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方程式의 近似解

Approximate Solutions of Equations in Chosun Mathematics

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數學史의 길을 열어주신 後山 李昌九 박사님의 喜壽를 축하드리며
헌정합니다.

구장산술이래 동양의 전통 수학은 유리수체를 기본으로 이루어져 있다. 따라서 방정식의 무리수해는 허용되지 않으므로 근사해를 구하는 방법은 방정식론에서 매우 중요한 과제가 되었다. 중국의 사료에 나타나는 근사해에 관한 역사를 먼저 기술하고, 이를 조선 산학에 나타나는 근사해에 관한 사료와 비교한다. 조선의 근사해에 대한 이론은 박율(朴繘, 1621 - 1668)의 산학원본(算學原本)과 조태구(趙泰耉, 1660-1723)의 주서관견(籌書管見)에 이미 정립되었다. 중국의 이론과 달리 두 산학자 모두 근사해의 오차에 관심을 가지고 더 좋은 근사해를 구하는 방법을 얻어내었음을 밝힌다.

Since JiuZhang SuanShu(九章算術), the basic field of the traditional mathemtics in Eastern Asia is the field of rational numbers and hence irrational solutions of equations should be replaced by rational approximations. Thus approximate solutions of equations became a very important subject in theory of equations. We first investigate the history of approximate solutions in Chinese sources and then compare them with those in Chosun mathematics. The theory of approximate solutions in Chosun has been established in SanHakWonBon(算學原本) written by Park Yul(朴繘, 1621 - 1668) and JuSeoGwanGyun(籌書管見, 1718) by Cho Tae Gu(趙泰耉, 1660-1723). We show that unlike the Chinese counterpart, Park and Cho were concerned with errors of approximate solutions and tried to find better approximate solutions.

Keywords: Polynomial equations, KaiFangFa(開方法), Approximate soulutions, Park Yul(朴繘), SanHakWonBon(算學原本), Cho Tae Gu(趙泰耉), JuSeoGwanGyun(籌書管見).

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0 Introduction

The traditional mathematics in Eastern Asia has been based on the field $\mathbb{Q}$ of rational numbers since JiuZhang SuanShu(九章算術). Consequently, irrational solutions of equations were not allowed([9]). In Song - Yuan era, a perfect method, namely the ZengCheng KaiFangFa(增乘開方法) to find rational approximate solutions of polynomial equations has been established. Even in cases that equations have rational solutions, mathematicians are quite satisfied with just getting significant digits of solutions or approximate ones. Although analytic geometry has never introduced in eastern mathematics, the linear interpolation and its variations have been introduced in the 3rd century for extracting radicals $\sqrt[n]{a}$. Further, mathemticians in Song - Yuan era extend the linear interpolation for $\sqrt[n]{a}$ to general equations.

The purpose of this paper is to investigate the history of approximate solutions of polynomial equations.

The paper divides into two parts. In the first part, we are concerned with the history of approximate solutions in Chinese mathematics up to the 16th century. We also discuss their validity for readers to understand the reason why they work. In these cases, we use freely modern mathematics for their proofs. In the second part, we deal with the history of approximate solutions in Chosun mathematics. The theory of approximate solutions in Chosun appeared first in SanHakWonBon (算學原本, 1700, [1]) written by Park Yul(朴繘, 1621 - 1668) and then in JuSeoGwanGyun(籌書管見, 1718, [2]) by Cho Tae Gu(趙泰耇, 1660 - 1723). Furthermore their theory is so complete that there is no more development of the theory in Chosun after their results. Thus we investigate mainly their results on errors of approximate solutions and better approximations of soultions of equations and conclude that they obtain quite remarkable and unique results.

For the Chinese mathematics, we refer to ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan(中國科學技術典籍通彙 數學卷, [4]) and ZhongGuo LiDai SuanXue JiCheng (中國歷代算學集成, [5]). Those books appeared in them will not be numbered as an indivisual reference.

1 History of approximate solutions

In this section, we present the theory of approximate solutions in Chinese sources which had important influences for the development of the theory in Chosun mathematics.

We first briefly describe the structure of ZengCheng KaiFangFa(增乘開方法) for solving polynomial equations as follows([11]).

