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THE CONNECTED SUBGRAPH OF THE TORSION GRAPH OF A MODULE

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Abstract. In this paper, we will investigate the concept of the torsion-graph of an R-module M, in which the set $T(M)^*$ makes up the vertices of the corresponding torsion graph, $\Gamma(M)$, with any two distinct vertices forming an edge if $[x : M][y : M]M = 0$. We prove that, if $\Gamma(M)$ contains a cycle, then $gr(\Gamma(M)) \leq 4$ and $\Gamma(M)$ has a connected induced subgraph $\bar{\Gamma}(M)$ with vertex set $\{m \in T(M)^* \mid \mathrm{Ann}(m)M \neq 0\}$ and $\mathrm{diam}(\bar{\Gamma}(M)) \leq 3$. Moreover, if M is a multiplication R-module, then $\bar{\Gamma}(M)$ is a maximal connected subgraph of $\Gamma(M)$. Also $\bar{\Gamma}(M)$ and $\bar{\Gamma}(S^{-1}M)$ are isomorphic graphs, where $S = R \setminus Z(M)$. Furthermore, we show that, if $\bar{\Gamma}(M)$ is uniquely complemented, then $S^{-1}M$ is a von Neumann regular module or $\bar{\Gamma}(M)$ is a star graph.

1. Introduction

In [11], Beck introduced and investigated the zero-divisor graph of a commutative ring. He let all elements of the ring be vertices of the graph. In [8], Anderson and Livingston introduced and studied a zero-divisor graph, whose vertices are non-zero zero-divisors while, $x{-}y$ is an edge whenever $xy = 0$. Since then, the concept of zero-divisor graphs has been studied extensively by many authors; see [2, 6, 7, 10]. The concept of a zero-divisor graph has been extended to non-commutative rings by Redmond [21]. This concept also has been introduced and studied for semigroups by DeMeyer, McKenzie and Schneider in [13], and for near-rings by Cannon et al, in [12]. For recent developments on graphs of commutative rings see [3, 4, 5, 17].

Let R be a commutative ring with identity element and M be a unitary R-module. In this research, we will investigate the concept of the torsion-graph of an R-module M, which has been defined in [16]. The torsion graph $\Gamma(M)$ of M is a simple graph, whose vertices are the non-zero torsion elements of M, and two distinct elements x, y are adjacent if and only if $[x : M][y : M]M = 0$. The residual of Rx by M, is denoted by $[x : M]$, is the set of elements $r \in R$

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such that $rM \subseteq Rx$ for $x \in M$. The annihilator of an R-module M, denoted by $\mathrm{Ann}_R(M)$, is $[0 : M]$.

An R-module M is called a multiplication module if for every submodule K of M there exists an ideal I of R such that $K = IM$ (Barnard [10]). A proper submodule N of M is called a prime submodule of M, whenever $rm \in N$ (where $r \in R$ and $m \in M$) implies that $m \in N$ or $r \in [N : M]$.

An R-module M is called a cancellation module if $IM = JM$ for any ideals I and J of R implies that $I = J$. Also, an R-module M is a weak-cancellation module if $IM = JM$ for any ideals I and J of R implies that $I + \mathrm{Ann}(M) = J + \mathrm{Ann}(M)$. Finitely generated multiplication modules are weak cancellation, Theorem 3 [1].

Let $T(M)$ be the set of elements of M such that $\mathrm{Ann}(m) \neq 0$. It is clear that if R is an integral domain, then $T(M)$ is a submodule of M, which is called the torsion submodule of M. If $T(M) = 0$, then the module M is said to be torsion-free, and it is called the torsion module if $T(M) = M$. Thus $\Gamma(M)$ is an empty graph if and only if M is a torsion-free R-module. We use the symbol $\bar{\Gamma}(M)$ to show the induced subgraph $\Gamma(M)$ with vertex set $\{m \in T(M)^* \mid \mathrm{Ann}(m)M \neq 0\}$. In this paper, we will also investigate the interplay of module properties of M in relation to the properties of $\bar{\Gamma}(M)$. We believe that this study helps to illuminate the structure of $T(M)$. For example, if M is a multiplication R-module, we show that M is finite if and only if $\bar{\Gamma}(M)$ is finite. Recall that a graph is finite if both its vertices set and edges set are finite. We know that a graph G is connected if there is a path between any two distinct vertices. The distance $d(x, y)$ between connected vertices x, y is the length of the shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of G is the diameter of a connected graph, which is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex. The girth of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G ($gr(G) = \infty$ if G contains no cycles).

A ring R is called reduced if $Nil(R) = 0$. An R-module M is called a reduced module if $rm = 0$ for $r \in R$ and $m \in M$, implies that $rM \cap Rm = 0$. Also a ring R is von Neumann regular if for each $a \in R$ there exists an element $b \in R$ such that $a = a^2 b$. It is clear that every von Neumann regular ring is reduced. Recall that a ring R is called Bézout if every finitely generated ideal I of R is principal. We know that every von Neumann regular ring is Bézout.

A submodule N of M is called a pure submodule of M if $IM \cap N = IN$ for every ideal I of R (Ribenboim in [22]). In [18], Kash (p. 105) states that an R-module M is called a von Neumann regular module if and only if every cyclic submodule of M is a direct summand in M. If N is a direct summand in M, then N is pure but not conversely (see [20], Example 2, p. 54 and [22], Example 14, p. 100). Therefore every von Neumann regular module is reduced.

A complete graph is a simple graph whose vertices are pairwise adjacent, and the complete graph with n vertices is denoted by K_n. A bipartite graph is one whose vertex set can be partitioned into two subsets so that no edge has both ends in the same subset. A complete bipartite graph is one in which each vertex is joined to every vertex that is not in the same subset; the complete bipartite graph, with two parts of sizes m and n, is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a star.

Let G be a graph and $V(G)$ denote the vertices of G. Let $v \in V(G)$, as in [7]; $w \in V(G)$ is called a complement of v, if v is adjacent to w and no vertex is adjacent to both v and w. That is, the edge $v - w$ is not an edge of any triangle in G. In this case, we write $v \perp w$. In module-theoretic terms, for multiplication R-module M, this is the same as saying $v \perp w$ in $\Gamma(M)$ if and only if $v, w \in T^*(M)$ and $\mathrm{Ann}(w)M \cap \mathrm{Ann}(v)M \subset \{0, v, w\}$. Moreover, we will follow the authors in [7] and say that G is complemented if every vertex has a complement, and it is uniquely complemented if it is complemented and any two complements of the vertices set are adjacent to the same vertices. From Theorems 3.5 and 3.9 [7], we know that for a ring R with non-zero nilpotent elements, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is a star graph. If R is reduced and $\Gamma(R)$ is complemented, then $S^{-1}R$ is a von Neumann regular ring, where $S = R \setminus Z(R)$.

In Section 2, we give an example of non-isomorphic modules with the same torsion graph. We show that $\bar{\Gamma}(M)$ is always connected with $\mathrm{diam}(\bar{\Gamma}(M)) \leq 3$. Furthermore, we prove that if $\Gamma(M)$ contains a cycle, then $gr(\Gamma(M) \leq 4$. In this manner, we study some of the properties of $\bar{\Gamma}(M)$, when M is a multiplication R-module. An R-module M is a multiplication module if for every submodule K of M there exists an ideal I of R such that $K = IM$. It is clear that if M is a multiplication R-module, then $\bar{\Gamma}(M)$ is a maximal connected subgraph of $\Gamma(M)$.

In Section 3, we obtain $\bar{\Gamma}(M) \cong \bar{\Gamma}(S^{-1}M)$, where $S = R \setminus Z(M)$ if M is an R-module such that $\mathrm{Ann}(x) = \mathrm{Ann}([x : M]M)$ for all $x \in T(M)$.

In Section 4, we investigate complemented and uniquely complemented torsion graphs. We also extend Theorem 3.9 of [7] to the multiplication R-modules. Furthermore, for a multiplication R-module M when R is Bézout or cyclic R-module and prove that if $\bar{\Gamma}(M)$ is uniquely complemented, then either $\bar{\Gamma}(M)$ is a star graph or $S^{-1}M$ is a von Neumann regular module, where $S = R \setminus Z(M)$.

Throughout the paper, we use the symbol (x, y) or $x + y$ to denote the elements of $M = M_1 \oplus M_2$ and $T(M)^* = T(M) \setminus \{0\}$. Also, we use the symbol $(M)_R$ to denote M as an R-module. Let $Z(M) := \{r \in R \mid rm = 0$ for some $0 \neq m \in M\}$. $Nil(R)$ is an ideal consisting of nilpotent elements of R,

$$Nil(M) := \cap_{N \in Spec(M)} N.$$

$Spec(M)$ is a set of the prime submodules of M, and for submodule N of M, $D(N) := \{n \in N \mid [n : M][n' : M]M = 0$ for some non-zero $n' \in M\}$. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

2. Properties of $\bar{\Gamma}(M)$

In this section, we show that $\bar{\Gamma}(M)$ is connected and has a small diameter and girth, and for a multiplication R-module M with $|M| \geq 5$, we prove that if $\bar{\Gamma}(M)$ is complete, then $Nil(M) = V(\Gamma(\bar{M})) \cup \{0\}$. We begin with the following example which shows that non-isomorphic modules may have the same torsion graphs.

Example 2.1. Let $M = M_1 \oplus M_2$ be an R-module, where M_1 is a torsion-free module. So $T(M)^* = \{(0, m_2) \mid m_2 \in T(M_2)^*\}$ and $[(0, m_2) : M] = 0$. Hence $\Gamma(M)$ is a complete graph. Let $M = \mathbb{Z} \oplus \mathbb{Z}_n$ be a $\mathbb{Z}$-module, so $\Gamma(M) = K_{n-1}$ for $n \geq 2$.

