J. Korean Math. Soc. **49** (2012), No. 5, pp. 1083–1096
http://dx.doi.org/10.4134/JKMS.2012.49.5.1083

ON THE k-REGULAR SEQUENCES AND THE GENERALIZATION OF F-MODULES

Khadijeh Ahmadi-Amoli and Navid Sanaei

Abstract. For a given ideal I of a Noetherian ring R and an arbitrary integer $k \geq -1$, we apply the concept of k-regular sequences and the notion of k-depth to give some results on modules called k-Cohen Macaulay modules, which in local case, is exactly the k-modules (as a generalization of f-modules). Meanwhile, we give an expression of local cohomology with respect to any k-regular sequence in I, in a particular case. We prove that the dimension of homology modules of the Koszul complex with respect to any k-regular sequence is at most k. Therefore homology modules of the Koszul complex with respect to any filter regular sequence has finite length.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity, I denotes an arbitrary ideal, M denotes a finitely generated R-module, and $k \geq -1$ an arbitrary integer. The notion of k-regular sequence was introduced by Chinh and Nhan [4] which is an extension of the well-known notion of filter regular sequence introduced by Schenzel, Trung, and Cuong [10]. First we give some standard properties of these sequences basically, by a different method from [4]. Most of the properties are familiar results for the cases $k = -1, k = 0$, and $k = 1$, cf. [5], [10], [1], and [3]. In the local case, it is shown, ([8], Lemma 3.4), that the local cohomology with respect to the maximal ideal of R is concerned with the local cohomology with respect to an ideal generated by any filter regular sequence. In a Noetherian ring (not necessary local), we prove (in Theorem 3.2), that for a k-regular M-sequence in I, say $a_1, \ldots, a_n$, if $\big(\mathrm{Supp}(M/(a_1, \ldots, a_n)M) \setminus V(I)\big)_{\leq k} = \emptyset$, then $H^i_I(M) \cong H^i_{(a_1,a_2,\ldots,a_n)}(M)$ for all $i < n$. In Section 4, we show that for any k-regular sequence $a_1, \ldots, a_n$ the dimension of homology modules of the Koszul complex with respect to $a_1, \ldots, a_n$ is at most k. In local case and $k = 0$, i.e., if $a_1, \ldots, a_n$ is a filter regular sequence it is well-known that the

Received August 21, 2011; Revised January 30, 2012.
2010 *Mathematics Subject Classification.* 13D45, 13C15.
Key words and phrases. k-regular M-sequences, k-depth, k-ht, local cohomology modules, k-Cohen Macaulay modules, f-modules, k-modules, Koszul complexes.

homology modules of the correspondence Koszul complex has finite length, cf. Corollary 4.3. Filter regular sequences have been studied in [4] and [7], and for non-local ring in [1]. Next, in Section 5, we introduce the concept of k-modules as a generalization of f-modules introduced by P. Schenzel, N. V. Trung, and N. T. Cuong [10]. Then we give some characterizations for k-modules in Proposition 5.2, and as a consequence of it, we conclude that k-Cohen Macaulay modules which is introduced in [2], are the same as k-modules under a dimensional equation, cf. Corollary 5.3. Another application of k-depth is shown in Theorem 5.4, which gives a necessary and sufficient condition for k-modules in terms of prime ideals. At the end we show that if M is a k-module, for an ideal I of R

$$k\text{-depth}(I, M) = k\text{-ht}_M I = \dim M - \dim R/I.$$

In cases $k = -1$ or $k = 0$ (i.e., when M is C.M. or f-module), this is the well-known equality in commutative ring theory.

2. Preliminaries

In this section we remind the concept of k-regular M-sequence and their properties. For more details, the reader is referred to [2]. For a subset T of $\mathrm{Spec}(R)$ and an integer $i \geq -1$, we set

$$(T)_{>i} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} > i\},$$

$$(T)_{\leq i} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \leq i\}.$$

Definition. A sequence $a_1, \ldots, a_n$ of elements of R is called a *poor k-regular M-sequence* whenever $a_i \notin \mathfrak{p}$ for all

$$\mathfrak{p} \in \mathrm{Ass}(M/\sum_{j=1}^{i-1} a_j M), \qquad \dim R/\mathfrak{p} > k$$

for all $i = 1, \ldots, n$. Moreover, if $\dim(M/\sum_{i=1}^{n} a_i M) > k$, $a_1, \ldots, a_n$ is called a *k-regular M-sequence*. An element a of R is called a *k-regular M-element* if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$ satisfying $\dim R/\mathfrak{p} > k$.

Remarks 2.1. (i) It is easy to see that for a given finitely generated R-module M, an ideal I of R with $\big(\mathrm{Supp}(M/IM)\big)_{>k} = \emptyset$ and any positive integer n, we can find n elements of I which form a poor k-regular M-sequence.

(ii) Every regular (filter regular, generalized regular, resp.) M-sequence is a k-regular M-sequence for all $k \geq -1$ ($k \geq 0$, $k \geq 1$, resp.) (Generalized regular sequences have been studied in [9]).

The following theorem gives a useful necessary and sufficient condition for a poor k-regular sequence. A nice application of this theorem is a characterization of filter regular sequences by length of a quotient module, cf. Corollary 4.4.

Theorem 2.2 ([2], Theorem 2.1)**.** *A sequence $a_1,\dots,a_n$ of elements of R is a poor k-regular M-sequence if and only if*

$$\dim\Big(\big(\sum_{j=1}^{i-1} a_jM :_M a_i\big)/\sum_{j=1}^{i-1} a_jM\Big) \le k$$

for all $i = 1,\dots,n$.