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. For the equation $p(x) = 0$, let α be the first part of the solution of the equation $p(x) = 0$, called ChuShang(初商) which is the largest digit part of the solution in most of cases.

Let $y = x - \alpha$. Then using iterated synthetic divisions, one has

$$p(x) = b_n(x-\alpha)^n + b_{n-1}(x-\alpha)^{n-1} + \cdots + b_1(x-\alpha) + b_0.$$

Hence one has the equation $b_n y^n + b_{n-1}y^{n-1} + \cdots + b_1 y + b_0 = 0$ for y. The polynomial of the equation is denoted by $q(y)$. As above, let β be the first part of its solution which is called ChiShang(次商) of the original equation $p(x) = 0$ and then again by iterated synthetic divisions, one has

$$q(y) = c_n(y-\beta)^n + c_{n-1}(y-\beta)^{n-1} + \cdots + c_1(y-\beta) + c_0.$$

Thus we have the equation $c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 = 0$ where $z = y - \beta$ and its polynomial part is denoted by $r(z)$.

Repeating the above processes, one can have a solution or an approximate solution of the equation $p(x) = 0$.

It is clear that in the above processes, the coefficients of $r(z)$ are precisely those of the Taylor expansion of $p(x)$ at $\alpha + \beta$, i.e.,

$$p(x) = c_n(x-(\alpha+\beta))^n + c_{n-1}(x-(\alpha+\beta))^{n-1} + \cdots + c_1(x-(\alpha+\beta)) + c_0.$$

Although the method of extracting radicals in JiuZhang SuanShu is slightly different from the ZengCheng KaiFangFa, the basic ingredients in both methods are the same and hence we do not differentiate them for classifying methods of finding approximate solutions of equations.

[A] The first method is just terminating the process in ZengCheng KaiFangFa in a certain stage. In the first book(卷上) of XiaHouYang SuanJing(夏侯陽算經,

ca. 8th C.), the author finds the integral part 723 of $\sqrt{522,900}$ and then indicates the remainder $522,900-723^2 = 171$. The same kind of the method also appears in Chapter 12 of ShuShu JiuZhang(數書九章, 1247) written by Qin Jiu Shao(秦九韶, 1202 - 1261). Indeed, for the equation $16x^2 + 192x - 1,863.2 = 0$, Qin terminates the process at its solution 6.35 and for the equation $36x^2 + 360x - 13,068.8 = 0$ at 14.7 (see also [7]).

[B] For an equation $p(x) = 0$, let α be the integral part of the solution of the equation and let $p(x) = b_n(x-\alpha)^n + b_{n-1}(x-\alpha)^{n-1} + \cdots + b_1(x-\alpha) + b_0$ be the Taylor expansion at α. As mentioned above, the coefficents of the expansion are obtained by the iterating synthetic divisions for the last digit of α. Let $z = x-\alpha$ and $q(z)$ denote the polynomial $b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$. We now assume that a positivie solution of the equation $p(x) = 0$ is not a natural number. Then the solution of the equation $q(z) = 0$ which lies in the open interval $(0, 1)$ is precisely the decimal part of the solution of the original equation $p(x) = 0$. The linear interpolation is well known but we include the following for the completeness of this paper. The equation of the line through two points $(0, q(0))$ and $(1, q(1))$ is $y = (b_n + b_{n-1} + \cdots + b_1)z + b_0$ which is a secant line of the curve $y = q(z)$ and hence $\dfrac{-b_0}{b_n + b_{n-1} + \cdots + b_1}$ is a good approximation of the decimal part of the solution of $p(x) = 0$. This process is known to be the linear interpolation.

In particular, for $\sqrt{a}$, i.e., the equation $x^2 - a = 0$, let α be the integral part of the solution, then the corresponding polynomial to $q(z)$ in the above, is

$$z^2 + 2\alpha z + (\alpha^2 - a).$$

Hence one has the well known interpolation $\sqrt{a} = \alpha + \dfrac{a-\alpha^2}{1+2\alpha}$ which has already appeared in Liu Hui(劉徽)'s commentary in JiuZhang SuanShu([3, 8]). But there is no example of this formula in the book.

Similarly one has $\sqrt[3]{a} = \alpha + \dfrac{a-\alpha^3}{1+3\alpha+3\alpha^2}$.