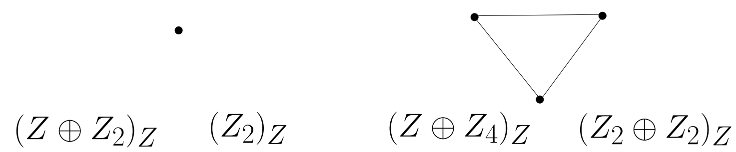


Figure 1

We know that $\Gamma(M)$ may be infinite (that is, the R-module M has infinitely torsion elements). An interesting case occurs when $\Gamma(M)$ is finite, because in the finite case a drawing of the graph is possible. The next theorem shows that for a multiplication R- module M, $\bar{\Gamma}(M)$ is finite (except when $\bar{\Gamma}(M)$ is empty) if and only if M is finite.

Theorem 2.2. *Let M be a multiplication R-module. Then $\bar{\Gamma}(M)$ is finite if and only if either M is finite or $V(\bar{\Gamma}(M)) = \emptyset$.*

Proof. Suppose that $\bar{\Gamma}(M)$ is finite and nonempty. Then there exists $x \in V(\bar{\Gamma}(M))$; let $N = Rx$ and $0 \neq sm \in Rx$, where $s \in [x : M]$ and $m \in M$, so $0 \neq \mathrm{Ann}(x)M \subseteq \mathrm{Ann}(n)M$ for all $n \in N$. Hence $N \subseteq V(\bar{\Gamma}(M))$. Therefore N is finite. Now if M is infinite, then there is an element $n \in N$ such that $H = \{m \in M \mid sm = n\}$ is infinite. For all distinct elements $m_1, m_2 \in H$, $sm \in \mathrm{Ann}(m_1 - m_2)M$. So $m_1 - m_2 \in V(\bar{\Gamma}(M))$ is a contradiction. Thus M is finite. $\square$

Corollary 2.3. *Let M be a multiplication R-module. Then $\Gamma(M)$ is finite if and only if either M is finite or M is a torsion-free R-module.*

Proof. If $\Gamma(M)$ is finite, then $\bar{\Gamma}(M)$ is finite. Therefore by Theorem 2.2 either M is finite or $V(\bar{\Gamma}(M)) = 0$. Thus for all $x \in M$, $\mathrm{Ann}(x)M = 0$; hence $\mathrm{Ann}(x) = \mathrm{Ann}(M)$ for all $x \in M$. Now if M is faithful, then M is torsion-free; otherwise $T(M) = M$. Consequently, either M is finite or M is a torsion-free R-module. $\square$

The following example shows that the multiplication condition is not superfluous.

Example 2.4. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}_n$. Clearly M is faithful and is not finite, but by Example 2.1, $\bar{\Gamma}(M) = K_{n-1}$ is finite.

Now we show that for all R-module M, $\bar{\Gamma}(M)$ is connected with diameter ≤ 3.

Theorem 2.5. *Let M be an R-module. Then*

$$\bar{\Gamma}(M) \text{ is connected with } \operatorname{diam}(\bar{\Gamma}(M)) \leq 3.$$

Moreover, if $\bar{\Gamma}(M)$ contains a cycle, then $gr(\bar{\Gamma}(M)) \leq 7$.

Proof. Let $x, y \in V(\bar{\Gamma}(M))$ be two distinct elements. If $[x : M]M$ or $[y : M]M$ or $[x : M][y : M]M$ is zero, then $d(x, y) = 1$. Therefore we suppose that $[x : M]M$, $[y : M]M$, and $[x : M][y : M]M$ is non-zero, so there are non-zero elements $\alpha \in [x : M][y : M]$ and $m \in M$ such that $\alpha m \neq 0$. If $[x : M]^2 = [y : M]^2 = 0$, then $\alpha m \in V(\bar{\Gamma}(M))$, and hence $x - \alpha m - y$ is a path of length 2. Hence suppose that $[x : M]^2 = 0$ and $[y : M]^2 \neq 0$; since $y \in V(\bar{\Gamma}(M))$, there exist non-zero elements $s \in \operatorname{Ann}(y)$ and $m_0 \in M$ such that $sm_0 \neq 0$. Now we consider the case $[x : M]\operatorname{Ann}(y)M = 0$. In this case $sm_0 \in V(\bar{\Gamma}(M))$, so $x - sm_0 - y$ is a path of length 2. In the other case, if $[x : M]\operatorname{Ann}(y)M \neq 0$, then $m_1 := \alpha_1 tm \in V(\bar{\Gamma}(M))$ for some non-zero elements $\alpha_1 \in [x : M], t \in \operatorname{Ann}(y), m \in M$, and $x - m_1 - y$ is a path of length 2. A similar argument holds if $[x : M]^2 \neq 0$, $[y : M]^2 = 0$. Thus we may assume that $[x : M]^2, [y : M]^2$ and $[x : M][y : M]$ are all non-zero. If $\operatorname{Ann}(x) \not\subseteq \operatorname{Ann}(y)$ and $\operatorname{Ann}(y) \not\subseteq \operatorname{Ann}(x)$, then there exist non-zero elements $r, s \in R$ such that $rx = 0, ry \neq 0$ and $sx \neq 0, sy = 0$, hence $ry, sx \in V(\bar{\Gamma}(M))$. Now if $ry \neq sx$, then $x - ry - sx - y$ is a path of length 3. In the other case, if $ry = sx$, then $x - ry - y$ is a path of length 2. Therefore $d(x, y) \leq 3$. Thus we may assume that $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(y)$ or $\operatorname{Ann}(y) \subseteq \operatorname{Ann}(x)$. If $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(y)$, then $rm \in V(\bar{\Gamma}(M))$ for some $r \in \operatorname{Ann}(x)$, $m \in M$ and $x - rm - y$ is a path of length 2. A similar argument holds if $\operatorname{Ann}(y) \subseteq \operatorname{Ann}(x)$. Hence $d(x, y) \leq 3$; thus $\operatorname{diam}(\Gamma(\bar{M})) \leq 3$. If $\bar{\Gamma}(M)$ contains a cycle, by Proposition 1.3 [14], then $gr(\bar{\Gamma}(M)) \leq 7$. □

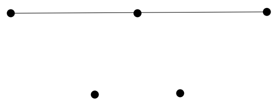
As an immediate consequence, we obtain the following result.

Corollary 2.6. *Let M be a faithful R-module. Then $\Gamma(M)$ is connected with* $\operatorname{diam}(\Gamma(M)) \leq 3$.

The following example shows that the faithful condition is not superfluous.

Example 2.7. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$; then $\Gamma(M)$ is not connected.

Proposition 1.3 [14] and Corollary 2.6 show that $gr(\Gamma(M)) \leq 7$, when $\Gamma(M)$ contains a cycle. We next improve this sentence to $gr(\Gamma(M)) \leq 4$.



$\Gamma(M)$

FIGURE 2

Theorem 2.8. *Let M be a multiplication R-module. If $\Gamma(M)$ contains a cycle, then $gr(\Gamma(M) \leq 4$.*

Proof. Let $m_0 - m_1 - m_2 - \cdots - m_n - m_0$ be the shortest cycle of $\Gamma(M)$ for $n \geq 4$. If $[m_1 : M][m_{n-1} : M]M = 0$, then $\Gamma(M)$ contains a cycle $m_1 - m_2 - \cdots - m_{n-1} - m_1$, which is a contradiction. So there exist non-zero elements $\alpha \in [m_1 : M]$, $\beta \in [m_{n-1} : M]$, and $m \in M$ such that $\alpha\beta m \in V(\Gamma(M))$. If $\alpha\beta m \neq m_0$ and $\alpha\beta m \neq m_n$, then $\Gamma(M)$ contains a cycle $m_0 - \alpha\beta m - m_n - m_0$ is a contradiction. Therefore $\alpha\beta m = m_0$ or $\alpha\beta m = m_n$. So without loss of generality, assume $\alpha\beta m = m_0$; thus $[m_0 : M]m_0 = 0$. Now we show that $Rm_0 = \{0, m_0\} \subset Rm_1$. If there exists a non-zero element $x \in Rm_0$ such that $x \neq m_0$ and $x \neq m_1$, then $m_0 - m_1 - x - m_0$ is a cycle of length 3, which is a contradiction. Hence $Rm_0 = \{0, m_0\} \subset Rm_1$. Therefore there exists $y \in Rm_1$ such that $y \neq 0$ and $y \neq m_1$. By a routine argument we obtain $y \neq m_0$ and $y \neq m_2$; therefore $m_0 - m_1 - m_2 - y - m_0$ is a cycle of length 4, which is a contradiction. Consequently, $gr(\Gamma(M) \leq 4$. □

Theorem 2.9. *Let M be a multiplication R-module. If $\bar{\Gamma}(M)$ is complete, then either $|M| = 4$ or $Nil(M) = V(\bar{\Gamma}(M)) \cup \{0\}$.*

Proof. First suppose that $[x : M]^2 M \neq 0$ for some $x \in V(\bar{\Gamma}(M))$, so $x \notin \mathrm{Ann}(x)M$. In this case, we show that $|M| = 4$. Put $N := \mathrm{Ann}(x)M$. We divide the proof of the theorem into 6 claims, which are of some interest in their own right.

Claim 1 : N is a prime submodule of M. Since $x \notin \mathrm{Ann}(x)M$, N is a proper submodule of M. Let $rm \in N$ and $m \notin N$; here r and m denote elements of R and M, respectively. Accordingly, $r[m : M][x : M]M = 0$, so $rkx = 0$ for all $k \in [m : M]$ and $r \in \mathrm{Ann}(kx)$. But there exists $k_0 \in [m : M]$ such that $k_0 x \in V(\bar{\Gamma}(M))$; consequently, $\mathrm{Ann}(k_0 x)M \subseteq \mathrm{Ann}(x)M$. Thus $rM \subseteq N$ and $r \in [N : M]$. Therefore N is a prime submodule, and as a consequence $[N : M]$ will be a prime ideal.

Claim 2 : $[x : M]M = [x : M]^2 M$. If $[x : M]M \neq [x : M]^2 M$, then $x \notin [x : M]^2 M$, so $x \neq \alpha x$ for all $\alpha \in [x : M]$. Since $\alpha x = 0$ or $\alpha x \in V(\bar{\Gamma}(M))$, we have αx adjacent to x. Therefore $\alpha^2 \in [N : M]$. We know that N is a prime submodule, so $\alpha \in [N : M]$ for all $\alpha \in [x : M]$. Thus $[x : M]M \subseteq N$, which is a contradiction with $x \notin \mathrm{Ann}(x)M$. Therefore $[x : M]M = [x : M]^2 M$.

