GENERAL RELATIVISTIC RADIATION HYDRODYNAMICS: FREQUENCY-INTEGRATED RADIATION MOMENT FORMALISM

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ABSTRACT

I present here one approach to general relativistic radiation hydrodynamics. It is based on covariant tensor conservation equations and considers only the frequency-integrated total energy and momentum exchange between matter and the radiation field. It is also a mixed-frame formalism in the sense that, the interaction between radiation and matter is described with quantities in the comoving frame in which the interaction is often symmetric in angle while the radiation energy and momentum equations are expressed in the fixed frame quantities in which the derivatives are simpler. Hence, this approach is intuitive enough to be applied straightforwardly to any spacetime or coordinate. A few examples are provided along with caveats in this formalism.

Key words : accretion, accretion disks — hydrodynamics — radiative transfer — relativity

1. INTRODUCTION

There is an increasing need for a correct treatment of matter and radiation that are interacting with each other, in general relativistic regime. Astronomical objects such as gamma-ray bursts, supernovae, neutron stars, stellar or supermassive black holes, are expected to involve interactions between high-velocity matter and strong radiation field under strong gravity.

Particles in astrophysical matter generally have either very short mean free paths to the collision or behave as such due to strong coupling with the magnetic field. The physical properties of the matter, therefore, can generally be described by ideal hydrodynamics and their thermal or statistical properties can also be easily specified. Photons, however, can have either short or long mean free paths against matter particles. When the mean free path is short, photons assume approximately isotropic distribution and diffuse out through the matter, which can be relatively easily described. When the mean free path is long, photons may assume highly anisotropic distribution and interact directly with particles far away. Therefore, correct description of radiation should contain the information on directional distribution as well as the total energy and the spectral distribution, which makes solving the radiative transfer equation highly non-trivial and difficult. Yet, radiative transfer for a medium that is minimally affected by the radiative force and energy is still manageable. But when the dynamics and energetics of matter are significantly affected by the radiation itself, such as in accretion flows or jets, radiative transfer becomes even more difficult.

It gets worse in relativisic situations: photons get red- or blue-shifted, length and time depend on the

fiducial frame chosen, radiation suffers relativistic beaming and various relativistic corrections, such as bulk Comptonizaton, become important. Gravity adds additional difficulties: photon trajectories become curved and their frequencies are not conserved anymore owing to the gravitational redshift, time runs differently depending on the position, photons suffer a loss cone effect or Penrose process around black holes. These relativistic effects may be added to the non-relativistic radiative transfer formalism as additional perturbations. However, this can cause confusion and inconsistency at times when other effects of similar order are not properly taken into account or when the effects are applied inconsistently. The best approach, naturally, would be using full general relativistic formalism.

Solving the full frequency- and angle-dependent radiative transfer equation would be most desirable, but it is a formidable task, especially in a curved spacetime. A more manageable approach is to use frequency-integrated radiation moments. Frequency-integrated radiation moments constitute the covariant stress-energy tensor, and can be easily incorporated into the covariant equations. The first relativistic radiation moment equations were derived by Lindquist (1966) for spherically symmetric diffusion regime. The most general and complete radiation moment equations were obtained by Thorne (1981) using projected symmetric trace-free tensor formalism. It was built within the comoving frame in which matters are at rest and has been applied to one-dimensional problems (Turoalla & Nobili 1988; Zampieri et al. 1993). However, since the velocity of matter depends on the temporal and spatial positions, the covariant derivatives become complicated. The derivatives become simpler in the fixed frame while the interaction of radiation and mat-

ter become simpler in the comoving frame. It is possible to utilize both frames: interaction terms are described by comoving quantities while the derivatives are applied to the fixed-frame quantities. This is called mixed frame formalism (Mihalas & Mihalas 1984). I have generalized this mixed frame approach to general relativistic regime and applied to spherically symmetric (Park 1990, 1993) and axisymmetric accretion flows (Park 2006b). It has further been applied to arbitrary flows in the Schwarzschild spacetime (Park 2006a) and in the Kerr spacetime (Takahashi 2007, 2008). This mixed-frame, frequency-integrated radiation moment formalism can be easily applied to any coordinate system because it is based on covariant tensor equations, as has been emphasized in Park (1993, 2006a). In this review, I will explain how this formalism is built and show how to apply it to specific coordinate systems. The limitations and caveats of this formalism will also be noted. Throughout this paper, the velocity of light c is chosen to be unity.

2. TENSOR CONSERVATION EQUATIONS

The energy-momentum tensor of matter that is approximated by an ideal gas is

$$T^{\alpha\beta} \equiv \omega_g U^\alpha U^\beta + P_g g^{\alpha\beta}, \tag{1}$$

where U^α is the four-velocity of the gas and $\omega_g \equiv \varepsilon_g + P_g$ the gas enthalpy per unit proper volume which is the sum of the gas energy density ε_g and the gas pressure P_g. The enthalpy of the gas is a function of the gas temperature and density whose exact dependence on the temperature T in transrelativistic regime is rather complicated (Service 1986).

A similar tensor for radiation, called the radiation stress tensor, can be constructed,

$$R^{\alpha\beta} = \iint I(\mathbf{n}, \nu) n^\alpha n^\beta d\nu d\Omega, \tag{2}$$

where $n^\alpha \equiv p^\alpha / h\nu$ with p^α is the four-momentum of photons and $I(x^\alpha; \mathbf{n}, \nu)$ the specific intensity of photons moving in direction $\mathbf{n}$ on the unit sphere of the projected tangent space with the frequency ν measured by the fiducial observer. The photon distribution function, which is the analogue of the velocity distribution function of the particles, is equal to $c^2 h^{-4} \nu^{-3} I_\nu$ and the Lorentz invariant, i.e., the invariant intensity $\nu^{-3} I_\nu$ remains the same regardless of the frame in which it is measured (Mihalas & Mihalas 1984; see also Chan 2011 for general construction of the radiation moment tensor from the photon distribution function).

In relativity, the mass density reflects the internal energy as well the rest mass-energy density. Since the internal energy can change by heating, cooling, compressing, and decompressing, the particle number density rather than the mass density is conserved. The conservation of particle number density is given by the continuity equation,

$$(nU^\alpha)_{;\alpha} = 0, \tag{3}$$

where n is the proper number density of the matter and U^α the four-velocity of the matter.

The conservation of energy and momentum of matter and radiation is expressed by the conservation of the energy-momentum tensor of matter and radiation combined:

$$\left(T^{\alpha\beta} + R^{\alpha\beta}\right)_{;\beta} = 0. \tag{4}$$

When radiation is in thermal equilibrium, it can be described simply by the temperature of the matter. In such a special case, no more equations for radiative transfer are required.

In general, however, radiation field is not in thermal equilibrium, i.e., blackbody, and the four-force density that describes the interaction between radiation and matter is defined (Mihalas & Mihalas 1984):

$$G^\alpha \equiv \int d\nu \int d\Omega [\chi I(\mathbf{n}, \nu) - \eta] n^\alpha. \tag{5}$$

The absorption and scattering of photons by matter particles are described by the opacity per unit proper length, χ, and the emission of photons by matter the emissivity per proper unit volume, η.