The linear interpolations are widely used and we collect some sources which deal with them:

the first book of ZhangQiuJian SuanJing(張丘建算經, ca. 5 - 6th C.); the second book of WuJing SuanShu(五經算術) written by Zhen Luan(甄鸞, ca. 6th C.); Chapter 4, 6 of ShuShu JiuZhang, XuGu ZhaiQi SuanFa(續古摘奇算法, 1275) in YangHui SuanFa(楊輝算法, 1274 - 1275).

We now discuss the interpolations in YangHui SuanFa. Yang Hui introduced the method in the first book of XuGu ZhaiQi SuanFa as one in BianGu TongYuan (辯古通源) and he included $\sqrt{1300} = 36\frac{4}{73}$ and then took $\sqrt{2.205}$ as another example in the second book of XuGu ZhaiQi SuanFa. Incidentally, when Seki Kowa (關孝和, ? - 1708) copied and corrected YangHui SuanFa published in Chosun in 1433, he put the statement in the first book after the example in the second book. Yang Hui took the example from a problem in the first book of ZhangQiuJian SuanJing. It asks the length of a square inscribed in a circle with the diameter 2 chi 1 cun(2尺 1寸). Using $\sqrt{2} \approx 1.4$, called FangWu XieQi(方五斜七), Zhang has an approximation $2.1 \times \frac{5}{7} = 1.5$ and Lee Chun Feng(李淳風, ca. 7th C.) added 1 chi $4\frac{21}{25}$ cun as another approximation. Yang Hui transforms this problem to $2x^2 = 2.1^2$, that is, $x^2 = 2.205$. He states that using the method in BianGu TongYuan, he obtains 1 chi $4\frac{245}{281}$ cun(1尺 4寸 281 分寸之 245). Yang Hui made a fumble for the denominator 281 for 1 chi 4 cun is not the integral part but 1.4 and we now give a detail to get the appropriate approximation by the linear interpolation. The first part (初商) of the solution is 1 and one has the equation $y^2 + 2y - 1.205 = 0$ for the next part (次商) by the iterated synthetic divisions. Since the next part of the solution lies in between 0 and 1, one can transform the equation by $y = \frac{z}{10}$ to $0.01z^2 + 0.2z - 1.205 = 0$ and then one has the next solution 4. Further the equation for the next solution(次次商) is $0.01u^2 + 0.28u - 0.245 = 0$. Thus the approximation by the linear interpolation is 1 chi $4\frac{0.245}{0.01+0.28} = 4\frac{245}{290}$ cun. Because of the numerator, Yang Hui should have the same equation for the approximation. For his denominator 281, the equation of the line through (0, -0.245) and (0.1, r(0.1)) is indeed $y = (0.001 + 0.28)x - 0.245$ where r(u) denotes the polynomial of the last equation and hence one can have Yang Hui's solution. But we don't take this for validating his approximation but gather that simply he made a wrong calculation.

Yang Hui also stated the linear interpolation for $\sqrt[3]{a}$ in the first book of XuGu ZhaiQi SuanFa when he quoted the one in BianGu TongYuan as follows:

> 校正 辯古通源 開方不盡之法 開方除不盡之數 命爲分子 術曰 倍隅數入
> 廉 一退 平方二因 立方三因 得入 下法一算 總爲分母 以命分子之數

Yang Hui did not give any example for $\sqrt[3]{a}$ but by his notations of equations, it is most likely that "三因" means $3\alpha + 3\alpha^2$.

Examples for the cube roots, one can find them in Book 35 of SuanXue BaoJian(算學寶鑑, 1524) written by Wang Wen Su(王文素) and Book 6 of SuanFa TongZong(算法統宗, 1592) by Cheng Da Wei(程大位, 1533 - 1606).

[C] Finally some variations of [B].