Claim 3 : $M = Rx \oplus M_2$. Since $[x : M]M = [x : M]^2 M$, we have $Rx = [x : M]x$. We know that Rx is a weak-cancellation R-module, and so $R =$

$[x : M] + \mathrm{Ann}(x)$. A simple check yields $M = Rx \oplus \mathrm{Ann}(x)M$. Hence we may assume that $M = Rx \oplus M_2$ with x adjacent to every other vertex and $M_2 = \mathrm{Ann}(x)M$.

Claim 4 : $Rx = \{0, x\}$. Let $x \neq c \in Rx$. Then $c \in V(\bar{\Gamma}(M))$ and $[c : M][x : M]M = 0$; hence $[c : M]x = 0$, so $c \in Rx \cap M_2 = \{0\}$.

Claim 5 : $D(M_2) = 0$. Let $D(M_2) \neq 0$. Then there exists a non-zero element $m_2 \in M_2$ such that $[m_2 : M][m_2' : M]M = 0$ for some $0 \neq m_2' \in M_2$. Thus $x + m_2'$ is a vertex of $\bar{\Gamma}(M)$, which is adjacent to x. Therefore $[x : M](x + m_2') = 0$, so

$$[x : M]x = [x : M]m_2'.$$

Thus $[x : M]x \subseteq Rx \cap M_2 = \{0\}$. Hence $x \in \mathrm{Ann}(x)M$, which is a contradiction; consequently, $D(M_2) = 0$.

Claim 6 : $M_2 = \{0, y\}$. Since $D(M_2) = 0$, we have $[y : M]y \neq 0$. On the other hand, $0 \neq x \in \mathrm{Ann}(y)M$, so $y \in V(\bar{\Gamma}(M))$. From the above argument we have $[y : M]^2M = [y : M]M$. Therefore

$$Ry \subseteq [y : M]y \subseteq (\mathrm{Ann}(x) \cap [y : M_2])y \subseteq [y : M_2]y.$$

Hence $Ry = [y : M_2]y$ and $y = sy$ for some $s \in [y : M_2]$. Let $m_2 \in M_2$, so

$$[y : M][(1 - s)m_2 : M]M = 0.$$

Thus $y = 0$ or $m_2 = sm_2 \in Ry$. Hence $M_2 = Ry$. Let $m_2 \in M_2$ and $m_2 \neq y$, so $m_2 \in V(\bar{\Gamma}(M))$ and $[m_2 : M][y : M]M = 0$. Therefore $m_2 = 0$ and Ry has exactly two elements. Consequently, $|M| = 4$.

Next, we may assume that $[x : M]^2M = 0$ for all $x \in V(\bar{\Gamma}(M))$. So $x \in Nil(M)$ and $V(\bar{\Gamma}(M)) \subseteq Nil(M)$. Now let $0 \neq x \in Nil(M)$. We can write $x = \sum_{i=1}^{n} \alpha_i m_i$, where $\alpha_i \in [x : M], m_i \in M$ such that $\alpha_i m_i \neq 0$ for $1 \leq i \leq n$. On the other hand, $\alpha_i^2 m_i = 0$, so $0 \neq \alpha_i m_i \in \mathrm{Ann}(\alpha_i m_i)M$. Therefore $\alpha_i m_i \in V(\bar{\Gamma}(M))$. One can easily check that $V(\bar{\Gamma}(M))$ is a submodule of M; hence $x \subseteq V(\bar{\Gamma}(M))$. Consequently, $Nil(M) = V(\bar{\Gamma}(M)) \cup \{0\}$. □

Example 2.10. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^2}$, where $p > 2$ is a prime number. It is clear that $\bar{\Gamma}(M) = K_{p-1}$ is a complete graph. So by Theorem 2.9, $Nil(M) = (\bar{p})$.

3. Isomorphisms

Recall that two graphs G and H are isomorphic, denoted by $G \cong H$, whenever there exists a bijection, say φ from $V(G)$ to $V(H)$, of vertices such that the vertices x and y are adjacent in G if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in H.

Let $S = R \setminus Z(M)$. It is clear that the well-defined map

$$\chi : M \longrightarrow S^{-1}M,$$

defined by

$$\chi(m) = \frac{ms}{s},$$

is a monomorphism. So we can identify M with its image in $S^{-1}M$. Thus if m denotes an element of M, then the same symbol is also used to denote the fraction $\frac{m}{1}$. In this manner, M become a submodule of $S^{-1}M$.

Let M be an R-module. For $m, m' \in V(\bar{\Gamma}(M))$, we define $m \sim_M m'$ if and only if $\operatorname{Ann}(m) = \operatorname{Ann}(m')$. Clearly $\sim_M$ is an equivalence relation on $V(\bar{\Gamma}(M))$. Let $S = R \setminus Z(M)$. For $m \in M$, denote the equivalence classes of $\sim_M$ and $\sim_{M_S}$ containing m, $\frac{m}{1}$ by $[m]_M$ and $[\frac{m}{1}]_{M_S}$, respectively, so

$$[m]_M = \{m' \in V(\bar{\Gamma}(M)) \mid m \sim_M m'\},$$

and let

$$[\frac{m}{1}]_{M_S} = \left\{\frac{m'}{t'} \in V(\bar{\Gamma}(M_S)) \mid \frac{m'}{t'} \sim_{M_S} \frac{m}{1}\right\}.$$

Next, we prove that $\bar{\Gamma}(S^{-1}M)$ and $\bar{\Gamma}(M)$ are isomorphic by showing that there is a bijection map between equivalence classes of vertex sets $\bar{\Gamma}(S^{-1}M)$ and $\bar{\Gamma}(M)$ such that the corresponding equivalence classes have the same cardinality.

Theorem 3.1. *Let M be an R-module such that* $\operatorname{Ann}(x) = \operatorname{Ann}([x : M]M)$ *for all $x \in T(M)$ and $S = R \setminus Z(M)$. Then $\bar{\Gamma}(M)$ and $\bar{\Gamma}(S^{-1}M)$ are isomorphic.*

Proof. (Our proof is quite similar to the proof in [7] applied for a ring) Let $S = R \setminus Z(M)$, $M_S = S^{-1}M$, $R_S = S^{-1}R$. A simple check yields that for all $N \leq M$, we have $S^{-1}\operatorname{Ann}_R(N) = \operatorname{Ann}_{S^{-1}R}(S^{-1}N)$. Hence

$$V(\bar{\Gamma}(M_S)) = \left\{\frac{m}{s} \mid m \in V(\bar{\Gamma}(M)), s \in S\right\},$$

and $([m]_M)_S = ([\frac{m}{1}])_{M_S}$. On the other hand,

$$V(\bar{\Gamma}(M)) = \bigcup_{\lambda \in \Lambda} [m_\lambda]_M, \text{ so } V(\bar{\Gamma}(M_S)) = \bigcup_{\lambda \in \Lambda} [\frac{m_\lambda}{1}]_{M_S}$$

(both are disjoint unions). Next we show that $|[x]_M| = |[\frac{x}{1}]_{M_S}|$ for all $x \in V(\bar{\Gamma}(M))$. It is clear that $[x]_M \subseteq [\frac{x}{1}]_{M_S}$. For the reverse inclusion, assume $\frac{m}{s} \in [\frac{x}{1}]_{M_S}$. We can suppose that $m \in [x]_M, s \in S$, so $\operatorname{Ann}(m) = \operatorname{Ann}(x)$. Therefore $\{s^n m \mid n \geq 1\} \subseteq [x]_M$. If $|[x]_M|$ is finite, then there exists $i \in I$ such that $s^i m = s^{i+1} m$. So

$$\frac{m}{s} = \frac{ms^i}{s^{i+1}} = \frac{ms^{i+1}}{s^{i+1}} = m \in [x]_M;$$

therefore $|[x]_M| = |[\frac{x}{1}]_{M_S}|$. Now suppose that $|[x]_M|$ is infinite. We define an equivalence relation $\approx$ on S by $s \approx t$ if and only if $sx = tx$. It is easily verified that the map

$$[x]_M \times S/\approx \longrightarrow [\frac{x}{1}]_{M_S}$$

$$(b, [s]) \longrightarrow \frac{b}{s},$$

is well-defined, because if $(b, [s]) = (a, [t])$, then $a = b$ and $[s] = [t]$. Hence

$$(s - t)M \subseteq \operatorname{Ann}(x)M = \operatorname{Ann}(a)M = \operatorname{Ann}(b)M;$$

by the hypothesis $sa = ta$ and $sb = tb$, therefore $\frac{a}{t} = \frac{b}{s}$. Also, it is clear that this map is surjective. Thus

$$|[\frac{x}{1}]| \leq |[x]_M||S/\approx|.$$

The map

$$S/\approx\longrightarrow [x]_M$$
$$[s] \longrightarrow sa.$$

Clearly, it is well-defined and injective. Hence $|S/\approx| \leq |[x]_M|$, and

$$|[\frac{x}{1}]_{M_S}| \leq |[x]_M|^2 = |[x]_M|,$$

since $|[x]_M|$ is infinite, $|[x]_M| = |[\frac{x}{1}]_{M_S}|$. Thus there is a bijection map

$$\varphi_\alpha : [x_\alpha] \longrightarrow [\frac{x_\alpha}{1}]$$

for each $\alpha \in \Lambda$. Therefore we define

$$\varphi : V(\bar{\Gamma}(M)) \longrightarrow V(\bar{\Gamma}(M_S))$$

by $\varphi(m) = \varphi_\alpha(m)$, if $m \in [x_\alpha]_M$. Clearly, φ is a bijection map. Thus we need only to show that m and n are adjacent in $\Gamma(M)$ if and only if $\varphi(m)$ and $\varphi(n)$ are adjacent in $\Gamma(M_S)$; that is, $[m : M][n : M]M = 0$ if and only if $[\varphi(m) : M_S][\varphi(n) : M_S]M_S = 0$. Let $m \in [x]_M$, $n \in [y]_M$, $w \in [\frac{x}{1}]_{M_S}$, and $z \in [\frac{y}{1}]_{M_S}$. It is sufficient to show that $[m : M][n : M]M = 0$ if and only if $[w : M_S][z : M_S]M_S = 0$. If m is adjacent to n, then

$$\begin{aligned}
&[m : M][n : M]M = 0\\
\Longrightarrow &[m : M] \subseteq \mathrm{Ann}_R(n) = \mathrm{Ann}_R(y)\\
\Longrightarrow &[m : M]_S \subseteq \mathrm{Ann}_{R_S}(\tfrac{y}{1}) = \mathrm{Ann}_{R_S}(z)\\
\Longrightarrow &[z : M_S] \subseteq \mathrm{Ann}_{R_S}(([m : M]M)_S)\\
\Longrightarrow &[z : M_S] \subseteq \mathrm{Ann}(\tfrac{m}{1}) = \mathrm{Ann}(\tfrac{x}{1}) = \mathrm{Ann}(w)\\
\Longrightarrow &[z : M_S][w : M_S]M_S = 0.
\end{aligned}$$