Eq. 4 can now be split into two separate conservation equations: one for the matter energy-momentum tensor

$$T^{\alpha\beta}_{;\beta} = G^\alpha, \tag{6}$$

and the other for the radiation stress-energy tensor,

$$R^{\alpha\beta}_{;\beta} = -G^\alpha. \tag{7}$$

We need to remember that a simple conservation law exists for the radiation stress tensor because all radiation quantities are frequency integrated quantities. We cannot define similar frequency-specific radiation stress-energy tensor because they will not be covariant. We obtain the covariance by sacrificing frequency information.

3. RADIATION MOMENTS AND TETRADS

In a flat spacetime, the time-time components of the radiation stress tensor R^{tt} are called the radiation energy density E, the time-space components R^{ti}, the radiation flux F^i, and the space-space components R^{ij}, the radiation pressure tensor P^{ij}, with $i, j = 1, 2, 3$. These three are the main radiation moments, i.e., moments in angle, that are widely used in astrophysical radiative transfer and radiation hydrodynamics. These definitions can be extended to any tetrad: for any fiducial tetrad, we can define

$$E = \iint I_\nu d\nu d\Omega,$$

$$F^i = \iint I_\nu n^{\hat{i}} d\nu d\Omega, \qquad (8)$$

$$P^{ij} = \iint I_\nu n^{\hat{i}} n^{\hat{j}} d\nu d\Omega,$$

where $n^{\hat{i}}$ is the spatial tetrad component of n^α and ν and Ω are the frequency and solid angle measured by the fiducial observer of the tetrad. In terms of these radiation moments, the tetrad components of the radiation stress tensor for a given fiducial tetrad are simply

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & F^1 & F^2 & F^3 \\ F^1 & P^{11} & P^{12} & P^{13} \\ F^2 & P^{21} & P^{22} & P^{23} \\ F^3 & P^{31} & P^{32} & P^{33} \end{pmatrix}. \qquad (9)$$

For a given spacetime, the two most useful tetrads are the fixed tetrad that is fixed with respect to the chosen coordinate system and the comoving frame that is comoving with the matter flow, and therefore at rest with respect to the matter flow. The derivatives of radiation quantities are simple to calculate in the fixed frame while the matter-radiation interaction is easily described in the comoving frame. The comoving tetrad is moving with respect to the fixed tetrad with a proper three-velocity

$$v^i = v_i = \frac{U^{\hat{i}}}{U^{\hat{t}}} = \frac{U_\alpha e_{\hat{i}}{}^\alpha}{-U_\alpha e_{\hat{t}}{}^\alpha}, \qquad (10)$$

where $U^{\hat{i}}$ and $U^{\hat{t}}$ are the spatial and temporal parts of the fixed tetrad components of the four-velocity of the matter flow, U_α, and $e_{\hat{\beta}}{}^\alpha$ is the coordinate components of the fixed tetrad base. From this, we define the energy parameter

$$y \equiv -U_t \qquad (11)$$

and the Lorentz factor between the fixed frame and the comoving frame

$$\gamma \equiv (1 - v^2)^{-1/2} \qquad (12)$$

where $v^2 = \mathbf{v} \cdot \mathbf{v} = \sum_{i=1}^{3} v_i v^i$.

We denote the bases of the fixed tetrad as $\partial/\partial x_{fx}^{\hat{\alpha}}$ and those of the comoving tetrad as $\partial/\partial x_{co}^{\hat{\alpha}}$. Since tetrads are locally inertial frames, they are related to each other by Lorentz transformation:

$$\frac{\partial}{\partial x_{co}^{\hat{\alpha}}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}} \frac{\partial}{\partial x_{fx}^{\hat{\beta}}}, \qquad (13)$$

where

$$\Lambda^{\hat{t}}{}_{\hat{t}} = \gamma, \quad \Lambda^{\hat{i}}{}_{\hat{t}} = \gamma v^i, \quad \Lambda^{\hat{t}}{}_{\hat{j}} = \gamma v_j, \qquad (14)$$

$$\Lambda^{\hat{i}}{}_{\hat{j}} = \delta^i{}_j + v^i v_j \frac{\gamma - 1}{v^2}.$$

Since $R^{\hat{\alpha}\hat{\beta}}$ is Lorentz tensor, the fixed-tetrad components $R_{fx}^{\hat{\alpha}\hat{\beta}}$ and the comoving-tetrad components are related by the Lorentz transformation:

$$R_{co}^{\hat{\alpha}\hat{\beta}} = \frac{\partial x_{co}^{\hat{\alpha}}}{\partial x_{fx}^{\hat{\mu}}} \frac{\partial x_{co}^{\hat{\beta}}}{\partial x_{fx}^{\hat{\nu}}} R_{fx}^{\hat{\mu}\hat{\nu}} = \Lambda^{\hat{\alpha}}{}_{\hat{\mu}}(-\mathbf{v}) \Lambda^{\hat{\beta}}{}_{\hat{\nu}}(-\mathbf{v}) R_{fx}^{\hat{\mu}\hat{\nu}}, \qquad (15)$$

where $\Lambda(-\mathbf{v})$ is the inverse of the Lorentz transformation (14). This leads to the transformation law between radiation moments in comoving and fixed frames (Mihalas & Mihalas 1984; Munier & Weaver 1986; Park 1993):

$$E_{co} = \gamma^2 \left[E_{fx} - 2v_i F_{fx}^i + v_i v_j P_{fx}^{ij} \right]$$

$$F_{co}^i = \left[\delta^i_j + (\frac{\gamma - 1}{v^2} + \gamma^2) v^i v_j \right] F_{fx}^i - \gamma^2 v^i E_{fx}$$

$$-\gamma v_j \left[\delta^i_k + \frac{\gamma - 1}{v^2} v^i v_k \right] P_{fx}^{ij} \qquad (16)$$

$$P_{co}^{ij} = \gamma^2 v^i v^j E_{fx}$$

$$-\gamma \left[v^i \delta^j_k + v^j \delta^i_k + 2\frac{\gamma - 1}{v^2} v^i v^j v_k \right] F_{fx}^i$$

$$+(\delta^i_k + \frac{\gamma - 1}{v^2} v^i v_k)(\delta^j_k + \frac{\gamma - 1}{v^2} v^j v_l) P_{fx}^{ij}.$$

Since the contravariant components of the radiation stress tensor are related to any tetrad components by

$$R^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^{\hat{\mu}}} \frac{\partial x^\beta}{\partial x^{\hat{\nu}}} R^{\hat{\mu}\hat{\nu}}, \qquad (17)$$

and the fixed tetrad components to the comoving components by Eq. 15, the contravariant components of the radiation stress tensor $R^{\alpha\beta}$ can be written in terms of the fixed-frame radiation moments, E_{fx}, F_{fx}^i, P_{fx}^{ij}, or the comoving-frame radiation moments, E_{co}, F_{co}^i, P_{co}^{ij}, as needed.