In the second book of SunZi SuanJing(孫子算經, ca. 3rd - 4th C.), $\sqrt{234,567} \approx 484\frac{311}{968}$, where $311 = 234,567 - 484^2$ and $968 = 2 \times 484$. Thus instead of the linear interpolation formula, he uses $\sqrt{a} \approx \alpha + \frac{a - \alpha^2}{2\alpha}$ which was also mentioned in Liu Hui's commentary in JiuZhang SuanShu([3]). We note that the equation of the tangent line at 0 to the function $f(x) = x^2+2\alpha x+(\alpha^2-a)$ is $y = 2\alpha x+(\alpha^2-a)$. Therefore Sun Zi's approximation can be obtained by the tangent line. Similary, Liu Hui states an approximation $\sqrt[3]{a} \approx \alpha + \frac{a - \alpha^3}{3\alpha^2}$ in his commentary without any example. In the third book(卷下) of ZhangQiuJian SuanJing, $\sqrt[3]{1,572,864} \approx 116\frac{11,968}{40,369}$, where $11,968 = 1,572,864 - 116^3$ and $40,369 = 1 + 3 \times 116^2$. Thus Zhang uses a variation $\sqrt[3]{a} \approx \alpha + \frac{a - \alpha^3}{1 + 3\alpha^2}$. Here Zhang disregards the term 3α of the denominator in the linear interpolation. Zhang adds another example which gives an approximation of $\sqrt[3]{1,293,732}$ by the same method. Zhang might have these approximations from the one for the square roots, for $3\alpha^2$ is the linear coefficent in the equation for the next solution as 2α for the square roots.

In [8], there is a claim that Liu Xiao Sun(劉孝孫, ca. 576 - 625) has an approximation $116\frac{11,968}{40,368}$, i.e., $\sqrt[3]{a} \approx \alpha + \frac{a - \alpha^3}{3\alpha^2}$ which is precisely an approximation by the tangent line as that in Sun Zi's formula.

2 Approximate Solutions in Chosun mathematics

In this section, we deal with development of approximate solutions in Chosun mathematics. Since the theory of approximate solutions was mainly established in Park Yul's SanHak WonBon and Cho Tae Gu's JuSeoGwanGyun, we will discuss their theory on approximate solutions in them.

Park Yul passed the national examination for the civil services(文科 科擧) in 1654 and served as governors of prefectures(縣監, 郡守). He had a mathematical collaboration with a governor of prefecture, Im Jun(任濬) who was known as an expert on TianYuanShu(天元術) but we do not have any work written by Im. Park's SanHakWonBon was posthumously published in 1700 by his son Park Du Se(朴斗世, 1650 - 1733). It is the first book dealing with TianYuanShu after Zhu Shi Jie(朱世傑)'s SiYuan YuJian(四元玉鑑, 1303) in Eastern Asia. It consists of three books. The first book is devoted to equations $ax^2 = b$, the second one to equations $ax^n = b$ and finally the third book to constructions of equations by TianYuanShu([12]).

Park studied Yang Hui's comment on BianGu TongYuan for the study of interpolations which was quoted in Section 1. The first book in SanHakWonBon deals with equations related to the right triangles.

It begins with the problem $\sqrt{97} \approx 9\frac{16}{19}$. The problem is to find the Xian(弦) of the right triangle with Gou(句) 4 chi and Gu(股) 9 chi. He obtains the approximation using the method [B] in previous section. From the Yang Hui's quotation of BianGu TongYuan in the previous section, he solves the problem as follows:

> 幷句股羃得九十七爲實 開方得九尺 餘實十六尺 即開方不盡之數二因方
> 法 添入下法 共一十九 命爲分母 餘實十六尺爲分子合問

The very next problem is to find Gu of the right triangle with Gou 4 chi and Xian $9\frac{16}{19}$ chi. Let b, c denote Gou and Xian, respectively, then one has the equation $x^2 = c^2 - b^2$ for Gu. The author does not take $c^2 = (9\frac{16}{19})^2$ but the original $c^2 = 97$. For the reverse of the linear interpolation $\sqrt{a} \approx \alpha + \frac{a-\alpha^2}{1+2\alpha}$, Yang Hui quoted the following after the quotation mentioned in Section 1.