Conversely, if z is adjacent to w, then

$$\begin{aligned}
&[z : M_S][w : M_S]M_S = 0\\
\Longrightarrow &[z : M_S] \subseteq \mathrm{Ann}_{R_S}([w : M_S]M_S) \subseteq \mathrm{Ann}_{R_S}(w)\\
\Longrightarrow &[z : M_S] \subseteq \mathrm{Ann}_{R_S}(w) = \mathrm{Ann}_{R_S}(\tfrac{x}{1}) = \mathrm{Ann}_{R_S}(\tfrac{m}{1})\\
\Longrightarrow &[z : M_S][\tfrac{x}{1} : M_S]M_S = 0,
\end{aligned}$$

implies that

$$\begin{aligned}
&[\frac{m}{1} : M_S] \subseteq \mathrm{Ann}_{R_S}([z : M_S]M_S) \subseteq \mathrm{Ann}_{R_S}(z) = \mathrm{Ann}_{R_S}(\frac{y}{1}) = \mathrm{Ann}_{R_S}(\frac{n}{1})\\
\Longrightarrow &[\frac{n}{1} : M_S][\frac{m}{1} : M_S]M_S = 0\\
\Longrightarrow &[m : M][n : M]M = 0,
\end{aligned}$$

hence $\bar{\Gamma}(M)$ and $\bar{\Gamma}(M_S)$ are isomorphic graphs. □

Theorem 3.2. *Let M be a multiplication R-module and $S = R \setminus Z(M)$. Then $\Gamma(M)$ and $\Gamma(S^{-1}M)$ are isomorphic.*

Proof. It is similar to the proof of Theorem 3.1. □

Corollary 3.3. *Let M and N be multiplication R-modules with $S^{-1}M \cong S^{-1}N$. Then $\Gamma(M) \cong \Gamma(N)$. In particular, $\Gamma(M) \cong \Gamma(N)$ when $S^{-1}M = S^{-1}N$.*

4. Complemented graph $\bar{\Gamma}(M)$ and multiplication module

In this section we prove that, if M is a reduced multiplication R-module and $\bar{\Gamma}(M)$ is uniquely complemented, $S^{-1}M$ is von Neumann regular. Furthermore, we show that if M is a multiplication R-module with $Nil(M) \neq 0$, then $\Gamma(M)$ is uniquely complemented if and only if $\bar{\Gamma}(M)$ is a star graph such that $\bar{\Gamma}(M)$ has at most six edges or is an infinite star graph. Finally, we show that if M is a multiplication R-module, and $\bar{\Gamma}(M)$ is uniquely complemented, then either $\bar{\Gamma}(M)$ is a star graph or $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.

Let G be a (undirected) graph. We will follow the authors in [6] and define that $a \leq b$ if a and b are not adjacent and each vertex of G adjacent to b is also adjacent to a; we define $a \sim b$ if and only if $a \leq b$ and $b \leq a$. Thus $a \sim b$ if and only if a and b are adjacent to exactly the same vertices. Clearly $\sim$ is an equivalence relation on G. Let M be a multiplication R-module and $m, n \in T(M)^*$; then $m \sim n$ if and only if $\mathrm{Ann}(m)M \setminus \{m\} = \mathrm{Ann}(n)M \setminus \{n\}$. We also know that if $m \perp n$, then $[m : M][n : M]M = 0$ and $\mathrm{Ann}(m)M \cap \mathrm{Ann}(n)M \subseteq \{0, m, n\}$. Now if $\mathrm{Ann}(m)M \cap \mathrm{Ann}(n)M = \{0, m, n\}$, then $[m : M]^2M = [n : M]^2M = [m : M][n : M]M = 0$ and so $m + n$ is adjacent to m and n, since $m \perp n$, $m + n \in \{0, m, n\}$, which is a contradiction. Therefore $m \perp n$ if and only if $\mathrm{Ann}(m)M \cap \mathrm{Ann}(n)M \subset \{0, m, n\}$ and $[m : M][n : M]M = 0$.

Proposition 4.1. *Let M be a multiplication R-module. Then M is von Neumann regular if and only if every cyclic submodule of M is pure in M.*

Proof. Let every cyclic submodule of M be pure in M. Hence $Rm = [m : M]m$ for all $m \in M$ and so $m = \alpha m$ for some $\alpha \in [m : M]$. Therefore, $M = \mathrm{Ann}(m)M + Rm$ so that $1 \in \mathrm{Ann}(m) + [m : M]$. On the other hand, if $x \in \mathrm{Ann}(m)M \cap Rm$, then $x = sm = rm_0$ for some $r \in \mathrm{Ann}(m)$ and $s \in R$. Thus $\alpha x = s\alpha m = r\alpha m_0 = 0$, so $x = 0$. This implies that $M = \mathrm{Ann}(m)M \oplus Rm$. Thus M is von Neumann regular. The converse is obvious. □

Lemma 4.2. *Consider the following statements for a multiplication R-module M with $m, m' \in T(M)^*$.*

(a) $m \sim m'$,
(b) $Rm = Rm'$,
(c) $\mathrm{Ann}(m)M = \mathrm{Ann}(m')M$.

Then under the above conditions, we have:

(1) *If M is reduced, then statements* (a) *and* (c) *are equivalent.*
(2) *If M is von Neumann regular, then all three statements are equivalent.*

Proof. (1) Let M be reduced; one can easily check that (a)$\Longleftrightarrow$(c).

(2) (a)$\Longleftrightarrow$(c); since every von Neumann regular module is reduced.

(b)$\Longrightarrow$(c); this implication is clear.

(c)$\Longrightarrow$(b); Since M is von Neumann regular, $Rm \cap [m : M]M = [m : M]Rm$. So $m = sm$ for some $s \in [m : M]$, hence; $(1-s)m' \in \mathrm{Ann}(m)M = \mathrm{Ann}(m')M$. Therefore $[m' : M]m' \in Rm$. Moreover, since M is a von Neumann regular multiplication, module $[m' : M]m' = Rm'$. So $Rm' \subseteq Rm$ and, similarly, $Rm \subseteq Rm'$; consequently, $Rm = Rm'$. □

Lemma 4.3. *Let M be a reduced multiplication R-module with $m, m', m'' \in V(\bar{\Gamma}(M))$. If $m \perp m'$ and $m \perp m''$, then $m' \sim m''$. Thus $\bar{\Gamma}(M)$ is uniquely complemented if and only if $\bar{\Gamma}(M)$ is complemented.*

Proof. Let $m, m', m'' \in \bar{\Gamma}(M)$. Suppose $m \perp m'$ and $m \perp m''$. It is sufficient to show that $\mathrm{Ann}(m')M = \mathrm{Ann}(m'')M$. Suppose $x \in \mathrm{Ann}(m')M$, so $[x : M][m' : M]M = 0$. One can easily show that for all $\alpha \in [x : M]$,

$$[\alpha m'' : M][m' : M]M = 0 = [\alpha m'' : M][m : M]M.$$

So $\alpha m'' \in \{0, m, m'\}$. If $\alpha m'' = m$ or $\alpha m'' = m'$, then $m = 0$ or $m' = 0$ is a contradiction. Thus $\alpha m'' = 0$ for all $\alpha \in [x : M]$, and therefore $x \in \mathrm{Ann}(m'')M$ and $\mathrm{Ann}(m')M \subseteq \mathrm{Ann}(m'')M$. Similarly, $\mathrm{Ann}(m'')M \subseteq \mathrm{Ann}(m')M$. □

As an immediate consequence, we obtain the following result.

