CONTROLLABILITY RESULTS FOR IMPULSIVE NEUTRAL EVOLUTION DIFFERENTIAL SYSTEMS

S.SELVI[1] AND M. MALLIKA ARJUNAN[†]

[1]DEPARTMENT OF MATHEMATICS, MUTHAYAMMAL COLLEGE OF ARTS & SCIENCE, RASIPURAM- 637 408, TAMIL NADU, INDIA.
E-mail address: sselvimaths@yahoo.com

[†]DEPARTMENT OF MATHEMATICS, KARUNYA UNIVERSITY, KARUNYA NAGAR, COIMBATORE- 641 114, TAMIL NADU, INDIA.
E-mail address: arjunphd07@yahoo.co.in

ABSTRACT. In this paper, we consider the controllability of a certain class of impulsive neutral evolution differential equations in Banach spaces. Sufficient conditions for controllability are obtained by using the Hausdorff measure of noncompactness and Monch fixed point theorem under the assumption of noncompactness of the evolution system.

1. INTRODUCTION

Impulsive differential equations form an appropriate model for describing phenomena where systems instantaneously change their state. Because of this reason they have numerous applications in several fields of applied sciences, such as Biology, Economics and Physics. There has been a significant development in impulsive theory in recent years, especially in the area of impulsive differential equations with fixed moments, see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [17] and Samoilenko and Perestyuk [25].

The study of the existence and stability of the differential equations with delay was initiated by Travis and Webb [26] and Webb [28]. In many areas of science there has been an increasing interest in the investigation of functional differential equations, incorporating memory or after-effect, that is, there is an effect of infinite delay on state equations. Related to this, we refer the reader to Kolmanovskii and Myshkis [15, 16] and Wu [29]. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. For the literature relative to impulsive neutral differential systems with infinite delay, we refer the reader to [4, 5, 8, 12, 30, 31].

[†]Corresponding Author Present Address: Department of Mathematics, C.B.M. College, Kovaipudur, Coimbatore- 641 042, Tamil Nadu, India.

On the other hand, the concept of controllability is of great importance in mathematical control theory. Controllability for differential systems in Banach spaces under the assumption of compactness and noncompactness of the operator semigroups has been studied by many authors [1, 6, 7, 9, 10, 13, 18, 20, 21, 23, 24, 27] by using various fixed point theorems. In particular, by using Monch fixed point theorem, Guo et al. [10] established the sufficient conditions for the controllability of the following class of impulsive evolution inclusions with nonlocal conditions:

$$x'(t) - A(t)x(t) \in F(t, x(t)) + Bu(t), \text{ a.e. on } T = [0, b],$$
$$\Delta x(t_i) = I_i(x(t_i)), \ i = 1, 2, \cdots, s,$$
$$x(0) + M(x) = x_0,$$

under the assumption of noncompactness of the semigroup generated by the evolution system. Very recently, by using the same fixed point theorem, Ji et al. [13] extended the controllability results of Guo et al. [10] into the following impulsive differential systems:

$$x'(t) = A(t)x(t) + f(t, x(t)) + (Bu)(t), \text{ a.e. on } [0, b],$$
$$\Delta x|_{t=t_i} = I_i(x(t_i)), \ i = 1, 2, \ldots, s,$$
$$x(0) + M(x) = x_0,$$

under the assumption that the evolution system generated by $A(t)$ is equicontinuous.

Motivated by the above mentioned works [10, 13, 31], in this paper, we establish the sufficient conditions for controllability of the impulsive neutral evolution differential equations with infinite delay of the form:

$$\frac{d}{dt}\Big[x(t) - g(t, x_t)\Big] + A(t)x(t) = f(t, x_t) + (Bu)(t),$$
$$t \in J = [0, b], \ t \neq t_k, \ k = 1, 2, \cdots, m, \tag{1.1}$$
$$\Delta x|_{t=t_k} = I_k(x_{t_k}), \ k = 1, 2, \cdots, m, \tag{1.2}$$
$$x_0 = \varphi \in \mathcal{B}, \tag{1.3}$$

where $\{A(t)\}_{t \in J}$ is a family of linear operators in a Banach space X generating an evolution operator $U : \Delta = \{(t, s) \in [0, b] \times [0, b] : 0 \leq s \leq t \leq b\} \to \mathcal{L}(X)$, here X is a Banach space and $\mathcal{L}(X)$ is the Banach space of all bounded linear operators in X; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $f, g : J \times \mathcal{B} \to X$ are appropriate functions; the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are given and $I_k : \mathcal{B} \to X$, $k = 1, 2, \cdots, m$, are given impulsive functions; the control function $u(\cdot)$ is considered in the space $L^2(J, V)$, where V is a Banach space of controls and $B : V \to X$ is a bounded linear operator.