4. INTERACTION BETWEEN MATTER AND RADIATION

The tetrad components of the radiation four-force density for the comoving frame are

$$G_{co}^{\hat{\alpha}} = \int d\nu_{co} \int d\Omega_{co} [\chi_{co} I_{\nu_{co}} - \eta_{co}] n_{co}^{\hat{\alpha}}. \qquad (18)$$

In most astrophysical situations, absorption, emission, and scattering are isotropic or effectively isotropic in the comoving frame, and the energy exchange rate $G_{co}^{\hat{t}}$ is simply the heating function Γ_{co} minus the cooling function Λ_{co}, both per unit proper volume:

$$G_{co}^{\hat{t}} = \Gamma_{co} - \Lambda_{co} \qquad (19)$$

where

$$\Gamma_{co} \equiv \int d\nu_{co} \int d\Omega_{co} \kappa_{co} I_{\nu_{co}}, \qquad (20)$$

$$\Lambda_{co} \equiv \int d\nu_{co} \int d\Omega_{co} \eta_{co}, \qquad (21)$$

and κ_{co} is the true absorption coefficient.

The momentum exchange rate per unit proper volume is

$$G_{co}^{\hat{i}} = \bar{\chi}_{co} F_{co}^i, \qquad (22)$$

where

$$\bar{\chi}_{co} F_{co}^i \equiv \int d\nu_{co} \int d\Omega_{co} \chi_{co} I_{\nu_{co}} n_{co}^i. \qquad (23)$$

Using the transformation between two tetrads (Eq. 13), the fixed-frame components $G_{fx}^{\hat{\alpha}}$ can be calculated, and the coordinate transformation between the tetrad $\partial/x_{fx}^{\hat{\alpha}}$ and the coordinate ∂/x_{fx}^{α} enables to express the covariant vector G^α in terms of $G_{co}^{\hat{\alpha}}$,

$$G^\alpha = \frac{\partial x^\alpha}{\partial x_{co}^{\hat{\mu}}} G_{co}^{\hat{\mu}\hat{\nu}}. \qquad (24)$$

Therefore, if we can specify the heating and cooling functions, i.e., Γ_{co} and Λ_{co}, and the mean opacity, $\bar{\chi}_{co}$, as functions of density and temperature of matter, the interaction between matter and radiation can be determined. In reality however, the opacity χ_{co} is often a function of photon frequency and the mean opacity and heating function can be accurately specified only when the specific radiation energy density, $E_{\nu,co}$, and flux, $F_{\nu,co}$, are known. This is a fundamental difficulty of frequency-integrated radiation moment formalism. To overcome this difficulty, one may make an educated guess for the radiation spectrum or perform iterative frequency-specific radiative transfer calculations in-between the radiation hydrodynamic calculations.

5. HYDRODYNAMIC EQUATIONS

If we define a projection tensor

$$P_\alpha{}^\beta = g_\alpha{}^\beta + U_\alpha U^\beta = \delta_\alpha{}^\beta + U_\alpha U^\beta, \qquad (25)$$

relativistic Euler equation can be obtained by projecting Eq. 6 with $P_\alpha{}^\beta$ to get $P_\beta{}^\alpha T^{\beta\lambda}{}_{;\lambda} = P_\beta{}^\alpha G^\beta$, which becomes the covariant relativistic Euler equation:

$$\omega_g U^\alpha{}_{;\beta} U^\beta + g^{\alpha\beta} P_{g,\beta} + U^\alpha U^\beta P_{g,\beta} = G^\alpha + U^\alpha U_\beta G^\beta. \qquad (26)$$

Since the interaction between radiation and matter becomes simpler, i.e., more symmetric, in the comoving frame, it is more convenient to express G^α in terms of comoving tetrad components $G_{co}^{\hat{t}}$ (Eqs. 19 and 22).

The energy equation for the matter is obtained by projecting equation (6) along the four-velocity, i.e., $U_\alpha T^{\alpha\beta}{}_{;\beta} = U_\alpha G^\alpha$:

$$-nU^t \frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i \frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right)$$
$$+U^t \frac{\partial P_g}{\partial t} + U^i \frac{\partial P_g}{\partial x^i} \qquad (27)$$
$$= U_t G^t + U_i G^i = -G_{co}^{\hat{t}} = \Gamma_{co} - \Lambda_{co}.$$

6. RADIATION MOMENT EQUATIONS

The other half of the radiation hydrodynamic equations are the radiation moment equations (Eq. 7). Choosing $\alpha = t$ gives the radiation energy equation and the rest, $\alpha = 1, 2, 3$, the radiation momentum equation in each spatial direction. The left-hand side consists of the radiation moments in the fixed-frame and their coordinate derivatives while the right-hand side may be expressed with the comoving radiation moments, which simplifies the description of the matter-radiation interaction. Unlike the hydrodynamic equations, radiation moments equations become simpler in terms of the fixed-frame radiation moments because the derivatives appear simpler in the fixed frame.

To apply these hydrodynamic and radiation moment equations to a specific spacetime, all we need is the spacetime metric and the transformation rules among the coordinate frame and the two tetrad frames.

7. CYLINDRICALLY SYMMETRIC FLOW IN A FLAT SPACETIME

Many astrophysical systems are believed to have cylindrical symmetry. Rotating accretion flow, i.e., accretion disc, is one important example. For the cylindrical flat spacetime metric

$$\begin{aligned} d\tau^2 &= -g_{\alpha\beta}dx^\alpha dx^\beta \\ &= dt^2 - dR^2 - R^2 d\theta^2 - dz^2, \end{aligned} \qquad (28)$$

the fixed tetrad bases are simply

$$\frac{\partial}{\partial\hat{t}} = \frac{\partial}{\partial t}, \ \frac{\partial}{\partial\hat{R}} = \frac{\partial}{\partial R}, \ \frac{\partial}{\partial\hat{\theta}} = \frac{1}{R}\frac{\partial}{\partial\theta}, \ \frac{\partial}{\partial\hat{z}} = \frac{\partial}{\partial z}. \qquad (29)$$

The comoving tetrad, $\partial/\partial x_{co}^{\hat{\alpha}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}}(\mathbf{v})\partial/\partial x^{\hat{\beta}}$, is then given as (Park 2006b)

$$\begin{aligned} \frac{\partial}{\partial\hat{t}_{co}} &= \gamma\frac{\partial}{\partial t} + \gamma v^R \frac{\partial}{\partial R} + \gamma v^\theta \frac{1}{R}\frac{\partial}{\partial\theta} + \gamma v^z \frac{\partial}{\partial z} \\ \frac{\partial}{\partial\hat{R}_{co}} &= \gamma v^R \frac{\partial}{\partial t} + \left[1 + (\gamma-1)\frac{v^R v_R}{v^2}\right]\frac{\partial}{\partial R} \\ &+ (\gamma-1)\frac{v^R v_\theta}{v^2}\frac{1}{R}\frac{\partial}{\partial\theta} + (\gamma-1)\frac{v^R v_z}{v^2}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial\hat{\theta}_{co}} &= \gamma v^\theta \frac{\partial}{\partial t} + (\gamma-1)\frac{v^\theta v_R}{v^2}\frac{\partial}{\partial R} \qquad (30) \end{aligned}$$

$$+ \quad \left[1 + (\gamma - 1)\frac{v^\theta v_\theta}{v^2}\right]\frac{1}{R}\frac{\partial}{\partial \theta} + (\gamma - 1)\frac{v^\theta v_z}{v^2}\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \hat{z}_{co}} = \gamma v^z \frac{\partial}{\partial t} + (\gamma - 1)\frac{v^z v_R}{v^2}\frac{\partial}{\partial R}$$

$$+ \quad (\gamma - 1)\frac{v^z v_\theta}{v^2}\frac{1}{R}\frac{\partial}{\partial \theta} + \left[1 + (\gamma - 1)\frac{v^z v_z}{v^2}\right]\frac{\partial}{\partial z}.$$