> 再求積數還元術曰 置方面全步以分母通之幷入分子自乘於上
> 又以分子減分母 餘以分子乘之 得數幷入上位爲實 (商除還元無此一段)
> 以分母自乘爲法 實如法而一

Indeed, for $\sqrt{a} \approx \alpha + \frac{a-\alpha^2}{1+2\alpha} = \frac{n}{m}$, where $n = \alpha(1+2\alpha) + (a-\alpha^2)$ and $m = 1+2\alpha$, a is given by the following process:

$$a = \frac{n^2 + (m - (a-\alpha^2))(a-\alpha^2)}{m^2}.$$

One can easily prove the above identity although the formula itself is quite complicated. We note that in most of cases, the fraction part is given by the formula and hence a can be obtained by adding α^2 to the numerator of the fraction. Furthermore, suppose that reducing the fraction part in the formula of the interpolation, one has the irreducible fraction $\frac{p}{q}$. Starting from $\frac{p}{q}$ and applying the above processes for the reverse of the interpolation, we have also a. Even in this case, one has immediately $a = \alpha^2 + \frac{p}{q}(1+2\alpha)$, for $\frac{p}{q} = \frac{a-\alpha^2}{1+2\alpha}$.

We now quote Park's solution of the above problem.

> 弦通分內子自乘於上 分子減分母 餘乘分子 加入上位寄左
> 又以分母自乘三百六十一 乘句冪得數 以減寄左
> 餘得二萬九千二百四十一爲實 以三百六十一爲隅 開方得弦

Let r, s denote the numerator and denominator of the formula of the reverse of the interpolations, i.e., $s = m^2$, then Park's equation is $sx^2 = r - sb^2$ which is equivalent to $x^2 = \frac{r}{s} - b^2$, where b denotes the Gou. In this example, the author tries to include two processes, namely the reverse of the linear interpolation and that of converting an equation with rational coefficients into the equation with integral coefficients.

Further, he notes that $\sqrt{\frac{n}{m}} = \frac{\sqrt{mn}}{m}$.

After practicing 3 more sets of these kinds of problems, his next problem is to find Xian with Gou $4\frac{1}{2}$ and Gu 9. He obtains the equation $4x^2 - 405 = 0$ for Xian and has the solution $10\frac{5}{84}$.

Park Yul uses the formula $\alpha + \frac{a - b\alpha^2}{b + 2b\alpha}$ for the linear interpolation for an equation $bx^2 - a = 0$ which can be obtained in the process [B] in Section 1. We note that among the books quoted in [B], Yang Hui SuanFa had been read throughout in Chosun but the remaining books were brought into Chosun in the mid-19th century and hence his formula is quite remarkable. On the other hand, Park may have had the above formula by the approximate solution $\alpha + \frac{\frac{a}{b} - \alpha^2}{1+2\alpha}$ of the equivalent equation $x^2 - \frac{a}{b} = 0$.

As the first set of problems discussed above, he gives the next problem to find Gou with Gu(9) and Xian($10\frac{5}{84}$). Here Park again notices that the equation $bx^2 - a = 0$ is equivalent with $x^2 - \frac{a}{b} = 0$. Thus using the formula for the reverse of interpolations for $x^2 - \frac{a}{b} = 0$, one can have $\frac{a}{b}$ which is x^2. Thus as above, he obtains the equation $84^2x^2 - 142,884 = 0$ for Gou, which is equivalent to $4x^2 -$

$81 = 0$. He has two solutions, namely 4.5 and $4\frac{17}{36}$. Clearly the former is the exact solution and the latter is an approximate solution obtained by the interpolation. By these, he compares the approximate solution given by the interpolation with the exact solution. For further practices, he deals with 10 more problems. Among them, the first 3 problems are related to the above triangle and the remaining 7 problems to the right triangle with Gou $4\frac{1}{2}$ and Gu $6\frac{1}{3}$ so that the author also illustrates the reduction to common denominators.

Dealing with 12 problems related to the diagonals and sides of squares, Park discusses approximate solutions for equations of the type $bx^2 = a$. Indeed, these problems concern with $\sqrt{2}$ since for diagonal d and one side a of a square, $d = \sqrt{2}a$ and $a = \frac{d}{\sqrt{2}}$. But Park first solves these problems by $d = \sqrt{2a^2}$ and $a = \sqrt{\frac{d^2}{2}}$ and using interpolations and their reverses as above, he obtaines the approximate solutions and compares them with the exact solutions whenever one can have the exact solutions.

After these problems, he took problems related to FangWu XieQi(方五斜七), i.e., $\sqrt{2} \approx 1.4$. Indeed, he found the diagonal by $d = \frac{7a}{5}$ and one side by $a = \frac{5c}{7}$.