2. Preliminaries

Let $(X, \| \cdot \|)$ be a real Banach space. We denote $\mathcal{C}([0, b], X)$ the space of all X-valued functions on $[0, b]$ with norm $\|x\| = \sup\{\|x(t)\| : t \in [0, b]\}$ and by $L^1([0, b], X)$ the space of X-valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| dt$.

To describe appropriately our problems, we say that a function $u : [\sigma, \tau] \to X$ is a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. By the symbol $\mathcal{PC}([\sigma, \tau]; X)$, we denote the space of normalized piecewise continuous functions from $[\sigma, \tau]$ into X. In particular, we denote the space $\mathcal{PC}$ formed by all functions $u : [0, b] \to X$ such that u is continuous at $t \neq t_k$, $u(t_k^-) = u(t_k)$ and $u(t_k^+)$ exists, for all $k = 1, 2, \cdots, m$. It is easy to see that $\mathcal{PC}$ is a Banach space with the norm $\|x\|_{\mathcal{PC}} = \sup_{s \in [0, b]} \|x(s)\|$.

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [11] and it is appropriate to treat retarded impulsive differential equations.

Let $\mathcal{B}$ will be a linear space of functions mapping from $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$, and satisfies the following axioms:

(A) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x|_{[\sigma, \sigma+b]} \in \mathcal{PC}([\sigma, \sigma+b]; X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:
  (i) $x_t \in \mathcal{B}$,
  (ii) $\|x(t)\| \leq H \|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.
(B) The space $\mathcal{B}$ is complete.

For the family of linear operators $\{A(t) : t \in J\}$, we assume the following hypotheses.

(A1) The domain $D(A(t))$ of $A(t)$ is dense in X and independent of t.
(A2) For each $t \in J$, the resolvent $R(\lambda : A(t))$ of $A(t)$ exists for all λ with $Re\lambda \leq 0$ and there exists a constant $M > 0$ such that $\|R(\lambda : A(t))\| \leq M(|\lambda| + 1)^{-1}$
(A3) There exist constants $L > 0$ and $0 < \mu \leq 1$ such that $\|(A(t) - A(s))A^{-1}(\tau)\| \leq L|t - s|^\mu$ for $t, s, \tau \in J$.

Under the assumptions $(A1) - (A3)$, the family $\{A(t) : t \in J\}$ generates an unique evolution system $\{U(t, s) : 0 \leq s \leq t \leq b\}$ satisfying:

(a) There exists a positive constant M_0 such that $\|U(t, s)\| \leq M_0$ for $0 \leq s \leq t \leq b$.
(b) For every $v \in D(A(t))$ and $t \in J$, $U(t, s)v$ is differential with respect to s on $0 \leq s \leq t \leq b$ and $\frac{\partial}{\partial s} U(t, s)v = U(t, s)A(s)v$.

Definition 2.1. *A two parameter family of bounded linear operators $U(t, s)$, $0 \leq s \leq t \leq b$ on X is called an evolution system if the following two conditions are satisfied:*

(i) $U(s, s) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq b$;

(ii) $(t, s) \to U(t, s)$ *is strongly continuous on* Δ, *i.e., for each* $x \in X$,
    the function $(t, s) \in \Delta \to U(t, s)x$ *is continuous.*

More details about evolution system can be found in Pazy [22].

Definition 2.2. *([3]) Let* E^+ *be the positive cone of an order Banach space* $(E, \leq)$. *A function* Φ *defined on the set of all bounded subsets of the Banach space* X *with values in* E^+ *is called a measure of noncompactness(MNC) on* X *if* $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ *for all bounded subsets* $\Omega \subseteq X$, *where* $\overline{co}\Omega$ *stands for the closed convex hull of* Ω.