As an easy consequence of elementary properties of regular sequences and definition of k-regular sequences, we have the following:

Theorem 2.3. *Let $a_1,\dots,a_n$ be a sequence of elements of R. The following statements are equivalent*:

(i) *$a_1,\dots,a_n$ is a poor k-regular M-sequence*;
(ii) *$a_1/1,\dots,a_i/1$ is a poor regular $M_{\mathfrak p}$-sequence in $R_{\mathfrak p}$ for all $\mathfrak p \in (\mathrm{Supp}(M))_{>k}$ and all $i = 1,\dots,n$*;
(iii) *$a_1^{m_1},\dots,a_n^{m_n}$ is a poor k-regular M-sequence for all $m_1,\dots,m_n \in \mathbb{N}$.*

Lemma 2.4 ([2], Lemma 2.8)**.** *Each k-regular M-sequence has finite number of elements.*

Remarks 2.5. (i) Let $a_1,\dots,a_n$ be a k-regular M-sequence contained in I. Then we say that $a_1,\dots,a_n$ is a *maximal* k-regular M-sequence contained in I if there is not $a_{n+1} \in I$ such that $a_1,\dots,a_n,a_{n+1}$ is a k-regular M-sequence.

(ii) There exists a k-regular M-sequence contained in I, and each k-regular M-sequence contained in I can be extended to a maximal k-regular M-sequence (by Lemma 2.4) (Note that an empty sequence is considered a k-regular M-sequence of length 0).

Definition. Let I be an ideal of R with $(\mathrm{Supp}(M/IM))_{>k} \ne \emptyset$. Then we denote the length of any maximal k-regular M-sequence contained in I by k-depth(I,M).

Remark 2.6. Let I be an ideal with $(\mathrm{Supp}(M/IM))_{>k} \ne \emptyset$. For all $\mathfrak p \in \mathrm{Supp}(M/IM)_{>k}$ we have

$$k < \dim R/\mathfrak p \le \dim M/IM.$$

Therefore, whenever we are concerned with *k-depth*(I,M), we should know that $-1 \le k < \dim M/IM$.

Proposition 2.7 ([2], Proposition 3.3(ii))**.** *Let M be a finitely generated R-module and I be an ideal with $(\mathrm{Supp}(M/IM))_{>k} \ne \emptyset$. Let $a \in I$ be a k-regular M-element. Then*

$$k\text{-depth}(I,M/aM)=k\text{-depth}(I,M)-1.$$

Proposition 2.8 ([2], Proposition 3.4)**.** *Let the situation be as in Proposition 2.7. Then*

$$k\text{-depth}(I,M) = \min\{k\text{-depth}(\mathfrak p,M) \mid \mathfrak p \in V(I)\}.$$

In the following we generalize the notion of height.

Definition. Let I be an ideal of R with $(\mathrm{Supp}(M/IM))_{>k} \neq \emptyset$. The *$k$-height of I with respect to M* is defined by

$$k\text{-ht}_M I = \min\{\mathrm{ht}_M \mathfrak{p} \mid \mathfrak{p} \in (\mathrm{Supp}(M/IM))_{>k}\}.$$

For an ideal I of R with $(\mathrm{Supp}(M/IM))_{>k} = \emptyset$, we set $k\text{-ht}_M I = \infty$. In the case $k = -1$, the notion of $k\text{-ht}_M I$ is the same as $\mathrm{ht}_M I$, the height of ideal I with respect to M.

Remark 2.9 ([2], 4.2(i)). Let $\mathfrak{p}$ be a prime ideal of R with $(\mathrm{Supp}(M/\mathfrak{p}M))_{>k} \neq \emptyset$. Then

$$k\text{-ht}_M \mathfrak{p} = \mathrm{ht}_M \mathfrak{p}.$$

The following theorem is a generalization of ([6], Theorem 17.4) when we put $k = -1$.

Theorem 2.10. *Let M be a finitely generated R-module, with* $\dim M > k$. *Let $a_1, \ldots, a_n$ be a k-regular M-sequence. Then*

$$k\text{-ht}_M(a_1, \ldots, a_n) = n.$$

Proof. Let $a_1, \ldots, a_n$ be a k-regular M-sequence and $k\text{-ht}_M(a_1, \ldots, a_n) = \mathrm{ht}_M \mathfrak{p}$ for some $\mathfrak{p} \in (\mathrm{Supp}(M/\sum_{i=1}^n a_i M))_{>k}$. Then $\mathfrak{p}$ is a minimal element of $\mathrm{Supp}(M/\sum_{i=1}^n a_i M)$. Therefore, we have $\mathrm{ht}_M \mathfrak{p} = \mathrm{ht}_{M_\mathfrak{p}}(a_1/1, \ldots, a_n/1)$. On the other hand, since by Theorem 2.3, $a_1/1, \ldots, a_n/1$ is a regular $M_\mathfrak{p}$-sequence, we have

$$\mathrm{ht}_{M_\mathfrak{p}}(a_1/1, \ldots, a_n/1) = n,$$

by ([6], Theorem 17.4). Therefore, the assertion follows. □

By the above theorem, we get a relation between k-depth and k-ht as follows.