He then states the following:

> 方斜之於開方多少之差在尺寸則甚微
> 而面求弦 至七十尺 則方斜之不及 開方者幾滿一尺
> 弦求面至七十尺 則方斜之過開方亦已過半尺矣
> 故曰方五斜七僅可施於尺寸之問 其可用於百步之外

Park clearly notices that FangWu XieQi is an approximation of $\sqrt{2}$ as other authors but he precisely calculates errors corresponding to diagonals and sides.

For the second line of the above quotation, we have to solve the inequality $\sqrt{2}x - 1.4x \leq 1$. For this, we solve the equation $\sqrt{2}x - 1.4x = 1$, equivalently $x^2 - 70x - 25 = 0$ and its positive solution lies in between 70 and 71. Thus as he claims, when the side is up to 70, the corresponding error is less than 1.

For the third line of the quotation, one has to solve the inequality $\frac{5x}{7} - \frac{x}{\sqrt{2}} > 0.5$. As above the solution of the equation $\frac{5x}{7} - \frac{x}{\sqrt{2}} = 0.5$, lies in between 69 and 70. Thus when the diagonal is greater than 70, the corresponding error is larger than 0.5 as he claimed.

As far as we have gone through sources, we can not find any other source except Park's book which deals with the above results. Further, we don't know exactly how Park obtains the above important result because mathematicians in Eastern Asia have never dealt with inequalities.

Although he explained the meaning of "三因" in BianGu TongYuan in Section 1 and moreover he added ZengCheng KaiFangFa for $\sqrt[4]{a}$ in the third book, the author didn't take any problem with approximate solutions where the degrees of the equations is larger than 2.

We now turn to approximate solutions in JuSeoGwanGyun. We refer to [10] for the general information on the book and its author Cho Tae Gu.

In the section JabBub(雜法) of its Introductory Remark, Cho includes the approximations of $\sqrt{2}$ and $\sqrt{3}$ together with various approximations of π. For $\sqrt{2}$, he mentions the usual FangWu XieQi, i.e., 1.4. For the area of a regular triangle, one needs $\dfrac{\sqrt{3}}{2}$ for its height. First he uses $\dfrac{7}{8}$ as its approximation. Using this, he quotes formulas describing the rates of areas between inscribed regular hexagon or circumscribed hexagon and the circle or those triangles and the circle as follows:

圓容六角八分之七 六角容圓七分之六
三角容圓七分之四 圓容三角十六分之七

Cho states the above quotaion which appears in SuanFa TongZong. Incidentally, the first statement is Cho's correction of trvially incorrect one in the book.

Furthermore, he states ZhengLiu MianQi(正六面七). This means the rate of a side and height of a regular triangle is 7 : 6. In other words, he quotes another approximation $\dfrac{6}{7}$ of $\dfrac{\sqrt{3}}{2}$ in one paragraph.

Cho derives the above ZhengLiu MianQi from a problem also in SuanFa TongZong. The problem is to find the area of the right triangle with a side 14, where the height is given by $14 \times 6 \div 7$ and it was transmitted from the same problem in JiuZhang SuanFa BiLei DaQuan(九章算法比類大全, 1450) written by Wu Jing(吳敬) through Wang's SuanXue BaoJian.

We will discuss more on these approximations later.

The method [B] in JuSeoGwanGyun appears first in problem 6 of Chapter ShaoGuang (少廣). His problem is to find $\sqrt{356}$ and he obtains the approximation $18\frac{32}{37}$ by the method in BianGu TongYuan. Further, he adds a comment $356+(37-32) = 19^2$. In the interpolation formula $\sqrt{a} \approx \alpha + \dfrac{a-\alpha^2}{1+2\alpha}$, one has immediately $a + (1 + 2\alpha - (a - \alpha^2)) = 1 + 2\alpha + \alpha^2 = (\alpha + 1)^2$, but this fact clearly gives some insight to the interpolations. Further, he deals with $\sqrt{162\frac{9}{16}}$ in problems 8 and 9. In problem 8, he has the exact solution $\dfrac{51}{4}$ by $\sqrt{\dfrac{b}{a}} = \dfrac{\sqrt{b}}{\sqrt{a}}$. After this, using $162\frac{9}{16} = \dfrac{2,601}{16}$, he solves the equation $16x^2 = 2,601$. Using the ZhengCheng KaiFangFa, he has the integral part 12 of the solution and $16z^2 + 384z - 297 = 0$ for the decimal part of the solution by the iterated synthetic divisions and then concludes that the approximate solution for the problem is $12\dfrac{297}{16+384} = 12\dfrac{297}{400}$ as the method [B] in Section 1. Moreover, he states the error is merely $\dfrac{3}{400}$.