The MNC Φ *is said:*

(1) *Monotone if for all bounded subsets* Ω_1, Ω_2 *of* X *we have:*
    $(\Omega_1 \subseteq \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))$;
(2) *Nonsingular if* $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ *for every* $a \in X$, $\Omega \subset X$;
(3) *Regular if* $\Phi(\Omega) = 0$ *if and only if* Ω *is relatively compact in* X.

One of the most examples of MNC is the noncompactness measure of Hausdorff β *defined on each bounded subset* Ω *of* X *by*

$\beta(\Omega) = \inf\{\epsilon > 0; \ \Omega \text{ can be covered by a finite number of balls of radii smaller than } \epsilon\}$.
It is well known that MNC β *enjoys the above properties and other properties see* [3, 14]: *For all bounded subsets* $\Omega, \Omega_1, \Omega_2$ *of* X,

(4) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, *where* $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$;
(5) $\beta(\Omega_1 \cup \Omega_2) \leq max\{\beta(\Omega_1), \beta(\Omega_2)\}$;
(6) $\beta(\lambda\Omega) \leq |\lambda|\beta(\Omega)$ *for any* $\lambda \in \mathbb{R}$;
(7) *If the map* $Q : D(Q) \subseteq X \to Z$ *is Lipschitz continuous with constant* k, *then* $\beta_Z(Q\Omega) \leq k\beta(\Omega)$ *for any bounded subset* $\Omega \subseteq D(Q)$, *where* Z *is a Banach space.*

Lemma 2.1. *([3]) If* $W \subset C([a, b], X)$ *is bounded and equicontinuous, then* $\beta(W(t))$ *is continuous for* $t \in [a, b]$ *and*

$$\beta(W) = \sup\{\beta(W(t)), \ t \in [a, b]\}, \quad where \quad W(t) = \{x(t) : x \in W\} \subseteq X.$$

Lemma 2.2. *([14]) Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions in* $L^1([0, b], \mathbb{R}^+)$. *Assume that there exist* $\mu, \eta \in L^1([0, b], \mathbb{R}^+)$ *satisfying* $\sup_{n \geq 1}\|f_n(t)\| \leq \mu(t)$ *and* $\beta(\{f_n(t)\}_{n=1}^{\infty}) \leq \eta(t)$ *a.e.* $t \in [0, b]$, *then for all* $t \in [0, b]$, *we have*

$$\beta\left(\left\{\int_0^t U(t, s)f_n(s)ds : n \geq 1\right\}\right) \leq 2M_0 \int_0^t \eta(s)ds.$$

The following fixed-point theorem, a nonlinear alternative of Monch type, plays a key role in our proof of controllability of the system $(1.1) - (1.3)$.

Lemma 2.3. *([19, Theorem 2.2]) Let* D *be a closed convex subset of a Banach space* X *and* $0 \in D$. *Assume that* $F : D \to X$ *is a continuous map which satisfies Monch's condition, that is* ($M \subseteq D$ *is countable,* $M \subseteq \overline{co}(\{0\} \cup F(M)) \Rightarrow \overline{M}$ *is compact*). *Then* F *has a fixed point in* D.

3. CONTROLLABILITY RESULTS

In this section, we present and prove the controllability results for the system $(1.1) - (1.3)$. First, we give the mild solution of the problem $(1.1) - (1.3)$.

Definition 3.3. *A function* $x : (-\infty, b] \to X$ *is a mild solution of the initial value problem* $(1.1) - (1.3)$, *if* $x_0 = \varphi \in \mathcal{B}$, $x(\cdot)|_J \in \mathcal{PC}$ *and*

$$x(t) = U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, x_t) + \int_0^t U(t,s)A(s)g(s, x_s)ds$$

$$+ \int_0^t U(t,s)\Big[f(s, x_s) + (Bu)(s)\Big]ds + \sum_{0 < t_k < t} U(t, t_k)I_k(x_{t_k}), \ t \in J.$$

Definition 3.4. *The system* $(1.1) - (1.3)$ *is said to be controllable on the interval J if for every initial function* $\varphi \in \mathcal{B}$ *and* $x_1 \in X$, *there exists a control* $u \in L^2(J, V)$ *such that the mild solution* $x(\cdot)$ *of* $(1.1) - (1.3)$ *satisfies* $x(b) = x_1$.