Corollary 4.4. *Let M be a reduced multiplication R-module with $m, m', m'' \in T(M)^*$. If $m \perp m'$ and $m \perp m''$, then $m' \sim m''$. Thus $\Gamma(M)$ is uniquely complemented if and only if $\Gamma(M)$ is complemented.*

Theorem 4.5. *Let R be a Bézout ring and M be a reduced multiplication R-module. If $\bar{\Gamma}(M)$ is complemented, then $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*

Proof. Let $0 \neq \frac{x}{s} \in S^{-1}M$, where $x \in M$ and $s \in S$. Let $x \notin V(\bar{\Gamma}(M))$ and $x = \sum_{i=1}^{n} \alpha_i m_i \in [x : M]M$, where $\alpha_i \in [x : M]$ and $m_i \in M$. Since R is a Bézout ring $\sum_{i=1}^{n} R\alpha_i = R\alpha$ for some $\alpha \in R$. So $x = \alpha m$ for some $\alpha \in M$. If $\alpha \in Z(M)$, then $\alpha m_0 = 0$ for some non-zero element $m \in M$. So $[m_0 : M][x : M]M = 0$; hence $0 \neq m_0 \subseteq \mathrm{Ann}(x)M = 0$, which is a contradiction. Therefore $\alpha \in S = R \setminus Z(M)$. Thus one can easily check that

$$S^{-1}R(\frac{x}{s}) \cap S^{-1}M(\frac{r}{t}) = S^{-1}R(\frac{r}{t}\frac{x}{s}).$$

Therefore by Proposition 4.1, $S^{-1}M$ is von Neumann regular.
Next we assume that $x \in V(\bar{\Gamma}(M))$. By the hypothesis there is $y \in V(\bar{\Gamma}(M))$ such that $x \perp y$. Hence $y \in \mathrm{Ann}(x)M$ and so $y = \sum_{i=1}^{m} \beta_i m_i$, $m_i \in M$ and $\beta_i \in \mathrm{Ann}(x)$. Let $R\beta = \sum_{i=1}^{m} R\beta_i$ for some $\beta \in R$, so $y = \beta m'$ for some

$m' \in M$. We show that $\alpha + \beta \in S$. If $\alpha + \beta \in Z(M)$, then $(\alpha + \beta)m_1 = 0$ for some non-zero $m_1 \in M$. So $[\alpha m_1 : M][x : M]M = 0 = [y : M][\alpha m_1 : M]M$. Since M is a reduced module $x \neq \alpha m_1$ and $y \neq \alpha m_1$. Thus $\alpha m_1 = 0$, and hence $\beta m_1 = 0$. So

$$[x : M][m_1 : M]M = 0 = [y : M][m_1 : M]M.$$

By a similar argument we have $m_1 = 0$, a contradiction. Therefore $\alpha + \beta \in S$ and $\frac{x}{s} = \frac{\alpha}{\alpha+\beta}\frac{x}{s}$. A simple check yields that

$$S^{-1}R(\frac{x}{s}) \cap S^{-1}M(\frac{r}{t}) = S^{-1}R(\frac{r}{t}\frac{x}{s}).$$

Hence by Proposition 4.1, $S^{-1}M$ is von Neumann regular. □

Lemma 4.6. *Let R be a von Neumann regular ring. Then every multiplication R- module is reduced.*

Proof. Lemma 2.5 of [19]. □

As an immediate consequence, we obtain the following result.

Corollary 4.7. *Let R be a von Neumann regular ring and M be a multiplication R-module. If $\bar{\Gamma}(M)$ is complemented, then $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*

Corollary 4.8. *Let M be a reduced cyclic R-module. If $\bar{\Gamma}(M)$ is complemented, then $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*

Proof. It is similar to the proof of Theorem 4.5. □

Example 4.9. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Clearly M is reduced, and by Example 2.7, $\bar{\Gamma}(M)$ is complemented. So by Corollary 4.8, $S^{-1}M$ is von Neumann regular.

Lemma 4.10. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{pq}$ where p and q are distinct prime numbers. Then $\bar{\Gamma}(M)$ is complete bipartite.*

Proof. Let $x \in V(\bar{\Gamma}(M))$, so either $x = tp$ or $x = sq$ for some $s, q \in R$. Therefore $\bar{\Gamma}(M)$ may be partitioned into two disjoint vertex sets A and B, where $A = \{tp \mid t \in R, tp < n\}$ and $B = \{sq \mid s \in R \; sq < n\}$, and so $\bar{\Gamma}(M)$ is a complete bipartite graph. □

Corollary 4.11. *Let M be a cyclic reduced R-module. The following statements are equivalent*:

(1) *$S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*
(2) *$\bar{\Gamma}(M)$ is uniquely complemented.*
(3) *$\bar{\Gamma}(M)$ is complemented.*

Proof. (1) $\Longrightarrow$ (2). Let M be a von Neumann regular R-module and $m \in V(\bar{\Gamma}(M))$. So $[m : M]M \cap Rm = Rm[m : M]$. Since Rm is a weak-cancellation module, $R = [m : M] + \mathrm{Ann}(m)$. Say $M := Rx$ for some $x \in M$. Thus $Rx = Rm + \mathrm{Ann}(m)x$ and therefore $x = rm + y$ for some $r \in R, y \in \mathrm{Ann}(m)x$. One can easily check that $y \in V(\bar{\Gamma}(M))$ and $y \perp m$, so $\bar{\Gamma}(M)$ is complemented. Since M is a cyclic R-module, then $S^{-1}M$ is a cyclic $S^{-1}R$-module, and therefore by the above comments, $\bar{\Gamma}(S^{-1}M)$ is complemented. Moreover, by Theorem 3.2 $\bar{\Gamma}(M) \cong \bar{\Gamma}(S^{-1}M)$, so $\bar{\Gamma}(M)$ is complemented. Consequently, $\bar{\Gamma}(M)$ is uniquely complemented by Lemma 4.3.

(2) $\Longrightarrow$ (3). This is true for any graph.

(3) $\Longrightarrow$ (1). By Corollary 4.8. □

Corollary 4.12. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p_1 p_2 \dots p_n}$, where p_i, $1 \le i \le n$ are distinct prime numbers. Then $\bar{\Gamma}(M)$ is uniquely complemented and $S^{-1}M$ is von Neumann regular.*

Proof. Let $n = 3$ and $x \in V(\bar{\Gamma}(M))$. So there exist $i, j, k \in \{1, 2, 3\}$ such that either $x = t_i p_i$, where $t_i \in \mathbb{Z}$ and p_j is not divisible by t_i for $i \neq j$, or $x = s_i p_k p_j$ where $s_i \in \mathbb{Z}$ and p_j is not divisible by s_i for $i \neq j, i \neq k$. A routine argument shows that $x = t_i p_i \perp p_k p_j$ and $x = s_i p_k p_j \perp p_i$ for distinct i, j, k. Therefore by a similar argument we can show that $\bar{\Gamma}(M)$ is complemented, and by Corollary 4.11 $\bar{\Gamma}(M)$ is uniquely complemented and $S^{-1}M$ is von Neumann regular. □

The next example shows that $S^{-1}M$ is von Neumann regular, while M is not von Neumann regular in spite of $\Gamma(M) \cong \Gamma(S^{-1}M)$.

Example 4.13. (a) Let M_1 be an R_1-module and M_2 an R_2-module; then $M = M_1 \times M_2$ is an $R = R_1 \times R_2$ module with this multiplication $R \times M \longrightarrow M$, defined by $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$.

Now let $M = \mathbb{Z} \times n\mathbb{Z}$ and $R = \mathbb{Z} \times \mathbb{Z}$. Therefore the graph $\Gamma(M)$ is a complete bipartite graph (that is, $\Gamma(M)$ may be partitioned into two disjoint vertex sets, $V_1 = \{(m_1, 0) \mid m_1 \in (\mathbb{Z})^*\}$ and $V_2 = \{(0, m_2) \mid m_2 \in (n\mathbb{Z})^*\}$, and two vertices x and y are adjacent if and only if they are in distinct vertex sets). Therefore $\Gamma(M)$ is complemented. Also, M is a faithful multiplication R-module, since $M = R(1, n)$. A simple check yields that M is reduced. Thus $S^{-1}M$ is von Neumann regular by Corollary 4.8. But M is not von Neumann regular (use $N = R(2, 2n)$ and $I = [N : M]$).

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}$ and $M = R$ as an R-module. So M is a faithful multiplication R-module. Clearly, M is reduced and $\Gamma(M)$ is an infinite star graph with center $(\bar{1}, 0)$. Thus $\Gamma(M)$ is complemented; by Corollary 4.8, $S^{-1}M$ is von Neumann regular, but M is not von Neumann regular.

Lemma 4.14. *Let M be a multiplication R-module; if $x \in Nil(M)$, then there exists $n \in \mathbb{N}$ such that $\alpha^n x = 0$ for all $\alpha \in [x : M]$.*

Proof. By the proof of Lemma 3.7 Step (1) of [16]. □

Proposition 4.15. *Let M be a multiplication R-module with $Nil(M) \neq 0$. Then*

(a) *If $\bar{\Gamma}(M)$ is complemented, then either $|M| \leq 16$ or $|M| > 16$ and $Nil(M) = \{0, x\}$ for some $0 \neq x \in M$.*

(b) *If $\bar{\Gamma}(M)$ is uniquely complemented with $|M| > 16$, then any complement of the non-zero $x \in Nil(M)$ is an end.*

Proof. (a) We subdivide the proof of (a) into the following steps:

Let $\bar{\Gamma}(M)$ be complemented and $x \in Nil(M)$. Assume that $\alpha \in [x : M]$, by Lemma 4.14 $\alpha^n x = 0$ for some $n \in \mathbb{N}$. Choose n to be as small as possible, $\alpha^n x = 0$. Then $n \geq 1$ and $\alpha^{n-1} x \neq 0$.

Step 1: In this step we claim that $n \leq 3$. Suppose that $n > 3$, so $\alpha x \in V(\bar{\Gamma}(M))$. Since $\bar{\Gamma}(M)$ is complemented, there is a $y \in V(\bar{\Gamma}(M)$ such that y is a complement of αx. Then

$$[\alpha^{n-1}x : M][y : M]M = 0 = [\alpha^{n-1}x : M][\alpha x : M]M,$$

so $\alpha^{n-1}x = y$ will be the only possibility. Thus $\alpha x \perp \alpha^{n-1}x$. Similarly, $\alpha^i x \perp \alpha^{n-1}x$ for each $1 \leq i \leq n-2$. Let $m = \alpha^{n-2}x + \alpha^{n-1}x$. Then

$$[m : M][\alpha^{n-1}x : M]M = 0 = [m : M][\alpha^{n-2}x : M]M,$$

which is a contradiction, since $\alpha^{n-2}x \perp \alpha^{n-1}x$ and $\alpha^{n-2}x+\alpha^{n-1}x \notin \{0, \alpha^{n-1}x, \alpha^{n-2}x\}$. Thus $n \leq 3$.