Theorem 2.11 ([2], Remark 4.2(i)). *Let I be an ideal of R, with*

$$(\mathrm{Supp}(M/IM))_{>k} \neq \emptyset.$$

Then

$$k\text{-depth}(I, M) \leq k\text{-ht}_M I.$$

Definition. Let M be a finitely generated R-module. M is called a *k-Cohen-Macaulay module*, and denoted by k-C.M., if either $k\text{-depth}(I, M) = k\text{-ht}_M I$ for all ideal I of R with $(\mathrm{Supp}(M/IM))_{>k} \neq \emptyset$ or $(\mathrm{Supp}(M))_{>k} = \emptyset$.

Theorem 2.12 ([2], Theorem 4.4). *Let M be a finitely generated R-module. Then M is a k-C.M. if and only if $M_\mathfrak{p}$ is Cohen-Macaulay for all*

$$\mathfrak{p} \in (\mathrm{Supp}(M))_{>k}.$$

3. k-regular sequences and local cohomology modules

In this section, in a Noetherian ring, not necessary local, we give an expression of local cohomology with respect to any k-regular sequence. If $(R, \mathfrak{m})$ is a local ring, it is the result of ([8], Lemma 3.4), which shows that the local cohomology with respect to $\mathfrak{m}$ is isomorphic to the local cohomology with respect to any filter regular sequence in $\mathfrak{m}$. To prove the main theorem we need the following lemma.

Lemma 3.1. *Let I be an ideal of R, M be a finitely generated R-module and n be a non-negative integer. Let $x_1, x_2, \ldots, x_r$ be $r(\in \mathbb{N})$ elements of I. If $H^i_{(x_1,x_2,\ldots,x_r)}(M) = 0$ for all $i < n$, then*

$$\operatorname{Hom}_R\big(R/I, H^n_{(x_1,x_2,\ldots,x_r)}(M)\big) \cong \operatorname{Ext}^n_R(R/I, M).$$

Proof. Let $\underline{x} = (x_1, x_2, \ldots, x_r)$ and $E = E_R(M)$ be the injective envelope of M. From the exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$$

we get the following long exact sequences

$$\begin{aligned}
(1) \qquad & 0 \longrightarrow \Gamma_{\underline{x}}(M) \longrightarrow \Gamma_{\underline{x}}(E) \longrightarrow \Gamma_{\underline{x}}(E/M) \\
& \longrightarrow H^1_{\underline{x}}(M) \longrightarrow H^1_{\underline{x}}(E) \longrightarrow H^1_{\underline{x}}(E/M) \\
& \longrightarrow H^2_{\underline{x}}(M) \longrightarrow \cdots \\
& \vdots \\
& \longrightarrow H^i_{\underline{x}}(M) \longrightarrow H^i_{\underline{x}}(E) \longrightarrow H^i_{\underline{x}}(E/M) \\
& \longrightarrow H^{i+1}_{\underline{x}}(M) \longrightarrow \cdots \\
& \vdots \\
& \longrightarrow H^{n-1}_{\underline{x}}(M) \longrightarrow H^{n-1}_{\underline{x}}(E) \longrightarrow H^{n-1}_{\underline{x}}(E/M) \\
& \longrightarrow H^n_{\underline{x}}(M) \longrightarrow H^n_{\underline{x}}(E) \longrightarrow H^n_{\underline{x}}(E/M) \\
& \longrightarrow \cdots,
\end{aligned}$$

$$\begin{aligned}
(2) \qquad & \ldots \longrightarrow \operatorname{Ext}^{n-1}_R(R/I, E) \longrightarrow \operatorname{Ext}^{n-1}_R(R/I, E/M) \longrightarrow \operatorname{Ext}^n_R(R/I, M) \\
& \longrightarrow \operatorname{Ext}^n_R(R/I, E) \longrightarrow \cdots .
\end{aligned}$$

We use induction on n. Let $n = 0$. We have

$$\operatorname{Hom}_R\big(R/I, \Gamma_{\underline{x}}(M)\big) = \operatorname{Hom}_R(R/I, M).$$

Let $n = 1$. By assumption, $\Gamma_{\underline{x}}(M) = 0$. Thus since E is an essential extension of M and $\Gamma_{\underline{x}}(E) \cap M = 0$, we get $\Gamma_{\underline{x}}(E) = 0$ (and so $\Gamma_I(E) = 0$). Therefore by (1), $\Gamma_{\underline{x}}(E/M) \cong H^1_{\underline{x}}(M)$. Also $\Gamma_I(E) = 0$ implies that $\operatorname{Hom}_R(R/I, E) = 0$ and by using (2) we get

$$\operatorname{Ext}^1_R(R/I, M) \cong \operatorname{Hom}_R(R/I, E/M)$$

$$= \operatorname{Hom}_R\big(R/I, \Gamma_{\underline{x}}(E/M)\big)$$
$$\cong \operatorname{Hom}_R\big(R/I, H^1_{\underline{x}}(M)\big).$$

Now, suppose that $n > 1$ and for any finitely generated R-module N, such that $H^i_{\underline{x}}(N) = 0$ for all $i < n-1$, we have

$$\operatorname{Hom}_R\big(R/I, H^{n-1}_{\underline{x}}(N)\big) \cong \operatorname{Ext}^{n-1}_R(R/I, N).$$

Therefore since $H^i_{\underline{x}}(M) = 0 = H^{i+1}_{\underline{x}}(M)$ for all $i < n-1$, we get

$$H^i_{\underline{x}}(E/M) \cong H^i_{\underline{x}}(E) = 0$$

for all $i < n-1$. Thus, by inductive hypothesis,

$$\operatorname{Hom}_R\big(R/I, H^{n-1}_{\underline{x}}(E/M)\big) \cong \operatorname{Ext}^{n-1}_R(R/I, E/M). \tag{3}$$