From these, we can write the contravariant radiation stress tensor $R^{\alpha\beta}$ in terms of the fixed- and comoving-frame radiation moments,

$$R^{\alpha\beta} = \begin{pmatrix} E_{fx} & F_{fx}^R & R^{-1}F_{fx}^\theta & F_{fx}^z \\ F_{fx}^R & P_{fx}^{RR} & R^{-1}P_{fx}^{R\theta} & P_{fx}^{Rz} \\ R^{-1}F_{fx}^\theta & R^{-1}P_{fx}^{R\theta} & R^{-2}P_{fx}^{\theta\theta} & R^{-1}P_{fx}^{\theta z} \\ F_{fx}^z & P_{fx}^{Rz} & R^{-1}P_{fx}^{Rz} & P_{fx}^{zz} \end{pmatrix}. \quad (31)$$

The contravariant components of the radiation four-force density in terms of tetrad components (Eqs. 19 and 22) are

$$G^t = \gamma G_{co}^{\hat{t}} + \gamma v^i G_{co}^{\hat{i}},$$

$$G^R = G_{co}^{\hat{R}} + \gamma v^R G_{co}^{\hat{t}} + \frac{\gamma - 1}{v^2}v^R v_i G_{co}^{\hat{i}}, \quad (32)$$

$$RG^\theta = G_{co}^{\hat{\theta}} + \gamma v^\theta G_{co}^{\hat{t}} + \frac{\gamma - 1}{v^2}v^\theta v_i G_{co}^{\hat{i}},$$

$$G^z = G_{co}^{\hat{z}} + \gamma v^z G_{co}^{\hat{t}} + \frac{\gamma - 1}{v^2}v^z v_i G_{co}^{\hat{i}}.$$

The explicit equations for cylindrically symmetric flow in a flat spacetime are given in Park (2006b). The continuity equation from Eq. 3 is

$$\frac{\partial}{\partial t}(\gamma n) + \frac{1}{R}\frac{\partial}{\partial R}(RnU^R) + \frac{\partial}{\partial \theta}(nU^\theta) + \frac{\partial}{\partial z}(nU^z) = 0. \quad (33)$$

The relativistic Euler equations are for R-component,

$$\gamma \omega_g \frac{\partial U^R}{\partial t} + \omega_g U^i \frac{\partial U^R}{\partial x^i} - \omega_g R(U^\theta)^2$$

$$+ \quad \frac{\partial P_g}{\partial R} + \gamma U^R \frac{\partial P_g}{\partial t} + U^R U^i \frac{\partial P_g}{\partial x^i} \quad (34)$$

$$= \quad -\gamma U^R G^t + [1 + (U^R)^2]G^R + R^2 U^R U^\theta G^\theta$$

$$+ \quad U^R U^z G^z,$$

for θ-component,

$$\gamma \omega_g \frac{\partial U^\theta}{\partial t} + \omega_g U^i \frac{\partial U^\theta}{\partial x^i} + 2\omega_g \frac{U^R U^\theta}{R}$$

$$+ \quad \frac{1}{R^2}\frac{\partial P_g}{\partial \theta} + \gamma U^\theta \frac{\partial P_g}{\partial t} + U^\theta U^i \frac{\partial P_g}{\partial x^i} \quad (35)$$

$$= \quad -\gamma U^\theta G^t + U^\theta U^R G^R + [1 + R^2(U^\theta)^2]G^\theta$$

$$+ \quad U^\theta U^z G^z,$$

and for z-component

$$\gamma \omega_g \frac{\partial U^z}{\partial t} + \omega_g U^i \frac{\partial U^z}{\partial x^i}$$

$$+ \quad \frac{\partial P_g}{\partial z} + \gamma U^z \frac{\partial P_g}{\partial t} + U^z U^i \frac{\partial P_g}{\partial x^i} \quad (36)$$

$$= \quad -\gamma U^z G^t + U^z U^R G^R + R^2 U^z U^\theta G^\theta$$

$$+ \quad [1 + (U^z)^2]G^z.$$

The gas energy Eq. 27 in this cylindrical coordinates is

$$- \quad nU^t \frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i \frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right)$$

$$+ \quad U^t \frac{\partial P_g}{\partial t} + U^i \frac{\partial P_g}{\partial x^i} \quad (37)$$

$$= \quad -G_{co}^{\hat{t}} = \Lambda_{co} - \Gamma_{co}.$$

The time component of Eq. 7 is the radiation energy equation,

$$\frac{\partial E}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(RF^R) + \frac{1}{R}\frac{\partial F^\theta}{\partial \theta} + \frac{\partial F^z}{\partial z} \quad (38)$$

$$= -G^t = \gamma(\Lambda_{co} - \Gamma_{co}) - \gamma\bar{\chi}_{co}v_i.$$

The equations for the radiative flux are

$$\frac{\partial F^R}{\partial t} + \frac{\partial P^{RR}}{\partial R} + \frac{1}{R}\frac{\partial P^{R\theta}}{\partial \theta}$$

$$+ \frac{\partial P^{Rz}}{\partial z} + \frac{P^{RR} - P^{\theta\theta}}{R} \quad (39)$$

$$= \quad -\bar{\chi}_{co}F_{co}^R - \gamma v^R(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma - 1}{v^2}v^R v_i \bar{\chi}_{co}F_{co}^i$$

for R-direction,

$$\frac{\partial F^\theta}{\partial t} + \frac{\partial P^{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial P^{\theta\theta}}{\partial \theta}$$

$$+ \frac{\partial P^{\theta z}}{\partial z} + \frac{2P^{R\theta}}{R} \quad (40)$$

$$= \quad -\bar{\chi}_{co}F_{co}^\theta - \gamma v^\theta(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma - 1}{v^2}v^\theta v_i \bar{\chi}_{co}F_{co}^i$$

for θ-direction, and

$$\frac{\partial F^z}{\partial t} + \frac{\partial P^{Rz}}{\partial R} + \frac{1}{R}\frac{\partial P^{\theta z}}{\partial \theta}$$

$$+ \frac{\partial P^{zz}}{\partial z} + \frac{P^{Rz}}{R} \quad (41)$$

$$= \quad -\bar{\chi}_{co}F_{co}^z - \gamma v^z(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma - 1}{v^2}v^z v_i \bar{\chi}_{co}F_{co}^i$$

for z-direction.

