$[l_1] = [l_2] = [m_3], [l_3] = [m_1][m_2].$$

Here $[\ -\ ]$ denotes the homology class in $H_1(E_{\overline{L}})$. In the following, we also denote the homology class in $H_1(M_0)$ and $H_1(M)$ by the same symbol.

In $H_1(M_0)$, we have $[m_2'] = [m_2][l_2] = [m_2][m_3] = 1$. Hence we have

$$H_1(M_0) \cong \langle [m_1], [m_2], [m_3] \mid [m_2][m_3] = 1 \rangle \cong \langle [m_1], [m_2] \mid - \rangle \cong \mathbb{Z}^2.$$

Then by the surgery formula (Lemma 2.1.) and (6.2), we have

$$\begin{aligned} \tau(M_0) &\doteq \Delta_{\overline{L}}(t_1, t_2, t_2^{-1})(t_2 - 1)^{-1} \\ &\doteq n(t_1 t_2 - 1)(t_1 - 1) - t_2^{-1} g_n(t_1, t_2, t_2^{-1}) \end{aligned} \tag{6.3}$$

In $H_1(M)$, we have $[m_3'] = [m_3] = 1$. Hence we have

$$H_1(M) \cong \langle [m_1], [m_2], [m_3] \mid [m_2] = [m_3] = 1 \rangle \cong \langle [m_1] \mid - \rangle \cong \mathbb{Z}.$$

Then by the surgery formula (Lemma 2.1.), (6.3) and Lemma 3.2., we have

$$\begin{aligned} \tau(M) &\doteq \{n(t_1 - 1)^2 - g_n(t_1, 1, 1)\}(t_1 - 1)^{-1} \\ &\doteq \{n(t_1 - 1)^2 + \varepsilon_n t_1\}(t_1 - 1)^{-1}. \end{aligned}$$

Since $\tau(M)$ depends only on L (i.e. independent from the third component K_3), and characterizes ε_n, ε_n is uniquely determined as an invariant of L. □

We remark that if $n = 0$, then we cannot determine ε_0 uniquely. The value ε_n for $n > 0$ depends on the geometric shape of L (see Remark 6.4.).

Since the first term of the righthand side of (6.2) vanishes for the case $n = 0$, we may also assume Lemma 3.2 for $n = 0$. Hence Theorem 1.1. (1) and (2) also hold for $n \geq 0$. Computations of the Reidemeister torsions in the present setting is the same as that in Section 3 and Section 4 by replacing $\overline{W}$ with $\overline{L}$. Then we have an extension of Lemma 4.1.

Theorem 6.2. *Suppose that $M = (L; p_1/q_1, p_2/q_2)$ is a lens space. Then we have the following:*

(1) $n = 0$ *or* 1.

(2) *If $n = 0$, then $|p_1| = 1$ or $|p_2| = 1$. Moreover if $|p_1| = 1$, then $M = L(p_2, \pm q_2)$.*

(3) *If $n = 1$ and $|p_2| \geq 5$, then $q_1 = 1$ and $p_1 = \varepsilon_1, 2\varepsilon_1$ or $3\varepsilon_1$.*

(4) *If $n = 1$ and $|p_1| \geq 5$, then $q_2 = 1$ and $p_2 = \varepsilon_1, 2\varepsilon_1$ or $3\varepsilon_1$.*

In each case (3) *and* (4), $\varepsilon_1 = 1$ *or* -1 *which is determined uniquely depending on* L.

Proof. Firstly, we suppose $n > 0$. Since the arguments in the proof of Lemma 4.1. also work in the present setting, we have $n = 1$, and (3) and (4). Secondly, we suppose $n = 0$. Then by the value of the Reidemeister torsion, we have (2). □

An extensions of Theorem 1.2. (Lemma 4.2.) can also be obtained (cf. [5] for (1) and (4)).

Theorem 6.3. *Suppose that* $n = 1$ *and* $M = (L; p_1/q_1, p_2/q_2)$ *is a lens space. Then one of the following* (1), (2), (3), (4), (5) *or* (6) *holds:*

(1) $p_1/q_1 = \varepsilon_1$, $\gcd(p_2, 6) = 1$ *and* $6q_2 \equiv \pm 1 \pmod{p_2}$.