We will study the problem $(1.1) - (1.3)$ under the following hypotheses:

(H1) The evolution system $\{U(t,s)\}_{(t,s) \in \Delta}$ generated by the family of linear operators $\{A(t)\}_{t \in J}$ is equicontinuous. i.e., $(t,s) \to \{U(t,s)x : x \in E\}$ is equicontinuous for $t > 0$ and for all bounded subsets E.

(H2) The function $f : J \times \mathcal{B} \to X$ satisfies:

  (i) For a.e. $t \in J$, the function $f(t, \cdot) : \mathcal{B} \to X$ is continuous and for all $\varphi \in \mathcal{B}$, the function $f(\cdot, \varphi) : J \to X$ is strongly measurable.

  (ii) For each positive integer r, there exists an integrable function $\alpha_r : J \to [0, +\infty)$ such that

$$\sup_{\|\varphi\|_{\mathcal{B}} \leq r} \|f(t, \varphi)\| \leq \alpha_r(t), \text{ for a.e. } t \in J,$$

$$\text{and} \quad \liminf_{r \to \infty} \int_0^b \frac{\alpha_r(t)}{r}dt = \delta < +\infty.$$

  (iii) There exists integrable function $\eta : J \to [0, +\infty)$ such that

$$\beta(f(t, D)) \leq \eta(t) \sup_{-\infty < \theta \leq 0} \beta(D(\theta)) \quad \text{for a.e. } t \in J \text{ and } D \subset \mathcal{B},$$

  where $D(\theta) = \{v(\theta) : v \in D\}$ and β is the Hausdorff MNC.

(H3) The linear operator $W : L^2(J, V) \to X$ is defined by

$$Wu = \int_0^b U(b,s)Bu(s)ds \quad \text{such that}$$

  (i) W has an invertible operator W^{-1} which take values in $L^2(J, V)\backslash \ker W$, and there exist positive constants M_1, M_2 such that $\|B\| \leq M_1$ and $\|W^{-1}\| \leq M_2$.

  (ii) There is $K_W \in L^1(J, \mathbb{R}^+)$ such that, for every bounded set $Q \subset X$,

$$\beta(W^{-1}Q)(t) \leq K_W(t)\beta(Q).$$

(H4) There exists a positive constant $M_3 > 0$ such that $\|A(t)A^{-1}(0)\| \leq M_3$ for $t \in J$.

(H5) The function $g : J \times \mathcal{B} \to X$ is continuous and there exist positive constants L_0, C_1, C_2 such that,

(i)

$$\|A(0)g(t, \varphi_1) - A(0)g(t, \varphi_2)\| \leq L_0(\|\varphi_1 - \varphi_2\|_{\mathcal{B}}), \ \forall \, t \in J, \ \varphi_1, \varphi_2 \in \mathcal{B},$$

(ii)

$$\|A(0)g(t, \varphi)\| \leq C_1\|\varphi\|_{\mathcal{B}} + C_2, \ \forall \varphi \in \mathcal{B}, \ t \in J.$$

(H6) (i) There exist positive constants γ_k such that

$$\|I_k(\varphi_1) - I_k(\varphi_2)\| \leq \gamma_k(\|\varphi_1 - \varphi_2\|_{\mathcal{B}}), \ \ \forall \, \varphi_1, \varphi_2 \in \mathcal{B}$$

.

(ii) There exist continuous nondecreasing functions $L_k : [0, +\infty) \to (0, +\infty)$ such that

$$\|I_k(\varphi)\| \leq L_k(\|\varphi\|_{\mathcal{B}}), \ \forall \varphi \in \mathcal{B},$$

$$\text{and} \quad \liminf_{\rho \to \infty} \frac{L_k(\rho)}{\rho} = \lambda_k < +\infty, \text{ where } \sum_{k=1}^{m} \lambda_k = \lambda.$$

(H7) The following estimation holds true:
$N + \tilde{N} < 1$ where,

$$N = K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big[\|A^{-1}(0)\|L_0 + M_0 M_3 b L_0 + M_0 \sum_{k=1}^{m} \gamma_k\Big],$$

$$\tilde{N} = (2M_0 + 4M_0^2 M_1\|K_W\|_{L^1})\|\eta\|_{L^1}.$$

Remark 3.1. *From* $(A3)$, *we obtain* $\|A(t)A^{-1}(0)\| \leq L|b|^\mu + 1$. *Thus we can choose a positive constant* $M_3 = L|b|^\mu + 1$ *satisfying* $(H4)$.