The author treats the interpolations for $\sqrt[3]{a}$ in problems 20 and 21. In problem 20, he has $\sqrt[3]{12,500} = 23\dfrac{333}{657}$ by the exactly same process as that in problem 9, i.e., the process to find the linear equation in the final stage of ZhengCheng KaiFangFa. In problem 21 to find $\sqrt[3]{2,072\frac{43}{64}}$, he finds the exact solution $\sqrt[3]{\dfrac{132,651}{64}} = \dfrac{51}{4}$. As before, he solves the equation $64x^3 - 132,651 = 0$ by ZhengCheng KaiFangFa and obtains the integral part 12 of the solution and the equation $64z^3 + 2,304z^2 + 27,648z - 22,059 = 0$ for the decimal part of the solution. Thus by [B], he has the approximate solution $12\dfrac{22,059}{64+2,304+27,648} = 12\dfrac{22,059}{30,016}$. Finally he gives the error $\dfrac{453}{30,016}$ as before.

One can easily deduce that extending the above two problems, Cho Tae Gu may have had the result [B] of interpolations for arbitrary polynomial equations.

As we stated in [10], the most important feature of JuSeoGwanGyun is in the last part, GuJang MunDab(九章問答) of the book. In its 10th item, Cho discusses FangWu XieQi, i.e., $\sqrt{2} = 1.4$ and ZhengLiu MianQi, i.e., $\dfrac{\sqrt{3}}{2} = \dfrac{6}{7}$ as the well known approximate values of π in the 9th item. Indeed, for FangWu XieQi, he calculates the diagonal $\sqrt{50} \approx 7.07$ for the square with a side 5 and the side $\sqrt{\dfrac{49}{2}} \approx 4.95$弱 for the square with a diagonal 7 and hence 1.4 is not an exact value of $\sqrt{2}$. For ZhengLiu MianQi, he calculates the height $\sqrt{7^2 - 3.5^2} \approx 6.06$强 for the regular triangle with a side 7.

He then suggests more accurate approximations $\sqrt{2} \approx \dfrac{99}{77}$ and $\dfrac{\sqrt{3}}{2} \approx \dfrac{84}{97}$ as follows:

> 今欲改定 方七十斜九十九 正八十四面九十七 則庶爲精率
> 而此不過徽密求圓之依樣耳未足爲奇也

In the 12th item of GuJang MunDab, he also uses an approximation $\sqrt{2} \approx \dfrac{239}{169}$.

We have no idea how he gets the above approximations but they appear in the sequence $(\dfrac{y_n}{x_n})$, where $x_{n+1} = x_n + y_n, y_{n+1} = 2x_n + y_n$ and $\lim \dfrac{y_n}{x_n} = \sqrt{2}$ for any pair (x_1, y_1) of positive numbers([6]). Starting from (1, 1), the second term is (5, 7), fifth and sixth terms are (70, 99) and (169, 239). We include a sketch of the proof for the limit.

Since $|y_n^2 - 2x_n^2| = |y_{n-1}^2 - 2x_{n-1}^2|$, $\lim |\dfrac{y_n^2}{x_n^2} - 2| = \lim \dfrac{|y_1^2 - 2x_1^2|}{x_n^2} = 0$.

The other suggestion $\dfrac{\sqrt{3}}{2} \approx \dfrac{84}{97}$ is equivalent to $\sqrt{3} \approx \dfrac{168}{97}$ which is also a term of the sequence $(\dfrac{y_n}{x_n})$, where $x_{n+1} = x_n + y_n, y_{n+1} = 3x_n + y_n$ and $\lim \dfrac{y_n}{x_n} = \sqrt{3}$ for any positive x_1, y_1([6]).