Step 2: Let $n = 3$, so $\alpha^3 x = 0$ but $\alpha^2 x \neq 0$. We show that $|M| \leq 16$. Similar to step 1, $\alpha x \perp \alpha^2 x$. Also, $\mathrm{Ann}(x)M \subseteq \{0, \alpha^2 x\}$, since if $z \in \mathrm{Ann}(x)M$, then $[z : M][x : M]M = 0$; hence z is adjacent to the two elements αx and $\alpha^2 x$. Therefore $z = \alpha^2 x$, so $\mathrm{Ann}(x)M \subseteq \{0, \alpha^2 x\}$. In this case $R\alpha^2 x = \{0, \alpha^2 x\}$, because for all $r \in R$,

$$[r\alpha^2 x : M][\alpha x : M]M = 0 = [r\alpha^2 x : M][\alpha^2 x : M]M;$$

hence $r\alpha^2 x \in \{0, \alpha x, \alpha^2 x\}$. But if $r\alpha^2 x = \alpha x$, then $\alpha^2 x = 0$ is a contradiction, and so $R\alpha^2 x = \{0, \alpha^2 x\}$. Also,

$$\mathrm{Ann}(\alpha^2 x)M \subseteq \{0, x, \alpha x, \alpha^2 x, x + \alpha x, x + \alpha^2 x, \alpha x + \alpha^2 x, x + \alpha x + \alpha^2 x\},$$

since if $z \in \mathrm{Ann}(\alpha^2 x)M$, then $\alpha^2 z \in \mathrm{Ann}(x)M = \{0, \alpha^2 x\}$ and either $\alpha^2 z = 0$ or $\alpha^2 z = \alpha^2 x$. Thus either

$$[\alpha z : M][\alpha x : M]M = 0 = [\alpha z : M][\alpha^2 x : M]M$$

or

$$[(\alpha z - \alpha x) : M][\alpha x : M]M = 0 = [(\alpha z - \alpha x) : M][\alpha^2 x : M]M.$$

Since $\alpha x \perp \alpha^2 x$, we have either $\alpha z \in \{0, \alpha x, \alpha^2 x\}$ or $(\alpha z - \alpha x) \in \{0, \alpha x, \alpha^2 x\}$. Now let $\alpha^2 z = 0$, so $\alpha z \neq \alpha x$; therefore either $\alpha z = 0$ or $\alpha(z - \alpha x) = 0$. So

$$[z : M][\alpha x : M]M = 0 = [z : M][\alpha^2 x : M]M$$

or

$$[(z - \alpha x) : M][\alpha x : M]M = 0 = [(z - \alpha x) : M][\alpha^2 x : M]M;$$

hence $z \in \{0, \alpha x, \alpha^2 x, \alpha^2 x + \alpha x\}$. Thus we may assume that $\alpha^2 z = \alpha^2 x$; then $\alpha z - \alpha x \neq \alpha x$. On the other hand, $\alpha z - \alpha x \in \{0, \alpha x, \alpha^2 x\}$, so either $\alpha z - \alpha x = 0$ or $(\alpha z - \alpha x) = \alpha^2 x$, and by a similar argument, $z \in \{x, \alpha^2 x, x + \alpha x, x + \alpha x + \alpha^2 x\}$. Consequently,

$$\text{Ann}(\alpha^2 x)M \subseteq \{0, x, \alpha x, \alpha^2 x, x + \alpha x, x + \alpha^2 x, \alpha x + \alpha^2 x, x + \alpha x + \alpha^2 x\}.$$

Since $\alpha^2[x : M]M \neq 0$, there are $\gamma \in [x : M]$ and $m \in M$ such that $\alpha^2 \gamma m \neq 0$, and a simple check yields $\alpha^2 \gamma m = \alpha^2 x$. Let $m_0 \in M$, so $\alpha^2 \gamma m_0 \in R\alpha^2 x = \{0, \alpha^2 x\}$. If $\alpha^2 \gamma m_0 = 0$, then $m_0 \in \text{Ann}(\alpha^2 x)M$, and if $\alpha^2 \gamma m_0 = \alpha^2 x$, then $m_0 - m \in \text{Ann}(\alpha^2 x)$. Consequently, $|M| \leq 16$.

Step 3: In this step we show that $H = \text{Ann}(\alpha^2 x)M$ is a unique maximal submodule of M. Clearly, $H \neq M$ and $R\alpha^2 x \cong \frac{R}{\text{Ann}(\alpha^2 x)}$. Since $R\alpha^2 x = \{0, \alpha^2 x\}$, $\text{Ann}(\alpha^2 x)$ is a maximal ideal of R. Hence by Theorem 2.5 [15], $\text{Ann}(\alpha^2 x)M$ is a maximal submodule. Also,

$$\text{Ann}(\alpha^2 x)M \subseteq Rx \subseteq Nil(M) \subseteq \text{Ann}(\alpha^2 x)M.$$

Therefore $\text{Ann}(\alpha^2 x)M = Nil(M)$ is a unique maximal submodule of M. Also, a simple check yields that $Rm \neq M$ for all $m \in V(\bar{\Gamma}(M))$. Therefore by Theorem 2.5 [15] $Rm \subseteq H$, so $V(\bar{\Gamma}(M)) \subseteq H$. So $V(\bar{\Gamma}(M)) = \text{Ann}(\alpha^2 x)M$, so $\bar{\Gamma}(M)$ is a star graph with center $\alpha^2 x$ and at most 6 edges.

Step (4): Assume that $n = 2$; we show that $[x : M]^2 x = 0$. Let $[x : M]^2 x \neq 0$. There exist two elements $\alpha, \beta \in [x : M]$ such that $\alpha\beta x \neq 0$. Also, $\alpha\beta\gamma m \neq 0$ for some $m \in M$ and $\gamma \in [x : M]$. On the other hand, $\alpha^2 x = \beta^2 x = \gamma^2 x = 0$ and $\alpha x \perp y$ for some $y \in V(\bar{\Gamma}(M))$. A simple check yields that $R\alpha x \subseteq \{0, \alpha x, y\}$ and $y = \alpha\beta x$. Hence $\alpha x \perp \alpha\beta x$. So $R(\alpha x) = \{0, \alpha x, \alpha\beta x\}$ and $\text{Ann}(\alpha x)M = \{0, \alpha x, \alpha\beta x\}$. Also, $\alpha\beta\gamma m$ is adjacent to two vertices αx and $\alpha\beta x$, but $\alpha\beta\gamma m \neq \alpha x$. Thus $\alpha\beta\gamma m = \alpha\beta x$. We know that $\alpha\beta m$ is adjacent to two vertices, αx and $\alpha\beta x$, but $\alpha\beta m \neq \alpha\beta\gamma m = \alpha\beta x$, so $\alpha\beta m = \alpha x$, which is a contradiction. Thus $[x : M]^2 x = 0$.

Step (5): Assume that $n = 2$ and $[x : M]^2 x = 0$. We show that $|M| \leq 12$. By hypothesis, $\alpha^2 x = 0$ and $\alpha x \neq 0$; hence $\alpha[x : M]M \neq 0$. Thus $\alpha\beta m \neq 0$ for some $\beta \in [x : M]$ and $m \in M$. We know that $\Gamma(M)$ is complemented and $0 \neq \alpha\beta m \in \text{Ann}(x)M$, so $x \in V(\bar{\Gamma}(M))$. So there is $y \in V(\bar{\Gamma}(M))$ such that $x \perp y$, but αx is adjacent to two vertices, x and y. Hence either $\alpha x = x$ or $\alpha x = y$. If $\alpha x = x$, then multiplying by α we have $\alpha x = 0$, which is a contradiction, so $\alpha x = y$. Let $z \in \text{Ann}(x)M$. Hence $z \in \{0, x, \alpha x\}$, since $x \perp \alpha x = y$. If $z = x$, then $[x : M]x = 0$, which is a contradiction. Therefore $\text{Ann}(x)M = \{0, \alpha x\}$. Also, a simple check yields that $R(\alpha x) = \{0, \alpha x\}$. On the other hand, $\alpha\beta m \in \text{Ann}(\alpha m)M$, so $\alpha m \in V(\bar{\Gamma}(M))$, and there exists $w \in V(\bar{\Gamma}(M))$ such that $\alpha m \perp w$; but $\alpha\beta m$ is adjacent to two vertices, αm and w. Therefore $\alpha\beta m = w$ will be the only possibility, and so $\alpha\beta m \perp \alpha m$. Also, $\alpha\beta m$ is adjacent to two vertices, αx and x; hence $\alpha\beta m = \alpha x$. Now we show that $\text{Ann}(\alpha x)M = \{0, \alpha m, \alpha x, x, x + \alpha m, x + \alpha x\}$. Let $v \in \text{Ann}(\alpha x)M$, so $\alpha v \in \mathrm{Ann}(x)M = \{0, \alpha x\}$. If $\alpha v = 0$, then

$$[v : M][\alpha\beta m : M]M = 0 = [v : M][\alpha m : M]M,$$

and if $\alpha v = \alpha x$, then

$$[v - x : M][\alpha\beta m : M]M = 0 = [v - x : M][\alpha m : M]M.$$

Consequently, $\mathrm{Ann}(\alpha x)M = \{0, \alpha m, \alpha x, x, x+\alpha m, x+\alpha x\}$, and $|\mathrm{Ann}(\alpha x)M| \leq 6$. For all $m_0 \in M$, $\alpha\beta m_0 \in R(\alpha x) = \{0, \alpha x\}$. So either $m_0 \in \mathrm{Ann}(\alpha x)M$ or $m_0 - m \in \mathrm{Ann}(\alpha x)M$, since $\alpha\beta m = \alpha x$. Therefore $|M| \leq 12$. By a similar argument in Step (3), $\mathrm{Ann}(\alpha x)M = Nil(M)$ is a unique maximal submodule of M, and $\bar{\Gamma}(M)$ is a star graph with a center αx with at most 4 edges.