Theorem 3.1. *Assume that the hypotheses* $(H1) - (H7)$ *are satisfied. Then the system* $(1.1) - (1.3)$ *is controllable on J provided that,*

$$K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big[C_1(\|A^{-1}(0)\| + M_0 M_3 b) + M_0(\delta + \lambda)\Big] < 1. \tag{3.1}$$

Proof. Using the hypothesis $(H3)$ for an arbitrary function $x : (-\infty, b] \to X$, define the control

$$u_x(t) = W^{-1}\Big[x_1 - U(b, 0)[\varphi(0) - g(0, \varphi)] - g(b, x_b) - \int_0^b U(t, s)A(s)g(s, x_s)ds$$

$$- \int_0^b U(b, s)f(s, x_s)ds - \sum_{k=1}^{m} U(b, t_k)I_k(x_{t_k})\Big](t).$$

We shall now show that using this control the operator defined by

$$\Phi x(t) = \begin{cases} \varphi(t), \ t \in (-\infty, 0], \\ U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, x_t) + \int_0^t U(t,s)A(s)g(s, x_s)ds \\ \quad + \int_0^t U(t,s)\Big[f(s, x_s) + Bu_x(s)\Big]ds, \ t \in J \end{cases}$$

has a fixed point. This fixed point is then a solution of $(1.1) - (1.3)$. Clearly, $(\Phi x)(b) = x_1$, which implies that the system $(1.1) - (1.3)$ is controllable.

Suppose that $x(t) = z(t) + y(t)$, $t \in (-\infty, b]$, where $y : (-\infty, 0] \to X$ be a function defined by $y_0 = \varphi$ and $y(t) = U(t,0)\varphi(0)$ on J. Then by the axioms of phase space, it is easy to see that $\|z_t + y_t\|_{\mathcal{B}} \le (K_b M_0 H + M_b)\|\varphi\|_{\mathcal{B}} + K_b\|z\|_t$, where $\|z\|_t = \sup_{0 \le s \le t} \|z(s)\|$, $K_b = \sup_{0 \le t \le b} K(t)$ and $M_b = \sup_{0 \le t \le b} M(t)$.

Define $S(b) = \Big\{ z : (-\infty, b] \to X \text{ such that } z_0 = 0, \ z|_J \in \mathcal{PC} \Big\}$ be the space endowed with the supremum norm $\|\cdot\|_b$. Then $(S(b), \|\cdot\|_b)$ is a Banach space. Let $\Gamma : S(b) \to S(b)$ be the operator defined by

$$(\Gamma z)(t) = \begin{cases} 0, \ t \in (-\infty, 0], \\ -U(t,0)g(0,\varphi) + g(t, z_t + y_t) + \int_0^t U(t,s)A(s)g(s, z_s + y_s)ds \\ \quad + \int_0^t U(t,s)\Big[f(s, z_s + y_s) + Bu_z(s)\Big]ds + \sum_{0 < t_k < t} U(t, t_k)I_k(z_{t_k} + y_{t_k}), \\ t \in J, \end{cases}$$

where $u_z(\cdot) \in L^2(J, V)$,

$$u_z(t) = W^{-1}\Big[x_1 - U(b,0)[\varphi(0) - g(0,\varphi)] - g(b, z_b + y_b)$$

$$- \int_0^b U(t,s)A(s)g(s, z_s + y_s)ds - \int_0^b U(b,s)f(s, z_s + y_s)ds$$

$$- \sum_{k=1}^m U(b, t_k)I_k(z_{t_k} + y_{t_k})\Big](t).$$

Clearly, Γ is well defined and with values in $S(b)$. It is easy to see that if z is a fixed point of Γ, then $z + y$ is a fixed point of Φ. So our aim is to find a fixed point of Γ.

Set $B_q = \{z \in S(b) : \|z\|_b \le q\}$ for some $q > 0$. Clearly, B_q is a nonempty, closed, convex and bounded set in $S(b)$. Then for any $z \in B_q$,

$$\|z_t + y_t\|_{\mathcal{B}} \le (K_b M_0 H + M_b)\|\varphi\|_{\mathcal{B}} + K_b q = q'. \tag{3.2}$$

For better readability, we break the proof into sequence of steps.

Step 1: There exists $q \ge 1$ such that $\Gamma(B_q) \subseteq B_q$.