Also by (1)

$$H^{n-1}_{\underline{x}}(E/M) \cong H^n_{\underline{x}}(M). \tag{4}$$

Now, by using (2), (3), and (4), we get

$$\begin{aligned}\operatorname{Ext}^n_R(R/I, M) &\cong \operatorname{Ext}^{n-1}_R(R/I, E/M)\\ &\cong \operatorname{Hom}_R\big(R/I, H^{n-1}_{\underline{x}}(E/M)\big)\\ &\cong \operatorname{Hom}_R\big(R/I, H^n_{\underline{x}}(M)\big)\end{aligned}$$

as required. □

Now, by using above lemma, we obtain one of the main results of this paper as follows.

Theorem 3.2. *Let I be an ideal of R and $a_1, a_2, \ldots, a_n$ be a k-regular M-sequence in I such that $\big(\operatorname{Supp}(M/(a_1, \ldots, a_n)M) \smallsetminus V(I)\big)_{\leq k} = \emptyset$. Then*

(i) $H^i_I(M) \cong H^i_{(a_1,a_2,\ldots,a_n)}(M)$ *for all* $i < n$,

(ii) $\big(\Gamma_I(H^n_{(a_1,a_2,\ldots,a_n)}(M))\big)_{\mathfrak{p}} \cong \big(H^n_I(M)\big)_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \big(\operatorname{Supp}(M/(a_1, a_2, \ldots, a_n)M)\big)_{>k}$.

Proof. (i) Let

$$0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^i \xrightarrow{d^i} \cdots$$

be a minimal injective resolution for M. For all $i \in \mathbb{N}_0$ we have

$$E^i = \bigoplus_{\mathfrak{p}} \mu_i(\mathfrak{p}, M) E(R/\mathfrak{p})$$

in which $\mu_i(\mathfrak{p}, M)$ is the *i-th* Bass number of M at the prime ideal $\mathfrak{p}$ of R. Let $i < n$ and $\mathfrak{p} \in \big(\operatorname{Supp}(M/(a_1, a_2, \ldots, a_n)M)\big)_{>k}$. By Theorem 2.3(ii), since $a_1/1, \ldots, a_n/1$ is a regular $M_{\mathfrak{p}}$-sequence, we have

$$\mu_i(\mathfrak{p}, M) = 0. \tag{5}$$

Next, we have

$$\Gamma_I(E^i) = \bigoplus_{\mathfrak{p}} \mu_i(\mathfrak{p}, M)\Gamma_I(E(R/\mathfrak{p})).$$

Let $\mathfrak{q} \in \mathrm{Supp}(M)$. If $\mathfrak{q} \not\supseteq I$, there exists $x \in I \setminus \mathfrak{q}$. Hence, $E(R/\mathfrak{q}) \xrightarrow{x.} E(R/\mathfrak{q})$ is an automorphism. We show that $\Gamma_I(E(R/\mathfrak{q})) = 0$. Let $y \in \Gamma_I(E(R/\mathfrak{q}))$. Then, since $x \in I$, $x^t y = 0$ for some $t \in \mathbb{N}$. Thus $y = 0$, and so $\Gamma_I(E(R/\mathfrak{q})) = 0$. Now, Let $\mathfrak{q} \supseteq I$. Let $y \in E(R/\mathfrak{q})$. Then $I^t y = 0$ for some $t \in \mathbb{N}$. Hence $\Gamma_I(E(R/\mathfrak{q})) = E(R/\mathfrak{q})$. Therefore, we have

$$\Gamma_I(E^i) = \bigoplus_{\mathfrak{p} \in \mathrm{Supp}(M/IM)} \mu_i(\mathfrak{p}, M)E(R/\mathfrak{p}) \tag{6}$$

for all $i < n$. Similarly, for the ideal $(a_1, \ldots, a_n)$ we get

$$\Gamma_{(a_1,\ldots,a_n)}(E^i) = \bigoplus_{\mathfrak{p} \in \mathrm{Supp}(M/(a_1,\ldots,a_n)M)} \mu_i(\mathfrak{p}, M)E(R/\mathfrak{p}) \tag{7}$$

for all $i < n$. From the hypothesis that $\big(\mathrm{Supp}(M/(a_1, \ldots, a_n)M) \setminus V(I)\big)_{\leq k} = \emptyset$ we have by (5), (6), (7) that

$$\Gamma_I(E^i) = \Gamma_{(a_1,\ldots,a_n)}(E^i)$$

for all $i < n$. It therefore follows that

$$H_I^i(M) = H^i_{(a_1,\ldots,a_n)}(M)$$

for all $i < n$.

(ii) Let $\mathfrak{p} \in \big(\mathrm{Supp}(M/(a_1, \ldots, a_n)M)\big)_{>k}$. Again, by Theorem 2.3(ii), since $a_1/1, \ldots, a_n/1$ is a regular $M_{\mathfrak{p}}$-sequence in $IR_{\mathfrak{p}}$,

$$H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$$

for all $i < n$. Now, by Lemma 3.1,

$$\mathrm{Hom}_{R_{\mathfrak{p}}}\big(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, H^n_{(a_1/1,\ldots,a_n/1)}(M_{\mathfrak{p}})\big) \cong \mathrm{Ext}^n_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}})$$

and so the assertion immediately follows. □

4. *k*-regular sequences and Koszul complexes

In this section, we obtain a result which shows that dimension of homology modules of the Koszul complex with respect to any k-regular M-sequence is less than or equal to k. In particular case, when R is local, the length of such homology modules with respect to any filter regular sequence is finite. For any $x_1, \ldots, x_n \in R$, i-th homology module of the Koszul complex $K_{\cdot}(x_1, \ldots, x_n; M)$ of M with respect to $x_1, \ldots, x_n$ is denoted by $H_i(x_1, \ldots, x_n; M)$.