(2) $p_1/q_1 = 2\varepsilon_1$ *and* $|\varepsilon_1 p_2 - 4q_2| = 1$.

(3) $p_1/q_1 = 3\varepsilon_1$ *and* $|\varepsilon_1 p_2 - 3q_2| = 1$.

(4) $p_2/q_2 = \varepsilon_1$, $\gcd(p_1, 6) = 1$ *and* $6q_1 \equiv \pm 1 \pmod{p_1}$.

(5) $p_2/q_2 = 2\varepsilon_1$ *and* $|\varepsilon_1 p_1 - 4q_1| = 1$.

(6) $p_2/q_2 = 3\varepsilon_1$ *and* $|\varepsilon_1 p_1 - 3q_1| = 1$.

In each case (1), (2), (3), (4), (5) *and* (6), $\varepsilon_1 = 1$ *or* -1 *which is determined uniquely depending on* L.

Proof. Since the arguments in the proof of Lemma 4.2. also work in the present setting, we have (2), (3), (5) and (6). By the values of the Reidemeister torsions, we have (1) and (4). □

Remark 6.4. The number ε_n may be understood from several viewpoints. We remark here one of them. The forms of the Reidemeister torsions in Theorem 1.1. show that both W_n and L in this section for $n > 0$ are not amphicheiral. For the case of W_1, Theorem 1.2. shows its chirality more clearly. They motivate a work of the first author [8] on the conditions for the Alexander polynomials of algebraically split component-preservingly amphicheiral links.

Let $\overline{L}$ ($n \geq 0$) be an oriented 3-component link in this section which is also expressed as $(\Sigma, \overline{L})$. We set its mirror imaged manifold pair as $(\Sigma', \overline{L'}) = (\Sigma, \overline{L})!$ where Σ' is the orientation-reversed Σ, and $\overline{L'} = K_1' \cup K_2' \cup K_3'$ has the induced orientation from $\overline{L}$, and set the 2-component sublink of $\overline{L'}$ corrsponding to L as $L' = K_1' \cup K_2'$ (K_i' $(i = 1, 2, 3)$ corresponds to K_i). Then the Alexander polynomial of $\overline{L'}$ is the same as that of $\overline{L}$ (up to trivial units). Though $\overline{L'}$ looks satisfying the same conditions as $\overline{L}$, only $\mathrm{lk}\,(K_i', K_3') = -1$ $(i = 1, 2)$ is different. Thus we re-set as $\overline{L'} = K_1' \cup K_2' \cup (-K_3')$ where $(-K_3')$ is the orientation-reversed component of K_3'. The Alexander polynomials of both $\overline{L}$ and $\overline{L'}$ satisfy (6.2) where we set the

$g_n(t_1,t_2,t_3)$-part for $\overline{L'}$ as $g'_n(t_1,t_2,t_3) \in \mathbb{Z}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]$. Then we have

$$\begin{aligned} \Delta_{\overline{L'}}(t_1,t_2,t_3) &\doteq \Delta_{\overline{L}}(t_1,t_2,t_3^{-1}) \\ &\doteq n(t_1t_2-1)(t_1-1)(t_2-1)+(t_3^{-1}-1)g_n(t_1,t_2,t_3^{-1}) \\ &\doteq n(t_1t_2-1)(t_1-1)(t_2-1)+(t_3-1)(-t_3^{-1})g_n(t_1,t_2,t_3^{-1}) \\ &\doteq n(t_1t_2-1)(t_1-1)(t_2-1)+(t_3-1)g'_n(t_1,t_2,t_3). \end{aligned}$$

We define an integer ε_n by $-\varepsilon_n = g_n(1,1,1)$. Since we can take $g'_n(t_1,t_2,t_3) = -t_3^{-1}g_n(t_1,t_2,t_3^{-1})$ and Lemma 6.1., we have $\varepsilon'_n = -\varepsilon_n$ for $n>0$. Therefore L cannot be amphicheiral in this case (i.e. only the case $n=0$ can be amphicheiral), and the statements of Theorem 6.2. and Theorem 6.3. have symmetries of this kind. In [8], it is conjectured that the Alexander polynomial of an algebraically split component-preservingly amphicheiral link with even components is zero.

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