Suppose the contrary. Then for each positive integer q, there exists $z \in B_q$ such that $\|(\Gamma z)(t)\| > q$ for some $t \in J$. It follows from the hypotheses $(H1) - (H6)$ and (3.2)

we have

$$q < \|(\Gamma z)(t)\|$$
$$\leq M_0\|A^{-1}(0)\|(C_1\|\varphi\|_{\mathcal{B}} + C_2) + \|A^{-1}(0)\|(C_1 q' + C_2) + M_0 M_3 b(C_1 q' + C_2)$$
$$+ M_0 \int_0^t \alpha_{q'}(s)ds + M_0 \sum_{k=1}^m L_k(q') + M_0 M_1 b^{\frac{1}{2}}\|u_z\|_{L^2} \tag{3.3}$$

where
$$\|u_z\|_{L^2} \leq M_2\Big[\|x_1\| + M_0[\|\varphi(0)\| + \|A^{-1}(0)\|(C_1\|\varphi\|_{\mathcal{B}} + C_2)]$$
$$+ \|A^{-1}(0)\|(C_1 q' + C_2) + M_0 M_3 b(C_1 q' + C_2)$$
$$+ M_0 \int_0^b \alpha_{q'}(s)ds + M_0 \sum_{k=1}^m L_k(q')\Big] \tag{3.4}$$

Hence by using (3.4) in (3.3), we have

$$q \leq \tilde{L} + (1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big[\|A^{-1}(0)\|C_1 q' + M_0 M_3 b C_1 q'$$
$$+ M_0 \int_0^b \alpha_{q'}(s)ds + M_0 \sum_{k=1}^m L_k(q')\Big], \tag{3.5}$$

where $\tilde{L}$ is independent of q.

Noting that $q' = (K_b M_0 H + M_b)\|\varphi\|_{\mathcal{B}} + K_b q \to +\infty$ as $q \to +\infty$, we obtain by hypotheses $(H2)(ii)$ and $(H6)(ii)$,

$$\lim_{q\to+\infty}\inf\Big(\frac{\int_0^b \alpha_{q'}(s)ds}{q}\Big) = \lim_{q\to+\infty}\inf\Big(\frac{\int_0^b \alpha_{q'}(s)ds}{q'}\cdot\frac{q'}{q}\Big) = \delta K_b,$$
$$\lim_{q\to+\infty}\inf\Big(\frac{\sum_{k=1}^m L_k(q')}{q}\Big) = \lim_{q\to+\infty}\inf\Big(\frac{\sum_{k=1}^m L_k(q')}{q'}\cdot\frac{q'}{q}\Big) = \lambda K_b.$$

Dividing both sides of (3.5) by q and employing the above two equalities, we have that

$$1 \leq K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big[C_1(\|A^{-1}(0)\| + M_0 M_3 b) + M_0(\delta + \lambda)\Big].$$

This contradicts (3.1). Thus, there exists $q \geq 1$ such that $\Gamma(B_q) \subseteq B_q$.

Step 2: $\Gamma : S(b) \to S(b)$ is continuous.

Let $(z^n)_{n\in\mathbb{N}}$ be a sequence in $S(b)$ such that $z^n \to z$ in $S(b)$. Then by hypotheses $(H2)(i)$, $(H5)(i)$ and $(H6)(i)$, we can prove that $f(s, z_s^n + y_s) \to f(s, z_s + y_s)$, $g(s, z_s^n + y_s) \to g(s, z_s + y_s)$ and $I_k(z_{t_k}^n + y_{t_k}) \to I_k(z_{t_k} + y_{t_k})$ uniformly on J.

Then by hypotheses $(H2)(i, ii)$ and $(H5)(i)$ with Dominated convergence theorem, we conclude that

$$\int_0^t U(t,s)f(s, z_s^n + y_s)ds \to \int_0^t U(t,s)f(s, z_s + y_s)ds,$$

and

$$\int_0^t A(s)U(t,s)g(s,z_s^n + y_s)ds \to \int_0^t A(s)U(t,s)g(s,z_s + y_s)ds, \text{ as } n \to \infty.$$

Which implies together with the continuity of the operators B, W^{-1} that,
we have $\|\Gamma z^n - \Gamma z\| \to 0$, as $n \to \infty$. Hence Γ is continuous on $S(b)$.