Theorem 4.1. *Let $a_1, \ldots, a_n$ be a k-regular M-sequence. Then, for all $i > 0$,*

$$\dim(H_i(a_1, \ldots, a_n; M)) \leq k.$$

Proof. We use induction on n. Let $n = 1$. Since

$$H_1(a_1; M) = (0 :_M a_1)$$

and, by Theorem 2.2, $\dim(0 :_M a_1) \leq k$, we have $\dim(H_1(a_1; M)) \leq k$ (Note that for all $i \geq 2$, $H_i(a_1; M) = 0$).

Now, suppose that $n > 1$ and the assertion is true for all k-regular M-sequences with length of less than n. Considering the following long exact sequence

$$\cdots \longrightarrow H_i(a_1, \ldots, a_{n-1}; M) \longrightarrow H_i(a_1, \ldots, a_n; M) \longrightarrow H_{i-1}(a_1, \ldots, a_{n-1}; M)$$
$$\overset{(-1)^{i-1}a_n}{\longrightarrow} H_{i-1}(a_1, \ldots, a_{n-1}; M) \longrightarrow \cdots \longrightarrow H_0(a_1, \ldots, a_n; M) \longrightarrow 0$$

and using the inductive hypothesis, for all $i > 1$, we get

$$\dim(H_{i-1}(a_1, \ldots, a_{n-1}; M)) \leq k,$$

and

$$\dim(H_i(a_1, \ldots, a_{n-1}; M)) \leq k.$$

Hence $\dim(H_i(a_1, \ldots, a_n; M)) \leq k$ for all $i > 1$.

Next, let $i = 1$ and again consider the above exact sequence. Since a_n is a k-regular $M/(a_1, \ldots, a_{n-1})M$-sequence, by Theorem 2.2,

$$\dim(0 :_{M/(a_1,\ldots,a_{n-1})M} a_n) \leq k.$$

Also by inductive hypothesis $\dim(H_1(a_1, \ldots, a_{n-1}; M)) \leq k$. On the other hand $H_0(a_1, \ldots, a_{n-1}; M) = M/(a_1, \ldots, a_{n-1})M$. Hence we obtain

$$\dim(H_1(a_1, \ldots, a_n; M)) \leq k$$

and the theorem is proved by induction. □

As an application of Theorem 4.1, we conclude the similar result for filter regular sequences that Lü and Tang showed in ([5], Proposition 2.2) about the length of homology modules of correspondence Koszul complex. For this purpose we need the following lemma.

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a local ring and N be a finitely generated R-module. Then following conditions are equivalent*:

(i) $\dim(N) \leq 0$;
(ii) $\ell(N) < \infty$.

Proof. It is an easy consequence of definitions of dimension and the length of N. □

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Let $a_1, \ldots, a_n \in \mathfrak{m}$ be a filter regular M-sequence. Then*

$$\ell(H_i(a_1, \ldots, a_n; M)) < \infty$$

for all $i > 0$.

Proof. Let $k = 0$ in Theorem 4.1 and use Lemma 4.2. □

Again by using Lemma 4.2 and Theorem 2.2, we obtain another necessary and sufficient condition for poor filter regular sequences as follows.

Corollary 4.4. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be a finitely generated* R*-module. Then* $a_1, \ldots, a_n \in \mathfrak{m}$ *is a poor filter regular* M*-sequence if and only if*

$$\ell\Big(\sum_{j=1}^{i-1} a_j M :_M a_i / \sum_{j=1}^{i-1} a_j M\Big) < \infty$$

for all $i = 1, \ldots, n$.

5. *k*-modules

In this section we generalize the concept of f-modules introduced by P. Schenzel, N. V. Trung, and N. T. Cuong in [10] as a generalization of Cohen-Macaulay modules. In this section, all rings are local.

Definition. Let $(R, \mathfrak{m})$ be a local ring. A finitely generated R-module M is called a *k-module* if every system of parameters for M is a poor k-regular M-sequence.

Remarks 5.1. (i) In the case $k = -1$, (-1)-modules are exactly Cohen-Macaulay modules.

(ii) In the case $k = 0$, 0-modules are the same as f-modules.

Now, we give some characterizations of k-modules as follows.