8. FLOW IN SCHWARZSCHILD SPACETIME

Spacetime around non-rotating black holes is described by the Schwarzschild coordinate,

$$d\tau^2 = -g_{\alpha\beta}dx^\alpha dx^\beta$$

$$= \Gamma^2 dt^2 - \frac{dr^2}{\Gamma^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (42)$$

where M is the mass of the central object, $m \equiv GM/c^2$, and $\Gamma \equiv (1 - 2m/r)^{1/2}$. The fixed tetrad in this metric is simply

$$\frac{\partial}{\partial \hat{t}} = \frac{1}{\Gamma}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \hat{r}} = \Gamma\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{43}$$

The comoving tetrad, moving with four velocity U^α with respect to the coordinate, is

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}_{co}} &= \frac{\gamma}{\Gamma}\frac{\partial}{\partial t} + \gamma\Gamma v_r\frac{\partial}{\partial r} + \gamma v_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \gamma v_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial \hat{r}_{co}} &= \frac{\gamma}{\Gamma}v_r\frac{\partial}{\partial t} + \Gamma\left[1 + (\gamma-1)\frac{v_r^2}{v^2}\right]\frac{\partial}{\partial r} \\
&\quad + (\gamma-1)\frac{v_r v_\theta}{v^2}\frac{1}{r}\frac{\partial}{\partial \theta} + (\gamma-1)\frac{v_r v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial \hat{\theta}_{co}} &= \frac{\gamma}{\Gamma}v_\theta\frac{\partial}{\partial t} + \Gamma(\gamma-1)\frac{v_r v_\theta}{v^2}\frac{\partial}{\partial r} \\
&\quad + \left[1 + (\gamma-1)\frac{v_\theta^2}{v^2}\right]\frac{1}{r}\frac{\partial}{\partial \theta} + (\gamma-1)\frac{v_\theta v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial \hat{\phi}_{co}} &= \frac{\gamma}{\Gamma}v_\phi\frac{\partial}{\partial t} + \Gamma(\gamma-1)\frac{v_r v_\phi}{v^2}\frac{\partial}{\partial r} \\
&\quad + (\gamma-1)\frac{v_\theta v_\phi}{v^2}\frac{1}{r}\frac{\partial}{\partial \theta} + \left[1 + (\gamma-1)\frac{v_\phi^2}{v^2}\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.
\end{aligned} \tag{44}$$

The contravariant components of the radiation stress tensor expressed in the fixed-frame radiation moments,

$$R^{\alpha\beta} = \begin{pmatrix} \frac{E_{fx}}{\Gamma^2} & F_{fx}^r & \frac{F_{fx}^\theta}{\Gamma r} & \frac{F_{fx}^\phi}{\Gamma r\sin\theta} \\ F_{fx}^r & \Gamma^2 P_{fx}^{rr} & \frac{\Gamma P_{fx}^{r\theta}}{r} & \frac{\Gamma P_{fx}^{r\phi}}{r\sin\theta} \\ \frac{F_{fx}^\theta}{\Gamma r} & \frac{\Gamma P_{fx}^{r\theta}}{r} & \frac{P_{fx}^{\theta\theta}}{r^2} & \frac{P_{fx}^{\theta\phi}}{r^2\sin\theta} \\ \frac{F_{fx}^\phi}{\Gamma r\sin\theta} & \frac{\Gamma P_{fx}^{r\phi}}{r\sin\theta} & \frac{P_{fx}^{\theta\phi}}{r^2\sin\theta} & \frac{P_{fx}^{\phi\phi}}{r^2\sin^2\theta} \end{pmatrix} \tag{45}$$

show all the curvature and coordinate specifics (Park 2006a). The contravariant components of the radiation four-force density are

$$\begin{aligned}
G^t &= \frac{\gamma}{\Gamma}\left[G_{co}^{\hat{t}} + v_i G_{co}^{\hat{i}}\right] \\
G^r &= \Gamma\left[G_{co}^{\hat{r}} + \gamma v_r G_{co}^{\hat{t}} + \frac{\gamma-1}{v^2}v_r v_i G_{co}^{\hat{i}}\right] \\
G^\theta &= \frac{1}{r}\left[G_{co}^{\hat{\theta}} + \gamma v_\theta G_{co}^{\hat{t}} + \frac{\gamma-1}{v^2}v_\theta v_i G_{co}^{\hat{i}}\right] \\
G^\phi &= \frac{1}{r\sin\theta}\left[G_{co}^{\hat{\phi}} + \gamma v_\phi G_{co}^{\hat{t}} + \frac{\gamma-1}{v^2}v_\phi v_i G_{co}^{\hat{i}}\right].
\end{aligned} \tag{46}$$

Continuity equation (3) in Schwarzschild metric is

$$\begin{aligned}
&\frac{1}{\Gamma^2}\frac{\partial}{\partial t}(yn) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n U^r) \\
&+ \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta n U^\theta) + \frac{\partial}{\partial \phi}(n U^\phi) = 0,
\end{aligned} \tag{47}$$

where $y = -U_t$. The r-component of Euler equation (26) is

$$\begin{aligned}
&\omega_g U^t\frac{\partial U^r}{\partial t} + \omega_g U^i\frac{\partial U^r}{\partial x^i} \\
&+ \omega_g\frac{m}{r^2}\left[\Gamma^2(U^t)^2 - \Gamma^{-2}(U^r)^2\right] \\
&- \omega_g\frac{\Gamma^2}{r}\left[(rU^\theta)^2 + (r\sin\theta U^\phi)^2\right] \\
&+ U^r U^t\frac{\partial P_g}{\partial t} + \Gamma^2\frac{\partial P_g}{\partial r} + U^r U^i\frac{\partial P_g}{\partial x^i} \\
&= -yU^r G^t + [1 + \Gamma^{-2}(U^r)^2]G^r + r^2 U^r U^\theta G^\theta \\
&\quad + r^2\sin^2\theta U^r U^\phi G^\phi,
\end{aligned} \tag{48}$$

the θ-component

$$\begin{aligned}
&\omega_g U^t\frac{\partial U^\theta}{\partial t} + \omega_g U^i\frac{\partial U^\theta}{\partial x^i} \\
&+ 2\omega_g\frac{1}{r}U^r U^\theta - \omega_g\sin\theta\cos\theta(U^\phi)^2 \\
&+ U^\theta U^t\frac{\partial P_g}{\partial t} + \frac{1}{r^2}\frac{\partial P_g}{\partial \theta} + U^\theta U^i\frac{\partial P_g}{\partial x^i} \\
&= -yU^\theta G^t + \Gamma^{-2}U^r U^\theta G^r + [1 + r^2(U^\theta)^2]G^\theta \\
&\quad + r^2\sin^2\theta U^\theta U^\phi G^\phi,
\end{aligned} \tag{49}$$

and the ϕ-part

$$\begin{aligned}
&\omega_g U^t\frac{\partial U^\phi}{\partial t} + \omega_g U^i\frac{\partial U^\phi}{\partial x^i} \\
&+ 2\omega_g\frac{1}{r}U^r U^\phi + 2\omega_g\cot\theta U^\theta U^\phi \\
&+ U^\phi U^t\frac{\partial P_g}{\partial t} + \frac{1}{r^2\sin^2\theta}\frac{\partial P_g}{\partial \phi} + U^\phi U^i\frac{\partial P_g}{\partial x^i} \\
&= -yU^\phi G^t + \Gamma^{-2}U^r U^\phi G^r + r^2 U^\theta U^\phi G^\theta \\
&\quad + [1 + r^2\sin^2\theta(U^\phi)^2]G^\phi.
\end{aligned} \tag{50}$$