Step 3: The Monch condition holds:

To prove this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, for $t \in J$, where

$$(\Gamma_1 z)(t)$$

$$= -U(t,0)g(0,\varphi) + g(t,z_t + y_t) + \int_0^t U(t,s)A(s)g(s,z_s + y_s)ds$$

$$+ \sum_{0 < t_k < t} U(t,t_k)I_k(z_{t_k} + y_{t_k}) + \int_0^t U(t,\zeta)BW^{-1}\Big[x_1 - U(b,0)[\varphi(0) - g(0,\varphi)]$$

$$- g(b, z_b + y_b) - \int_0^b U(b,s)A(s)g(s,z_s + y_s)ds \sum_{k=1}^m U(b,t_k)I_k(z_{t_k} + y_{t_k})\Big](\zeta)d\zeta,$$

and

$$(\Gamma_2)z(t)$$

$$= \int_0^t U(t,s)f(s,z_s + y_s)ds - \int_0^t U(t,\zeta)BW^{-1}\Big[\int_0^b U(b,s)f(s,z_s + y_s)ds\Big](\zeta)d\zeta.$$

Firstly, we prove that Γ_1 is Lipschitz continuous.
Take $z_1, z_2 \in S(b)$. Then by the axioms of phase space and hypotheses $(H5)\&(H6)$, we get that

$$\|\Gamma_1 z_1(t) - \Gamma_1 z_2(t)\|$$

$$\leq \|A^{-1}(0)\|L_0\|z_{1t} - z_{2t}\|_{\mathcal{B}} + M_0 M_3 b L_0\|z_{1s} - z_{2s}\|_{\mathcal{B}}$$

$$+ M_0 \sum_{k=1}^m \gamma_k\|z_{1t_k} - z_{2t_k}\|_{\mathcal{B}} + M_0 M_1 M_2 b^{\frac{1}{2}}\Big[\|A^{-1}(0)\|L_0\|z_{1b} - z_{2b}\|_{\mathcal{B}}$$

$$+ M_0 M_3 b L_0\|z_{1s} - z_{2s}\|_{\mathcal{B}} + M_0 \sum_{k=1}^m \gamma_k\|z_{1t_k} - z_{2t_k}\|_{\mathcal{B}}\Big]$$

$$\leq K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big(\|A^{-1}(0)\|L_0 + M_0 M_3 b L_0 + M_0 \sum_{k=1}^m \gamma_k\Big)\|z_1 - z_2\|_b.$$

That is, $\|\Gamma_1 z_1(t) - \Gamma_1 z_2(t)\|_b \leq N\|z_1 - z_2\|_b$, \hfill (3.6)

where $N = K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big(\|A^{-1}(0)\|L_0 + M_0 M_3 b L_0 + M_0 \sum_{k=1}^m \gamma_k\Big)$

Hence, Γ_1 is Lipschitz continuous with Lipschitz constant N.

Next we prove that, Γ_2 maps B_q into an equicontinuous family on J.

Indeed let $t_1, t_2 \in J$, $0 < t_1 < t_2$. Then for arbitrary $z \in B_q$, we have

$$
\begin{aligned}
&\|\Gamma_2 z(t_2) - \Gamma_2 z(t_1)\| \\
&\leq \int_0^{t_1} \|[U(t_2, s) - U(t_1, s)]f(s, z_s + y_s)\| ds + \int_{t_1}^{t_2} \|U(t_2, s)f(s, z_s + y_s)\| ds \\
&\quad + \int_0^{t_1} \|[U(t_2, \zeta) - U(t_1, \zeta)]BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s + y_s)\Big](\zeta)\| d\zeta \\
&\quad + \int_{t_1}^{t_2} \|U(t_2, \zeta)BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s + y_s)\Big](\zeta)\| d\zeta.
\end{aligned}
$$

Let $Y(\zeta) = BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s + y_s)\Big](\zeta)$, then

$$
\begin{aligned}
&\|\Gamma_2 z(t_2) - \Gamma_2 z(t_1)\| \\
&\leq \int_0^{t_1} \|U(t_2, s) - U(t_1, s)\|\alpha_{q'}(s) ds + \int_{t_1}^{t_2} \|U(t_2, s)\|\alpha_{q'}(s) ds \\
&\quad + \int_0^{t_1} \|U(t_2, \zeta) - U(t_1, \zeta)\|\|Y(\zeta)\| d\zeta + \int_{t_1}^{t_2} \|U(t_2, \zeta)\|\|Y(\zeta)\| d\zeta. \quad (3.7)
\end{aligned}
$$

By the equicontinuity property of $\{U(t, s) : (t, s) \in \Delta\}$ and the absolute continuity of the Lebesgue integral, we can see that the right hand side of (3.7) tends to zero and independent of z as $t_2 \to t_1$. Hence, $\Gamma_2(B_q)$ is equicontinuous on J.