Proposition 5.2. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be a finitely generated* R*-module in which* $(\mathrm{Supp}(M))_{>k} \neq \emptyset$ *and* $\dim M = d > 0$*. Then the following conditions are equivalent*:

(i) M *is a* k*-module*;

(ii) *for all subsets of systems of parameters* $a_1, \ldots, a_r$ *for* M *and all* $\mathfrak{p} \in \big(\mathrm{Ass}(M/(a_1, \ldots, a_r)M)\big)_{>k}$,

$$\dim R/\mathfrak{p} = d - r;$$

(iii) *for all* $\mathfrak{p} \in (\mathrm{Supp}(M))_{>k}$, $M_{\mathfrak{p}}$ *is a Cohen-Macaulay* $R_{\mathfrak{p}}$*-module and*

$$\dim M = \dim R/\mathfrak{p} + \dim M_{\mathfrak{p}}.$$

Proof. (i)$\Rightarrow$(ii): Let $a_1, \ldots, a_r$ be a subset of system of parameters for M such that $\dim R/\mathfrak{p} < d - r$ for some $\mathfrak{p} \in \big(\mathrm{Ass}(M/(a_1, \ldots, a_r)M)\big)_{>k}$. Then we have

$$\mathfrak{p} \not\subseteq \bigcup_{\substack{\mathfrak{q} \,\in\, \mathrm{Ass}(M/(a_1,\ldots,a_r)M) \\ \dim R/\mathfrak{q} = d-r}} \mathfrak{q}.$$

Therefore, there is $a \in \mathfrak{p}$ such that $a \notin \mathfrak{q}$ for all $\mathfrak{q} \in \big(\mathrm{Ass}(M/(a_1, \ldots, a_r)M)\big)$ with $\dim R/\mathfrak{q} = d - r$. We show that $a_1, \ldots, a_r, a$ form a subset of system of parameters for M. Since $k < \dim R/\mathfrak{p} < d - r$, we have $k + r < \dim R/\mathfrak{p} + r < d$. But $k > -1$, so $r < k + r + 1 < d$. Hence $r + 1 < d$. Therefore

$t = \dim M/(a_1,\ldots,a_r,a)M \geq d-(r+1) > 0$. Thus we can find t elements, say $b_1,\ldots,b_t$ in $\mathfrak{m}$ such that their residues modulo $(a_1,\ldots,a_r,a)$ form a subset of a system of parameters for M. Therefore, we have $d = \dim M \leq t+r+1$ and so

$$\begin{aligned} d-(r+1) &\leq \dim M/(a_1,\ldots,a_r,a)M \\ &\leq \dim M/(a_1,\ldots,a_r)M = d-r. \end{aligned}$$

Suppose that $\dim M/(a_1,\ldots,a_r,a)M = d-r$. Let $\mathfrak{q} \in \operatorname{Supp}\big(M/(a_1,\ldots,a_r,a)M\big)$ be such that

$$\dim M/(a_1,\ldots,a_r,a)M = \dim R/\mathfrak{q}.$$

By assumption, since $a \in \mathfrak{q}$, we have $\mathfrak{q} \notin \operatorname{Ass}\big(M/(a_1,\ldots,a_r)M\big)$. Therefore, $\mathfrak{q}$ is not a minimal element of $\operatorname{Supp}\big(M/(a_1,\ldots,a_r)M\big)$. Hence, there exists $\mathfrak{q}' \in \operatorname{Supp}\big(M/(a_1,\ldots,a_r)M\big)$ such that $\mathfrak{q}' \subsetneqq \mathfrak{q}$. Therefore

$$d-r = \dim R/\mathfrak{q} < \dim R/\mathfrak{q}' \leq \dim M/(a_1,\ldots,a_r)M = d-r,$$

which is a contradiction. Therefore $\dim M/(a_1,\ldots,a_r)M = d-(r+1)$, and so $a_1,\ldots,a_r,a$ is a subset of system of parameters for M. Now, since M is a k-module, $a_1,\ldots,a_r,a$ is a poor k-regular M-sequence which contradicts $a \in \mathfrak{p}$.

(ii)$\Rightarrow$(iii): Let $^-: R \longrightarrow R/\operatorname{Ann}M$ be the natural homomorphism and $\mathfrak{p} \in (\operatorname{Supp}(M))_{>k}$. Set $\bar{R} = R/\operatorname{Ann}M, \bar{\mathfrak{p}} = \mathfrak{p}/\operatorname{Ann}M$. Let $\operatorname{ht}_{\bar{R}}\bar{\mathfrak{p}} = r$. Then there exist $\bar{a}_1,\ldots,\bar{a}_r \in \bar{\mathfrak{p}}$ such that $\bar{\mathfrak{p}}$ is a minimal prime ideal of $(\bar{a}_1,\ldots,\bar{a}_r)$ and $\operatorname{ht}(\bar{a}_1,\ldots,\bar{a}_r) = r$. Thus

$$\dim\bar{R} - r \leq \dim\bar{R}/(\bar{a}_1,\ldots,\bar{a}_r) \leq \dim\bar{R} - \operatorname{ht}(\bar{a}_1,\ldots,\bar{a}_r) = \dim\bar{R} - r.$$

Therefore $\dim\bar{R}/(\bar{a}_1,\ldots,\bar{a}_r) = \dim\bar{R} - r$, and so $\bar{a}_1,\ldots,\bar{a}_r$ is a subset of a system of parameters for $\bar{R}$. Hence $a_1,\ldots,a_r$ is a subset of a system of parameters for M. On the other hand, $\mathfrak{p} \in \operatorname{Supp}(M)$ is a minimal prime ideal of $(a_1,\ldots,a_r)$; hence $\mathfrak{p} \in \operatorname{Ass}\big(M/(a_1,\ldots,a_r)M\big)$. Therefore, by hypothesis

$$\dim R/\mathfrak{p} = d-r = \dim \mathrm{M} - \operatorname{ht}_{\bar{R}}\bar{\mathfrak{p}} = \dim M - \dim M_{\mathfrak{p}}.$$