The gas energy equation is

$$\begin{aligned}
&-nU^t\frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i\frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right) + U^t\frac{\partial P_g}{\partial t} + U^i\frac{\partial P_g}{\partial x^i} \\
&= -G_{co}^{\hat{t}} = \Lambda_{co} - \Gamma_{co}.
\end{aligned} \tag{51}$$

The equation for the radiation energy is the time component of the conservation equation for the radiation stress tensor (Eq. 7),

$$\begin{aligned}
&\frac{1}{\Gamma^2}\frac{\partial E}{\partial t} + \frac{1}{\Gamma^2 r^2}\frac{\partial}{\partial r}(r^2\Gamma^2 F^r) \\
&+ \frac{1}{\Gamma r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F^\theta) + \frac{1}{\Gamma r\sin\theta}\frac{\partial}{\partial \phi}(F^\phi) \\
&= -G^t = -\frac{y}{\Gamma^2}\left(\Gamma_{co} - \Lambda_{co} + \bar{\chi}_{co} v_i F_{co}^i\right).
\end{aligned} \tag{52}$$

The equations for the radiative momentum are the spatial components of the conservation equation for the

radiation stress tensor (Eq. 7),

$$\frac{\partial F^r}{\partial t} + \Gamma^2 \frac{\partial P^{rr}}{\partial r}$$
$$+ \frac{\Gamma}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P^{r\theta}\right) + \frac{\Gamma}{r\sin\theta}\frac{\partial P^{r\phi}}{\partial\phi}$$
$$+ \quad \frac{m}{r^2}\left(E + P^{rr}\right) + \frac{\Gamma^2}{r}\left(2P^{rr} - P^{\theta\theta} - P^{\phi\phi}\right) \qquad (53)$$
$$= \quad -G^r$$
$$= \quad -\Gamma\bar\chi_{co}F^r_{co} - \Gamma\gamma v_r(\Gamma_{co} - \Lambda_{co}) - \Gamma\frac{\gamma-1}{v^2}v_r v_i \bar\chi_{co}F^i_{co},$$

$$\frac{1}{\Gamma}\frac{\partial F^\theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r P^{r\theta}\right)$$
$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P^{\theta\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial P^{\theta\phi}}{\partial\phi}$$
$$+ \frac{2\Gamma}{r}P^{r\theta} - \frac{1}{r\tan\theta}P^{\phi\phi} \qquad (54)$$
$$= \quad -rG^\theta$$
$$= \quad -\bar\chi_{co}F^\theta_{co} - \gamma v_\theta(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma-1}{v^2}v_\theta v_i \bar\chi_{co}F^i_{co},$$

and

$$\frac{1}{\Gamma}\frac{\partial F^\phi}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r P^{r\phi}\right)$$
$$+ \frac{1}{r}\frac{\partial P^{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial P^{\phi\phi}}{\partial\phi}$$
$$+ \quad \frac{2\Gamma}{r}P^{r\phi} + \frac{2}{r\tan\theta}P^{\theta\phi} \qquad (55)$$
$$= \quad -r\sin\theta G^\phi$$
$$= \quad -\bar\chi_{co}F^\phi_{co} - \gamma v_\phi(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma-1}{v^2}v_\phi v_i \bar\chi_{co}F^i_{co}.$$

In the case when the radiation field is spherically symmetric and time-independent with no interaction with matter, the Eq. 52 reduces to

$$4\pi r^2 \Gamma^2 F^r = L_\infty, \qquad (56)$$

where the constant L_∞ is the luminosity measured by an observer at infinity. However, a static observer at a given r finds that the luminosity he or she measures is a function of radius with

$$L_r \equiv 4\pi r^2 F^r = \frac{L_\infty}{1 - 2m/r}, \qquad (57)$$

which shows the gravitational redshift effect. Hence, in the presence of gravitational field, one always has to be careful about the definition of luminosity. For example, the usual Eddington luminosity at which radiative force balances the gravitational pull from the central mass becomes a function of radius and matter velocity (Park 1992).

9. FLOW IN KERR SPACETIME

This covariant radiation moment formalism can be straightforwardly extended to more complex spacetime. If we describe the Kerr spacetime with Boyer-Lindquist coordinate

$$
\begin{aligned}
d\tau^2 &= -g_{\alpha\beta}dx^\alpha dx^\beta \qquad (58)\\
&= \alpha^2 dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),
\end{aligned}
$$

where $i, j = r, \theta, \phi$, the lapse function $\alpha = (\Sigma\Delta/A)^{1/2}$, the shift vector β^i with a non-zero component $\beta^\phi = -\omega$, the spatial matrix γ_{ij} with non-zero components of $\gamma_{rr} = \Sigma/\Delta$, $\gamma_{\theta\theta} = \Sigma$, $\gamma_{\phi\phi} = A\sin^2\theta/\Sigma$, $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2mr + a^2$, and $A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$, the tetrad for the locally non-rotating reference frame, i.e., the fixed frame, is (Bardeen et al. 1972)

$$
\begin{aligned}
\frac{\partial}{\partial \hat t} &= \frac{1}{\alpha}\left(\frac{\partial}{\partial t} - \beta^\phi\frac{\partial}{\partial\phi}\right),\\
\frac{\partial}{\partial \hat r} &= \frac{1}{\gamma_{rr}^{1/2}}\frac{\partial}{\partial r}, \qquad (59)\\
\frac{\partial}{\partial \hat\theta} &= \frac{1}{\gamma_{\theta\theta}^{1/2}}\frac{\partial}{\partial\theta},\\
\frac{\partial}{\partial \hat\phi} &= \frac{1}{\gamma_{\phi\phi}^{1/2}}\frac{\partial}{\partial\phi}.
\end{aligned}
$$