To prove the Monch condition, let $W \subseteq B_q$ is countable and $W \subseteq \overline{co}(\{0\} \cup \Gamma(W))$. We shall show that $\beta(W) = 0$. Without loss of generality, we may suppose that $W = \{z^n\}_{n \in \mathbb{N}}$. Then by the hypothesis $(H2)(iii)$, $(H3)(ii)$ and Lemma 2.2, we have

$$
\begin{aligned}
\beta\Big(\Gamma_2 W(t)\Big) &= \beta\Big(\Big\{\Gamma_2 z^n(t)\Big\}_{n=1}^\infty\Big) \\
&\leq \beta\Big(\Big\{\int_0^t U(t, s)f(s, z_s^n + y_s) ds\Big\}_{n=1}^\infty\Big) \\
&\quad + \beta\Big(\Big\{\int_0^t U(t, \zeta)BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s^n + y_s) ds\Big](\zeta) d\zeta\Big\}_{n=1}^\infty\Big) \\
&\leq 2M_0 \int_0^b \eta(s) \sup_{-\infty < \theta \leq 0} \beta\Big(\Big\{z^n(s + \theta) + y(s + \theta)\Big\}_{n=1}^\infty\Big) ds \\
&\quad + 2M_0 M_1 \int_0^b \beta\Big(W^{-1}\Big[\Big\{\int_0^b U(b, s)f(s, z_s^n + y_s) ds\Big\}_{n=1}^\infty\Big](\zeta)\Big) d\zeta
\end{aligned}
$$

$$\leq 2M_0 \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} \beta(z^n(\tau)) + 2M_0 M_1 \Big(\int_0^b K_W(\zeta)d\zeta \Big)$$

$$\times \beta\Big(\Big\{ \int_0^b U(b,s)f(s,z_s^n + y_s)ds \Big\}_{n=1}^\infty \Big),$$

$$\leq 2M_0 \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} \beta(W(\tau)) + 4M_0^2 M_1 \Big(\int_0^b K_W(\zeta)d\zeta \Big)$$

$$\times \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} \beta(W(\tau)),$$

$$= (2M_0 + 4M_0^2 M_1 \|K_W\|_{L^1})\|\eta\|_{L^1} \sup_{0\leq\tau\leq s} \beta(W(\tau)).$$

That is, $\beta(\Gamma_2 W(t)) \leq \tilde{N} \sup_{0\leq\tau\leq s} \beta(W(\tau)),$ (3.8)

where $\tilde{N} = (2M_0 + 4M_0^2 M_1 \|K_W\|_{L^1})\|\eta\|_{L^1}.$

Since Γ_2 maps B_q into an equicontinuous family on J, $\Gamma_2(W)$ is equicontinuous on J and so W is equicontinuous on J. Then by Lemma 2.1, taking supremum on both sides of (3.8) over J, we have

$$\beta(\Gamma_2(W)) \leq \tilde{N}\beta(W). \tag{3.9}$$

By the property (7) of Definition 2.2,

$$\beta(\Gamma_1(W)) \leq N\beta(W). \tag{3.10}$$

Hence $\beta(\Gamma(W)) \leq \beta(\Gamma_1(W)) + \beta(\Gamma_2(W)) \leq (N + \tilde{N})\beta(W).$

From the Monch condition, we get that

$$\beta(W) \leq \beta(\overline{co}(\{0\} \cup \Gamma(W))) = \beta(\Gamma(W)) \leq (N + \tilde{N})\beta(W).$$

By $(H7)$, $N + \tilde{N} < 1$, which implies that $\beta(W) = 0$. In the view of Lemma 2.3, i.e., Monch fixed point theorem, we conclude that Γ has a fixed point z in W. Then $x = z + y$ is a fixed point of Φ and thus the system $(1.1) - (1.3)$ is controllable on $[0, b]$. $\square$

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