Thus $\dim R/\mathfrak{p} + \dim M_{\mathfrak{p}} = \dim M$ for all $\mathfrak{p} \in \big(\operatorname{Supp}(M)\big)_{>k}$. Now, we show that $M_{\mathfrak{p}}$ is a C.M. $R_{\mathfrak{p}}$-module. First we show that $a_1/1,\ldots,a_r/1$ is a regular $M_{\mathfrak{p}}$-sequence in $\mathfrak{p}R_{\mathfrak{p}}$. Let $1 \leq i \leq r$ and $\mathfrak{q} \in \operatorname{Ass}\big(M/(a_1,\ldots,a_{i-1})M\big)$ be such that $a_i/1 \in \mathfrak{q}R_{\mathfrak{p}}$. Since $\mathfrak{q} \subseteq \mathfrak{p}$, we have $\dim R/\mathfrak{q} > k$. Thus $\mathfrak{q} \in \Big(\operatorname{Ass}\big(M/(a_1,\ldots,a_{i-1})M\big)\Big)_{>k}$ and so by hypothesis, $\dim R/\mathfrak{q} = d-(i-1)$. Thus

$$d-(i-1) = \dim R/\mathfrak{q} \leq \dim M/(a_1,\ldots,a_i)M = d-i$$

which is a contradiction. Hence

$$r = \dim M_{\mathfrak{p}} \geq \operatorname{depth} M_{\mathfrak{p}} \geq r;$$

that is $\operatorname{depth} M_{\mathfrak{p}} = \dim M_{\mathfrak{p}}$. Therefore $M_{\mathfrak{p}}$ is C.M.

(iii)$\Rightarrow$(ii): We use induction on r. If $r = 0$, there is nothing to prove, since for all $\mathfrak{p} \in (\operatorname{Ass}(M))_{>k}$, $\dim M_{\mathfrak{p}} = \operatorname{ht}_M \mathfrak{p} = 0$. Suppose that $r > 0$ and the result has been proved for all non-negative integers less than r. Let $a_1, \dots, a_r$ be a subset of a system of parameters for M and $\mathfrak{p} \in \Big(\mathrm{Ass}\big(M/(a_1, \dots, a_r)M\big)\Big)_{>k}$ be such that $\dim R/\mathfrak{p} > d - r$. By hypothesis, $r < d - \dim R/\mathfrak{p} = \dim M_{\mathfrak{p}} = \mathrm{depth} M_{\mathfrak{p}}$. Now, we show that $a_1/1, \dots, a_r/1$ is a (maximal) regular $M_{\mathfrak{p}}$-sequence in $\mathfrak{p}R_{\mathfrak{p}}$. Let $1 \leq i \leq r$ be such that

$$a_i/1 \in \mathfrak{q}R_{\mathfrak{p}}$$

for some $\mathfrak{q} \in \Big(\mathrm{Ass}\big(M/(a_1, \dots, a_{i-1})M\big)\Big)_{>k}$. By inductive hypothesis, $\dim R/\mathfrak{p} = d - (i-1)$. But $a_i \in \mathfrak{q}$ implies that $\dim R/\mathfrak{p} \leq \dim M/(a_1, \dots, a_i)M = d - i$, which is a contradiction. Now, since

$$\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}\big(M_{\mathfrak{p}}/(a_1/1, \dots, a_r/1)M_{\mathfrak{p}}\big),$$

we conclude that $a_1/1, \dots, a_r/1$ is a maximal regular $M_{\mathfrak{p}}$-sequence. Thus $\mathrm{depthM}_{\mathfrak{p}} = r$ which is a contradiction.

(ii)$\Rightarrow$(i): Let $a_1, \dots, a_r$ be a subset of system of parameters for M. Let $1 \leq i \leq r$ be such that $a_i \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}\big(M/(a_1, \dots, a_{i-1})M\big)_{>k}$. Therefore $\mathfrak{p} \in \mathrm{Supp}\big(M/(a_1, \dots, a_i)M\big)$ and so

$$d - (i-1) = \dim R/\mathfrak{p} \leq \dim M/(a_1, \dots, a_i)M = d - i$$

which is a contradiction. □

As an immediate consequence of Proposition 5.2 and Theorem 2.12, we have the following result.

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $(\mathrm{Supp}(M))_{>k} \neq \emptyset$. Then the following conditions are equivalent*:

(i) *M is a k-module*;

(ii) *M is k-C.M. and for all $\mathfrak{p} \in (\mathrm{Supp}(M))_{>k}$,*

$$\dim M = \dim R/\mathfrak{p} + \dim M_{\mathfrak{p}}.$$

In the following, we give another characterization for k-modules by the notion of k-depth.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Then M is a k-module if and only if for all $\mathfrak{p} \in (\mathrm{Supp}(M))_{>k}$,*

$$k\text{-depth}(\mathfrak{p}, M) = \dim M - \dim R/\mathfrak{p}.$$

Proof. Let $d = \dim M$. We note that if $d = 0$, then for all $\mathfrak{p} \in (\mathrm{Supp}(M))_{>k}$, by Remark 2.6, $k < \dim R/\mathfrak{p} \leq \dim M = 0$ that implies $k = -1$. Thus k-modules are C.M. modules and the assertion is obvious.