Step (6): Assume that $n = 1$. If $[x : M]x \neq 0$, based on the above steps we have $6 \leq |M| \leq 16$. So we may assume that $[x : M]x = 0$. We show that $|M| = 9$ or $Nil(M) = \{0, x\}$ with $2x = 0$ and $|M| \neq 9$. Let $x \in [x : M]M$ so $x = \Sigma_{i=1}^{n} \alpha_i m_i$ where $\alpha_i \in [x : M]$ and $m_i \in M$ for all $1 \leq i \leq n$. Since $\bar{\Gamma}(M)$ is complemented, there is $y \in T(M)^*$ such that $x \perp y$, so $Rx \subseteq \{0, x, y\}$. If $x \neq \alpha_i m_i$ for all i, then $\alpha_i m_i \in Rx$, and so $\alpha_i m_i = y$ for all i. Suppose that $\alpha_i m_i = \alpha_1 m_1$; thus $x = \Sigma_{i=1}^{n} \alpha_1 m_1 = (\Sigma_{i=1}^{n} \alpha_1) m_1 = \beta m_1$ where $\beta = \Sigma_{i=1}^{n} \alpha_1 \in [x : M]$. Hence we may assume that $x = \alpha m$ for some $\alpha \in [x : M]$ and $m \in M$ such that $\alpha^2 m = 0$, but $0 \neq \alpha m$. We know that $x + x \in Rx \subseteq \{0, x, y\}$; if $x + x \neq 0$, then $Rx = \{0, x, 2x\}$, $x \perp 2x$, and $\mathrm{Ann}(x)M = \{0, x, 2x\}$. For all $m_0 \in M$, $\alpha m_0 \in Rx$; therefore

$$[m_0 : M][x : M]M = 0 = [m_0 : M][2x : M]$$

or

$$[m_0 - m : M][x : M]M = 0 = [m_0 - m : M][2x : M]$$

or

$$[m_0 - 2m : M][x : M]M = 0 = [m_0 - 2m : M][2x : M].$$

Hence $|M| = 9$, and by a similar argument in Step (3), $\mathrm{Ann}(x)M$ is a unique maximal submodule of M and $\bar{\Gamma}(M)$ is a star graph. Now let $|M| \neq 9$. So by the above argument, we must have $2x = 0$. We claim that $Nil(M) = \{0, x\}$. Suppose that z is another non-zero element of $Nil(M)$; hence $[z : M]z = 0$ and $z = \beta m'$ for some $\beta \in [z : M]$ and $m' \in M$, such that $\beta^2 m' = 0$. Also, $0 \neq x \in \mathrm{Ann}(x)M$ and $0 \neq z \in \mathrm{Ann}(z)M$, so $x, z \in V(\bar{\Gamma}(M))$. Since $\bar{\Gamma}(M)$ is complemented, there are $x', z' \in V(\bar{\Gamma}(M))$ such that $x \perp x'$ and $z \perp z'$. Therefore $Rx \subseteq \{0, x, x'\}$ and $Rz \subseteq \{0, z, z'\}$. Observe that $\alpha\beta m = 0$. Let $0 \neq \alpha\beta m \in Rx$ and $\alpha\beta m \in Rz$, if $\alpha\beta m = x \in Rz$. Thus $x = z'$, so $x \perp z$, and hence, $\alpha\beta m = 0$ is a contradiction. If $\alpha\beta m = x'$, then $Rx = \{0, x, \alpha\beta m\} = \mathrm{Ann}(x)M$, and, similar to the above argument, $|M| = 9$, which is a contradiction. So $\alpha\beta m = 0$. On the other hand, $x = \beta m' \in \mathrm{Ann}(x + z)M$, so $x + z \in V(\bar{\Gamma}(M))$. Let w be a complement of $x + z$; clearly, w is neither x nor z. It is clear that $\alpha w \in Rx \subseteq \{0, x, x'\}$, if $\alpha w = 0$. Then x is adjacent to two elements, w and $x + z$, which is a contradiction. If $\alpha w = x'$, then $Rx =$

$\{0, x, \alpha w\} = \mathrm{Ann}(x)M$, and it implies that $|M| = 9$, which is a contradiction. Hence we may assume that $\alpha w = x$ and similarly, $\beta w = z$. Then

$$Rz = [z : M]w, Rx = [x : M]w.$$

Since $w \perp x + z$,

$$[w : M]x = [w : M]z,$$

and $x, x + z \in Rz$. Hence $x + z = z' = x$ or $x + z = 0$. In both case, we have a contradiction. Consequently, $Nil(M) = \{0, x\}$.

(b) Let $0 \neq x \in Nil(M)$ and $|M| \geq 17$. By the proof (a) we have $Nil(M) = \{0, x\}$ for some $x \in M$ such that $2x = 0$ and $[x : M]x = 0$. Hence $x \in V(\bar{\Gamma}(M))$. Since $\bar{\Gamma}(M)$ is complemented, there is $y \in V(\bar{\Gamma}(M))$ such that $x \perp y$. We claim that y is an end. We first show that $x + y$ also is a complement for x. Clearly, $x + y \in V(\bar{\Gamma}(M))$ and $[x + y : M][x : M]M = 0$, because $[x : M]x = 0$ and $x \perp y$. If $w \in V(\bar{\Gamma}(M))$ is adjacent to both x and $x + y$, then

$$[x + y : M][w : M]M = 0 = [x : M][w : M]M.$$

Hence $[w : M]R(x + y) = 0$, so $[y : M][w : M]M = 0$. Moreover, $x \perp y$, thus either $w = x$ or $w = y$. If $w = y$, then $[y : M]y = 0$. Therefore $y \in Nil(M) = \{0, x\}$, which is a contradiction, so $x = w$. Thus $x + y$ is a complement for x. Since $\bar{\Gamma}(M)$ is uniquely complemented, $x + y \sim y$. Assume that $z \in V(\bar{\Gamma}(M)) \setminus \{x\}$ such that z is adjacent to y; hence, z is adjacent to $x + y$. So $[z : M][x : M]M = 0$. Thus $z = y$, since $x \perp y$. Consequently, y is an end. □

Remark 4.16. The proof of Proposition 4.15(a) shows that if M is a multiplication R-module such that $\bar{\Gamma}(M)$ is complemented and $|Nil(M)| > 2$, then $|M| \leq 16$. Also, M has a unique maximal submodule and $\bar{\Gamma}(M)$ is a star graph with at most six edges. Therefore $\bar{\Gamma}(M)$ is uniquely complemented. Also, it shows that if $\bar{\Gamma}(M)$ is not uniquely complemented, then $Nil(M) = \{0, x\}$, in which x is an element of M such that $x[x : M] = 0$. Hence $x = \beta m$ for some $m \in M$, and $\beta \in [x : M]$.

Example 4.17. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{20}$. Clearly, M is not reduced, so $Nil(M) \neq 0$ and $\Gamma(M)$ is complemented, but not uniquely complemented. So by the proof of Proposition 4.15, $Nil(M) = \{0, 10\}$.

Clearly, star graphs are uniquely complemented. The next theorem shows that for a multiplication R-module M with $Nil(M) \neq 0$, if $\bar{\Gamma}(M)$ is uniquely complemented, then $\bar{\Gamma}(M)$ is a star graph.

Theorem 4.18. *Let R be a Bézout ring and M be a multiplication R-module with $Nil(M) \neq 0$. If $\bar{\Gamma}(M)$ is a uniquely complemented graph, then either $\bar{\Gamma}(M)$ is a star graph with at most six edges or $\bar{\Gamma}(M)$ is an infinite star graph with center x, where $Nil(M) = \{0, x\}$.*

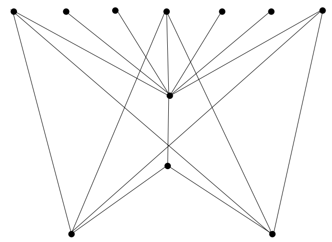


FIGURE 3

Proof. Suppose that $\bar{\Gamma}(M)$ is uniquely complemented and $Nil(M) \neq 0$. Let $|M| \leq 16$; then by Remark 4.16, $\bar{\Gamma}(M)$ is a star graph with at most six edges. Let $|M| > 16$. Hence by Step (7) of Proposition 4.15(a), $Nil(M) = \{0, x\}$ for some $0 \neq x \in M$ and $[x : M]x = 0$.

We first show that $\bar{\Gamma}(M)$ is an infinite graph. Let c be a complement of x, so $\mathrm{Ann}(c)M = \{0, x\} = Nil(M)$, by Proposition 4.15(b). Let $c = \Sigma_{i=1}^{n}(\alpha_i m_i) \in [c : M]M$, where $\alpha_i \in [c : M]$ and $m_i \in M$ for $1 \leq i \leq n$. Since R is a Bézout ring, $\Sigma_{i=1}^{n} R\alpha_i = R\alpha$ for some $\alpha \in R$. We claim that αc is also a complement of x. If z is adjacent to both vertices x and αc, then

$$[\alpha c : M][z : M]M = 0 = [x : M][z : M]M.$$

Therefore $\alpha z \in \mathrm{Ann}(c)M = \{0, x\}$. So either $\alpha z = 0$ or $\alpha z = x$. If $\alpha z = 0$, then $[z : M]c = 0$; so $z \in \mathrm{Ann}(c)M$, which is a contradiction, and $\alpha z = x$. Hence $\alpha[z : M]z = x[z : M] = 0$. Therefore $z[z : M] \subseteq \mathrm{Ann}(c)M = Nil(M)$, and hence $z \in Nil(M) = \{0, x\}$, which again is a contradiction. Consequently, $\alpha c \perp x$; so by Proposition 4.15(b), $\mathrm{Ann}(\alpha c)M = \{0, x\}$. By a similar argument, $\alpha^i c \perp x$ and $\mathrm{Ann}(\alpha^i c)M = \{0, x\}$ for $1 \leq i \leq n$. Hence each $\alpha^i c$ is an end. Next, note that $\alpha^i c$ are all distinct. If not, suppose that $\alpha^i c = \alpha^j c$ for some $1 \leq i < j$. Therefore $\alpha^i(1 - \alpha^{j-i})c = 0$, so $(1 - \alpha^{j-i}) \in \mathrm{Ann}(\alpha^i c)$. Using the proof of Proposition 4.15(a) Step 6, $x = \beta m$ for some $\beta \in [x : M]$ and $m \in M$, such that $\beta^2 m = 0$ but $\beta m \neq 0$. Hence $(1 - \alpha^{j-i})m \in \mathrm{Ann}(\alpha^i c)M = \{0, x\}$. So either $m - \alpha^{i-j}m = 0$ or $m - \alpha^{i-j}m = x$. If $m = \alpha^{i-j}m$, then

$$x = \beta m = \beta\alpha^{i-j}m \in \beta\alpha^{i-j-1}Rc \subseteq \alpha^{i-j-1}[x : M][c : M]M = 0,$$

which is a contradiction. Thus $m - \alpha^{i-j}m = x$. So

$$x - \alpha^{i-j}\beta m = \beta m - \alpha^{i-j}\beta m = \beta x = 0.$$

Hence $x \in \alpha^{i-j-1}\beta Rc = 0$, which again is a contradiction. Consequently, $\bar{\Gamma}(M)$ is infinite.