An observer in the fixed frame sees the matter moving with the proper three-velocity (Takahashi 2007)

$$
\begin{aligned}
v^r &= v_r = \frac{\gamma_{rr}^{1/2}}{\alpha}\frac{U^r}{U^t},\\
v^\theta &= v_\theta = \frac{\gamma_{\theta\theta}^{1/2}}{\alpha}\frac{U^\theta}{U^t}, \qquad (60)\\
v^\phi &= v_\phi = \frac{\gamma_{\phi\phi}^{1/2}}{\alpha}\left(\frac{U^\phi}{U^t} + \beta^\phi\right),
\end{aligned}
$$

the Lorentz factor of which is $\gamma \equiv (1 - v^2)^{-1/2} = \alpha U^t$. Now, the comoving tetrad can be expressed in terms of the coordinate base (Takahashi 2007),

$$
\begin{aligned}
\frac{\partial}{\partial \hat t_{co}} &= \frac{\gamma}{\alpha}\frac{\partial}{\partial t} + \frac{\gamma v_r}{\gamma_{rr}^{1/2}}\frac{\partial}{\partial r}\\
&\quad + \frac{\gamma v_\theta}{\gamma_{\theta\theta}^{1/2}}\frac{\partial}{\partial\theta}\\
&\quad + \gamma\left(\frac{v_\phi}{\gamma_{\phi\phi}^{1/2}} - \frac{\beta^\phi}{\alpha}\right)\frac{\partial}{\partial\phi},\\
\frac{\partial}{\partial \hat r_{co}} &= \frac{\gamma v_r}{\alpha}\frac{\partial}{\partial t} + \frac{1}{\gamma_{rr}^{1/2}}\left(1 + v_r^2\frac{\gamma^2}{\gamma+1}\right)\frac{\partial}{\partial r}\\
&\quad + \frac{v_r v_\theta}{\gamma_{\theta\theta}^{1/2}}\frac{\gamma^2}{\gamma+1}\frac{\partial}{\partial\theta}
\end{aligned}
$$

$$+ \left(\frac{v_r v_\phi}{\gamma_{\phi\phi}^{1/2}} \frac{\gamma^2}{\gamma+1} - \frac{\beta^\phi}{\alpha} \gamma v_r \right) \frac{\partial}{\partial\phi}, \qquad (61)$$

$$\frac{\partial}{\partial\hat{\theta}_{co}} = \frac{\gamma v_\theta}{\alpha} \frac{\partial}{\partial t} + \frac{v_r v_\theta}{\gamma_{rr}^{1/2}} \frac{\gamma^2}{\gamma+1} \frac{\partial}{\partial r}$$
$$+ \frac{1}{\gamma_{\theta\theta}^{1/2}} \left(1 + v_\theta^2 \frac{\gamma^2}{\gamma+1} \right) \frac{\partial}{\partial\theta}$$
$$+ \left(\frac{v_\theta v_\phi}{\gamma_{\phi\phi}^{1/2}} \frac{\gamma^2}{\gamma+1} - \frac{\beta^\phi}{\alpha} \gamma v_\theta \right) \frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial\hat{\phi}_{co}} = \frac{\gamma v_\phi}{\alpha} \frac{\partial}{\partial t} + \frac{v_r v_\phi}{\gamma_{rr}^{1/2}} \frac{\gamma^2}{\gamma+1} \frac{\partial}{\partial r}$$
$$+ \frac{v_\theta v_\phi}{\gamma_{\theta\theta}^{1/2}} \frac{\gamma^2}{\gamma+1} \frac{\partial}{\partial\theta}$$
$$+ \left[\frac{1}{\gamma_{\theta\theta}^{1/2}} \left(1 + v_\phi^2 \frac{\gamma^2}{\gamma+1} \right) - \frac{\beta^\phi}{\alpha} \gamma v_\phi \right] \frac{\partial}{\partial\phi}.$$

The coordinate base in terms of comoving tetrad is also given in Takahashi (2007). This allows the contravariant components of radiation stress tensor $R^{\alpha\beta}$ to be expressed with the fixed-frame radiation moments (Takahashi 2007),

$$R^{tt} = \frac{E_{fx}}{\alpha^2}$$

$$R^{tr} = R^{rt} = \frac{F_{fx}^r}{\alpha\gamma_{rr}^{1/2}}$$

$$R^{t\theta} = R^{\theta t} = \frac{F_{fx}^\theta}{\alpha\gamma_{\theta\theta}^{1/2}}$$

$$R^{t\phi} = R^{\phi t} = \frac{1}{\alpha} \left(\frac{F_{fx}^\phi}{\gamma_{\phi\phi}^{1/2}} - \beta^\phi \frac{E_{fx}}{\alpha} \right)$$

$$R^{rr} = \frac{P_{fx}^{rr}}{\gamma_{rr}} \qquad (62)$$

$$R^{r\theta} = R^{\theta r} = \frac{P_{fx}^{r\theta}}{\gamma_{rr}^{1/2}\gamma_{\theta\theta}^{1/2}}$$

$$R^{r\phi} = R^{\phi r} = \frac{1}{\gamma_{rr}^{1/2}} \left(\frac{P_{fx}^{r\phi}}{\gamma_{\phi\phi}^{1/2}} - \beta^\phi \frac{F_{fx}^r}{\alpha} \right)$$

$$RR^{\theta\theta} = \frac{P_{fx}^{\theta\theta}}{\gamma_{\theta\theta}}$$

$$R^{\theta\phi} = R^{\phi\theta} = \frac{1}{\gamma_{\theta\theta}^{1/2}} \left(\frac{P_{fx}^{\theta\phi}}{\gamma_{\phi\phi}^{1/2}} - \beta^\phi \frac{F_{fx}^\theta}{\alpha} \right)$$

$$R^{\phi\phi} = \frac{P_{fx}^{\phi\phi}}{\gamma_{\phi\phi}} - \frac{2\beta^\phi}{\alpha\gamma_{\phi\phi}^{1/2}} F_{fx}^\phi + \left(\frac{\beta^\phi}{\alpha} \right)^2 E_{fx}.$$

The contravariant components of the radiation force are

$$G^t = \frac{\gamma}{\alpha} \left[G_{co}^{\hat{t}} + v_i G_{co}^{\hat{i}} \right]$$

$$G^r = \frac{1}{\gamma_{rr}^{1/2}} \left[G_{co}^{\hat{r}} + \gamma v_r G_{co}^{\hat{t}} + \frac{\gamma^2}{\gamma+1} v_r v_i G_{co}^{\hat{i}} \right] (63)$$

$$G^\theta = \frac{1}{\gamma_{\theta\theta}^{1/2}} \left[G_{co}^{\hat{\theta}} + \gamma v_\theta G_{co}^{\hat{t}} + \frac{\gamma^2}{\gamma+1} v_\theta v_i G_{co}^{\hat{i}} \right]$$

$$G^\phi = \frac{G_{co}^{\hat{\phi}}}{\gamma_{\phi\phi}^{1/2}} + \gamma \left[\frac{v_\phi}{\gamma_{\phi\phi}^{1/2}} - \frac{\beta^\phi}{\alpha} \right] G_{co}^{\hat{t}}$$
$$+ \gamma \left[\frac{\gamma v_\phi}{\gamma_{\phi\phi}^{1/2}(\gamma+1)} - \frac{\beta^\phi}{\alpha} \right] v_i G_{co}^{\hat{i}}.$$

The explicit forms of continuity equation can be obtained from Eq. 3, relativistic Euler equations from Eq. 26, and energy equation from Eq. 27 with radiation-matter interaction expressed as functions of comoving-frame radiation moments, as shown in (Takahashi 2007). Radiation energy and momemtum equations can be expressed with coordinate derivatives and fixed-frame radiation momemnts (Takahashi 2007). Similarly, same equations can also be derived for the Kerr-Schild coordinate, which does not have a coordinate singularity at the event horizon (Takahashi 2008).