($\Longrightarrow$) Let $\mathfrak{p} \in (\mathrm{Supp}(M))_{>k}$ and $r = k\text{-depth}(\mathfrak{p}, M)$. Let $a_1, \dots, a_r \in \mathfrak{p}$ be a maximal k-regular M-sequence. Thus, there exists

$$\mathfrak{q} \in \Big(\mathrm{Ass}\big(M/(a_1, \dots, a_r)M\big)\Big)_{>k}$$

such that $\mathfrak{p} \subseteq \mathfrak{q}$. By Proposition 5.2, part (i)⇒(ii), $\dim R/\mathfrak{q} = d - r$ and again by the same Proposition, part (i)⇒(iii), and Remark 2.9 and Theorem 2.10, we have

$$\dim R/\mathfrak{p} = \dim M - \dim M_{\mathfrak{p}} = d - \mathrm{ht}_M \mathfrak{p} \leq d - (k\text{-}\mathrm{ht}_M(a_1, \ldots, a_r)) = d - r.$$

Hence $\mathfrak{p} = \mathfrak{q}$. Again by Proposition 5.2, part (i)⇒(ii), k-depth$(\mathfrak{p}, M) = r = d - \dim R/\mathfrak{p}$ as required.

($\Longleftarrow$) By induction on d we show that M is a k-module. Assume that the assertion is true for all R-modules with dimension less than d. Let $a_1, \ldots, a_d$ be a system of parameters for M. Then a_1 is a k-regular M-element. Because if there is $\mathfrak{p} \in (\mathrm{Ass}(M))_{>k}$ such that $a_1 \in p$, then

$$\dim M = k\text{-depth}(\mathfrak{p}, M) + \dim R/\mathfrak{p} = \dim R/\mathfrak{p} \leq \dim M/a_1 M = \dim M - 1$$

which is a contradiction.

Now, let $M_1 = M/a_1 M$. Then $\dim M_1 = d - 1$ and $a_2, \ldots, a_d$ is a system of parameters for M_1. For any $\mathfrak{p} \in (\mathrm{Supp}(M_1))_{>k}$, since $a_1 \in \mathfrak{p}$, we have

$$\begin{aligned} k\text{-depth}(\mathfrak{p}, M_1) &= k\text{-depth}(\mathfrak{p}, M) - 1 \\ &= \dim M - \dim R/\mathfrak{p} - 1 \\ &= \dim M_1 - \dim R/\mathfrak{p}. \end{aligned}$$

Therefore by inductive hypothesis, $a_2, \ldots, a_d$ is a k-regular M_1-sequence. Thus $a_1, \ldots, a_d$ is a k-regular M-sequence and the theorem is proved by induction. □

At the end, we show that for k-modules, the equality of Theorem 5.4, holds even for not necessary prime ones.

Lemma 5.5. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and I be an ideal of R such that $\big(\mathrm{Supp}(M/IM)\big)_{\leq k} = \emptyset$ and $I \supseteq \mathrm{Ann} M$. Then*

$$k\text{-}\mathrm{ht}_M I \leq \dim M - \dim R/I.$$

Proof. Since

$$k\text{-}\mathrm{ht}_M I = \min\{\mathrm{ht}_M \mathfrak{p} \mid \mathfrak{p} \in (\mathrm{Supp}(M/IM))_{>k}\}$$

and, by hypothesis

$$\dim R/I = \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in (\mathrm{Supp}(M/IM))_{>k}\},$$

it is enough to show that for all $\mathfrak{p} \in (\mathrm{Supp}(M/IM))$, k-$\mathrm{ht}_M \mathfrak{p} \leq \dim M - \dim R/\mathfrak{p}$. But by Remark 2.9, for all $\mathfrak{p} \in \big(\mathrm{Supp}(M/IM)\big)_{>k}$, k-$\mathrm{ht}_M \mathfrak{p} = \mathrm{ht}_M \mathfrak{p}$, thus the assertion is obvious. □

Theorem 5.6. *Let $(R, \mathfrak{m})$ be a local ring, M be a k-module, and I be an ideal of R such that $\big(\mathrm{Supp}(M/IM)\big)_{\leq k} = \emptyset$ and $I \supseteq \mathrm{Ann} M$. Then*

$$k\text{-depth}(I, M) = k\text{-}\mathrm{ht}_M I = \dim M - \dim R/I.$$

Proof. If $k = -1$, then the concepts of k-module, k-depth and k-ht coincide with the concepts C.M., depth and ht. Thus the assertion is trivial. Therefore, we assume that $k > -1$. By Proposition 2.8, the assumption, and Theorem 5.4 and Lemma 5.5,

$$\begin{aligned} k\text{-depth}(I, M) &= \min\{k\text{-depth}(\mathfrak{p}, M) \mid \mathfrak{p} \in V(I)\} \\ &= \min\{k\text{-depth}(\mathfrak{p}, M) \mid \mathfrak{p} \in (\mathrm{Supp}(M/IM))_{>k}\} \\ &= \dim M - \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in (\mathrm{Supp}(M/IM))_{>k}\} \\ &= \dim M - \dim R/I. \end{aligned}$$

On the other hand, by Theorem 2.11 and Lemma 5.5,

$$\dim M - \dim R/I = k\text{-depth}(I, M) \leq k\text{-ht}_M I \leq \dim M - \dim R/I,$$

as required. □

Acknowledgements. The authors are grateful to the referee for his or her careful reading of the manuscript. The authors also thank M. Y. Sadeghi for his helpful discussions especially on Lemma 3.1.

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Khadijeh Ahmadi-Amoli
Department of Mathematics
Payame Noor University
Tehran 19585-83133, Iran
E-mail address: khahmadi@pnu.ac.ir

Navid Sanaei
Department of Mathematics
Payame Noor University
Tehran 19585-83133, Iran
E-mail address: navidsanaei@yahoo.com