For the proof of the limit, consider the sequence of even(odd, resp.) terms of the original sequence which are given by $x_{n+2} = 4x_n + 2y_n, y_{n+2} = 6x_n + 4y_n$ and hence they reduce to the terms $a_{n+1} = 2a_n + b_n, b_{n+1} = 3a_n + 2b_n$. Since $|3a_n^2 - b_n^2| = |3a_{n-1}^2 - b_{n-1}^2|$, $\lim |\dfrac{b_n^2}{a_n^2} - 3| = \lim \dfrac{|b_1^2 - 3a_1^2|}{a_n^2} = 0$. Thus one has $\lim \dfrac{y_n}{x_n} = \lim \dfrac{b_n}{a_n} = \sqrt{3}$.

As the above proof shows, the sequences $a_{n+1} = 2a_n + b_n, b_{n+1} = 3a_n + 2b_n$ would involve much simpler calculations for they are reduced and represent just odd or even terms of the original sequences.

Cho's approximation $\dfrac{168}{97}$ of $\sqrt{3}$ is the third term of the sequence $(\dfrac{b_n}{a_n})$ with the first term $\dfrac{12}{7}$.

Cho also uses $\sqrt{3} \approx \dfrac{97}{56}$ in the 13th item of the MunDab which is the 8th term of the sequence $(\dfrac{y_n}{x_n})$ with the first term $\dfrac{1}{1}$([6]), or equivalently the 4th term of the sequence $(\dfrac{b_n}{a_n})$ beginning with $\dfrac{4}{2}$.

So far, we can not find any source where Cho could have obtained the above results, all the more, the sequence with the first term $\dfrac{12}{7}$.

Finally Cho discusses approximations of $\sqrt{a}$ and $\sqrt[3]{a}$ again in the 25th item of the MunDab. For $\sqrt[3]{a} \approx \alpha + \dfrac{a-\alpha^3}{1+3\alpha+3\alpha^2}$, he states $a+q-p=(\alpha+1)^3$, where p, q denote the numerator and denominator of the fraction. Clearly the above properties for square roots and cube roots do not hold when the fractions are reduced.

3 Conclusion

We first quote the following statement which appears at KaiFangShu(開方術) in Chapter 4 of JiuZhang SunaShu.

若開之不盡者 爲不可開 當以面命之

As soon as the method of extracting square roots were introduced, mathematicians have come across the cases where by their method, they can not have the exact values. Since they do not have any idea of irrational numbers, they just call the cases BuKeKai(不可開, i.e., unextractable) and $\sqrt{a}$ as a side(面) of a square, where a is understood its area. For the extracting cube roots, the same statement without "當以面命之" was included. Liu Hui made commentaries on approximations for these cases, but did not like these and urged to continue the process to get more digits([3, 8]). The authors after JiuZhang SuanShu didn't follow his suggestion probably because Liu Hui didn't give any example dealing with BuKeKai cases and were rather interested in approximate solutions. Liu Hui commented the inequality $\alpha + \dfrac{a-\alpha^2}{1+2\alpha} < \sqrt{a} < \alpha + \dfrac{a-\alpha^2}{2\alpha}$. Since his commentary, Chinese mathematicians have not given any attention to the errors of approximations.

Chosun mathematics has succeeded in its revival in the second half of the 17th century([10]). During the period, Chosun mathematicians could have studied only Yang Hui SuanFa, SuanXue QiMeng(1299), XiangMing SuanFa(詳明算法, 1373) and SuanFa TongZong for Chinese mathematics and TongWen SuanZhi(同文算指, 1613) for Chinese and wesetern mathematics. Among these, Yang Hui SuanFa and SuanFa TongZong deal with approximate solutions. Using these references, Park Yul and Cho Tae Gu have successfully built their theory of approximate solutions in SanHakWonBon and JuSeoGwanGyun respectively. They obtained theory of the interpolations which can be applied to any polynomial equations. Their most important contributions to theory of approximate solutions are their study of the errors of approximations. Although they did not have any idea of inequalities which are the essential tool to describe the errors, Park Yul accomplished a perfect theory of errors. Cho Tae Gu also succeeded in getting better approximations because of his understanding the errors.

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