10. CLOSURES AND CAVEATS

The number of unknown radiation moments in general is 10 (Eq. 9) while the number of radiation moment equations are 4 (Eq. 7). This is the general limitation of moment approach: equations at each order are not sufficient to determine all the moments of the same order. Astronomers routinely employ some kind of closure relations to close the equations, such as flux-limited diffusion, P_N approximation, the M_1 closure, and the variable Eddington factor (see Chan 2011, for review and references). One popular choice is the variable Eddington factor because it is easy to implement yet gives reasonable limit behaviour: it is defined, in a given frame, as $f_E \equiv P/E$ and has an asymptotic value $f_E = 1/3$ when the optical depth $\tau \gg 1$ and $f_E = 1$ when $\tau \ll 1$. The correct form of f_E can only be calculated by solving the full angle-dependent radiative transfer equation (see e.g., Auer & Mihalas 1970; Hummer & Rybicki 1971; Yin & Miller 1995) for a specific case. In a complex radiation hydrodynamic flow, this is impractical, and an educated guess of f_E as a function of the optical depth is sometimes tried (see, e.g., Tamazawa et al. 1975; Park 1990). In an arbitrary three-dimensional system, it is far less trivial to calculate the variable Eddington factor from the basic angle-dependent radiative transfer equations. One may apply some physical principles to derive the functional form. For example, Minerbo (1978) determined from statistical mechanics the maximum entropy Eddington

'tensor', $f_\nu^{ij} \equiv P_\nu^{ij}/E_\nu$, for a static flow in a flat space-time and provided the functional form of f_ν^{ij} in terms of E_ν and F_ν^i (Minerbo 1978), which may be generalized to frequency-integrated radiation moments in comoving tetrad frame (Park 2006c). Recently, Chan (2011) derived another physically motivated closures based on Grad's moment method, and provided a more sophisticated 14-field method that minimizes unphysical photon self-interaction.

It is important to remember that the current formalism with a simple closure such as variable Eddington factor may describe the real radiation field with reasonable accuracy, but can fail for certain conditions. Since any moment formalism terminates the higher order moments artificially, it can lead to pathological behaviours in special situations. For example, a very steep velocity gradient is known to produce unphysical effects (Turolla & Nobili 1988; Dullemond 1999; Fukue 2008a). Although careful velocity dependent Eddington factor may circumvent such problems (see e.g., Fukue 2008a, 2008b, 2009), it is still a specific and not a general remedy that is not guaranteed to work in different cases.

11. SUMMARY

I have shown one approach to general relativistic radiation hydrodynamics. It is a covariant formalism based on the frequency-integrated radiation moments and, therefore, can be straightforwardly applied to any spacetime or coordinate systems. Since it uses a finite number of radiation moments and a certain choice of closure relation, it is an approximate yet reasonable way to solve the radiation transport. It has an advantage of being simple and physically intuitive enough to be applicable to diverse astrophysical problems.

REFERENCES

Auer, L. H., & Mihalas, D. 1970, On The Use of Variable Eddington Factors in Non-Lte Stellar Atmospheres Computations, MNRAS, 149, 65

Bardeen, J. M., Press, W. H., & Teukolsky, S. A. 1972, Rotating Black Holes: Locally Nonrotating Frames, Energy Extraction, and Scalar Synchrotron Radiation, ApJ, 178, 347

Chan, C.-K. 2011, A Class of Physically Motivated Closures for Radiation Hydrodynamics, ApJ, 727, 67

Dullemond, C. P. 1999, Relativistic Moment Equations In Media With Strong Velocity Gradients, A&AS, 343, 1030

Fukue, J. 2008a, Relativistic Variable Eddington Factor, PASJ, 60, 377

Fukue, J. 2008b, Variable Eddington Factor in a Relativistic Plane-Parallel Flow, PASJ, 60, 1209

Fukue, J. 2009, Relativistic Variable Eddington Factor in a Relativistic Plane-Parallel Flow, PASJ, 61, 367

Hummer, D. G., & Rybicki, G. B. 1971, Radiative Transfer In Spherically Symmetric Systems. The Conservative Grey Case, MNRAS, 152, 1

Lindquist, R. W. 1966, Relativistic Transport Theory, Ann. Phys., 37, 487

Mihalas, D., & Mihalas, B. W. 1984, Foundations of Radiation Hydrodynamics (Oxford: Oxford Univ. Press)

Minerbo, G. N. 1978, Maximum Entropy Eddington Factors, J. Quant. Spectosc. Radiat. Transfer, 20, 541

Munier, A., & Weaver, R. 1986, Radiation Transfer in the Fluid Frame: A Covariant Formulation, Comp. Phys. Rep., 3, 127

Park, M.-G. 1990, Self-Consistent Models Of Spherical Accretion Onto Black Holes. I - One-Temperature Solutions, ApJ, 354, 64

Park, M.-G. 1992, General Relativistic Spherical Accretion onto Neutron Star, JKPS, 25, 294

Park, M.-G. 1993, Relativistic Theory Of Radiative Transfer: Time-Dependent Radiation Moment Equations, A&AS274, 642

Park, M.-G. 2006a, Equations Of General Relativistic Radiation Hydrodynamics From A Tensor Formalism, MNRAS, 367, 1739

Park, M.-G. 2006b, Relativistic Radiation Hydrodynamic Equations in Cylindrical Coordinates, JKPS, 49, 736

Park, M.-G. 2006c, Relativistic Radiation Hydrodynamics for Accretion, JKPS, 49, 1725

Service, A. T. 1986, Fitting Formulae for the Equation of State of a Perfect, Semirelativistic Boltzmann Gas, ApJ, 307, 60

Takahashi, R. 2007, Equations of General Relativistic Radiation Hydrodynamics in Kerr Space-Time, MNRAS, 382, 1041

Takahashi, R. 2008, Radiation Hydrodynamics in Kerr Space-Time: Equations without Coordinate Singularity at the Event Horizon, MNRAS, 383, 1155

Tamazawa, S., Toyama, K., Kaneko, N., & Ono, Y. 1975, Optically Thick Accretion onto Black Holes, Astrophys. Space Sci., 32, 403

Thorne, K. S. 1981, Relativistic Radiative Transfer - Moment Formalisms, MNRAS, 194, 439

Turolla, R., & Nobili, L. 1988, General Relativistic Radiative Transfer in Spherical Flows, MNRAS, 235, 1273

Yin, W.-W., & Miller, G. S. 1995, Exact and Approximate Radiative Transfer in Differentially Moving Media, ApJ, 449, 826

Zampieri, L., Turolla, R., & Treves, A. 1993, Dynamically Comptonized Spectra from Near-Critical Accretion onto Neutron Stars, ApJ, 19, 311