Next, we show that $\bar{\Gamma}(M)$ is a star graph with center x. By contradiction, suppose that $\bar{\Gamma}(M)$ is not a star graph. Let $c \in V(\bar{\Gamma}(M))$ be a complement of x, so there is a $a \in V(\bar{\Gamma}(M)) \setminus \{x, c\}$ such that $[a : M][x : M]M = 0$, but a is not an end. Hence there is $y \in V(\bar{\Gamma}(M)) \setminus \{a, x, c\}$ such that $y \perp a$. Let $c = \Sigma_{i=1}^{n}(\alpha_i m_i)$, where $\alpha_i \in [c : M]$ and $m_i \in M$, for $1 \leq i \leq n$, and let $R\alpha = \Sigma_{i=1}^{n} R\alpha_i$. We can check that $\alpha y \notin \{0, a, x, c, y\}$. If $\alpha y = 0$, then $[y : M]c = 0$, which is a contradiction with c is an end. If $\alpha y = x$, then $\alpha[y : M][c : M]M = 0$, so $y \in \mathrm{Ann}(\alpha c)M = \{0, x\}$, which is a contradiction. If $\alpha y = y$, then $\alpha y[x : M] \subseteq [x : M]Rc = 0$, which is a contradiction. If $\alpha y = c$, then a is adjacent to c, which is a contradiction. Last, if $\alpha y = a$, then $\alpha y[y : M] = 0$. So $y[y : M] \in \mathrm{Ann}(\alpha c)M = Nil(M)$, and therefore $y \in Nil(M)$, which is a contradiction. Thus $\alpha y \in V(\bar{\Gamma}(M)) \setminus \{a, x, c, y\}$. By the hypothesis, there is $z \in V(\bar{\Gamma}(M))$ such that z is a complement of αy. One can also verify that $z \notin \{0, \alpha y, a, x, c, y\}$ (Use $y \notin Nil(M)$ to show that $z \notin \{c, y\}$ and use $\alpha y \perp z$ to show that $z \notin \{a, x\}$). Clearly, $[x : M][z : M]M \neq 0$. Let $z = \Sigma_{i=1}^{s} r_i m_i$, where $r_i \in [z : M]$ and $m_i \in M$ for $1 \leq i \leq s$, and let $R\gamma = \Sigma_{i=1}^{n} Rr_i$. If $\gamma x = 0$, then $[x : M][z : M]M = 0$, which is a contradiction. So we must suppose that $\gamma x \neq 0$. Also, $[\gamma x : M][c : M]M = 0$; hence $\gamma x \in \mathrm{Ann}(c)M$. Thus $\gamma x = x$. On the other hand, $\alpha y \perp z$, so

$$[\gamma y : M][c : M]M = [y : M]R(\Sigma_{i=1}^{n}(\gamma\alpha_i m_i)) \subseteq [y : M]R\alpha z = 0.$$

Therefore $\gamma y \in \mathrm{Ann}(c)M$. Hence either $\gamma y = 0$ or $\gamma y = x$. So x is adjacent to both y and a, but this is a contradiction with $a \perp y$; consequently, $\bar{\Gamma}(M)$ is an infinite star graph with center x. □

Corollary 4.19. *Let M be a cyclic R-module with $Nil(M) \neq 0$. If $\bar{\Gamma}(M)$ is a uniquely complemented graph, then either $\bar{\Gamma}(M)$ is a star graph with at most six edges or $\bar{\Gamma}(M)$ is an infinite star graph with center x, where $Nil(M) = \{0, x\}$.*

Proof. It is similar to the proof of Theorem 4.18. □

Corollary 4.20. *Let R be a Bézout ring and M be a multiplication R module. If $\bar{\Gamma}(M)$ is uniquely complemented, then either $\bar{\Gamma}(M)$ is a star graph or $S^{-1}M$ is von Neumann regular. Moveover, for a cyclic R-module M, the converse is true.*

Proof. Let $\bar{\Gamma}(M)$ be uniquely complemented. If $Nil(M) = 0$, then M is reduced and by Theorem 4.5, $S^{-1}M$ is von Neumann regular. If $Nil(M) \neq 0$, then by Theorem 4.18, $\bar{\Gamma}(M)$ is a star graph. The converse is true by Corollary 4.11. □

Corollary 4.21. *Let M be a cyclic R module. Then $\bar{\Gamma}(M)$ is uniquely complemented if and only if either $\bar{\Gamma}(M)$ is a star graph or $S^{-1}M$ is von Neumann regular.*

Proof. Let $\bar{\Gamma}(M)$ be uniquely complemented. If $Nil(M) = 0$, then M is reduced, and by Corollary 4.8, $S^{-1}M$ is von Neumann regular. If $Nil(M) \neq 0$, then by Corollary 4.19, $\bar{\Gamma}(M)$ is a star graph. The converse is true by Corollary 4.11. □

Example 4.22. Let $R = \mathbb{Z}$, $M_1 = \mathbb{Z}_{33}$, and $M_2 = \mathbb{Z}_{30}$. $\Gamma(M_i)$, $i = 1, 2$, is a uniquely complemented graph. So by Corollary 4.20, $S^{-1}M_i$, $i = 1, 2$, is von Neumann regular.

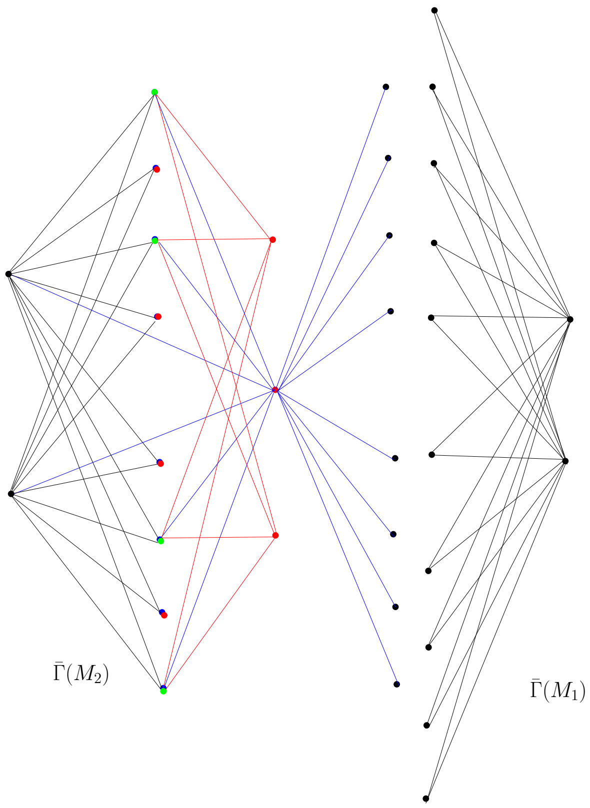


FIGURE 4

References

[1] D. D. Anderson, *Multiplication ideals, multiplication rings and the ring $R(X)$*, Canad. J. Math. **28** (1976), no. 4, 260–768.

[2] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative rings*, J. Algebra **159** (1993), no. 2, 500–514.

[3] D. F. Anderson, M. C. Axtell, and J. A. Stickles, *Zero-divisor graphs in commutative rings*, Commutative algebra, Noetherian and non-Noetherian perspectives, 23–45, Springer, New York, 2011.

[4] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra **320** (2008), no. 7, 2706–2719.

[5] ———, *On the zero-divisor graph of a ring*, Comm. Algebra **36** (2008), no. 8, 3073–3092.

[6] D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston, *The zero-divisor graph of a commutative ring. II*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), 61–72, Lecture Notes in Pure and Appl. Math., 220, Dekker, New York, 2001.

[7] D. F. Anderson, R. Levy, and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, J. Pure Appl. Algebra **180** (2003), no. 3, 221–241.

[8] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.
[9] A. Badawi and D. F. Anderson, *Divisibility conditions in commutative rings with zero-divisors*, Comm. Algebra **38** (2002), no. 8, 4031–4047.
[10] A. Barnard, *Multiplication modules*, J. Algebra **71** (1981), no. 1, 174–178.
[11] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.
[12] G. A. Cannon, K. Neuerburg, and S. P. Redmond, *Zero-divisor graphs of nearrings and semigroups*, Nearrings and nearfields, 189–200, Springer, Dordrecht, 2005.
[13] F. R. DeMeyer, T. McKenzie, and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum **65** (2002), no. 2, 206–214.
[14] R. Diestel, *Graph Theory*, Springer-Verlag, New York, 1997.
[15] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), no. 4, 755–779.
[16] SH. Ghalandarzadeh and P. Malakooti Rad, *Torsion graph over multiplication modules*, Extracta Math. **24** (2009), no. 3, 281–299.
[17] J. Han, *The zero-divisor graph under group actions in a noncommutative ring*, J. Korean Math. Soc. **45** (2008), no. 6, 1647–1659.
[18] F. Kash, *Modules and Rings*, London: Academic Press, 1982.
[19] P. Malakooti Rad, Sh. Ghalandarzadeh, and S. Shirinkam, *On the torsion graph and von Neumann regular rings*, Filomat **26**, 47–53. To appear.
[20] H. Matsumara, *Commutative Ring Theory*, Cambridge, UK: Cambridge University Press, 1986.
[21] S. P. Redmond, *The zero-divisor graph of a non-commutative ring*, Internat. J. Commutative Rings **1** (2002), no. 4, 203–211.
[22] P. Ribenboim, *Algebraic Numbers*, Wiley, 1972.

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