THE QUANTUM $\mathfrak{sl}(n, \mathbb{C})$ REPRESENTATION THEORY AND ITS APPLICATIONS

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ABSTRACT. In this paper, we study the quantum $\mathfrak{sl}(n)$ representation category using the web space. Specially, we extend $\mathfrak{sl}(n)$ web space for $n \geq 4$ as generalized Temperley-Lieb algebras. As an application of our study, we find that the HOMFLY polynomial $P_n(q)$ specialized to a one variable polynomial can be computed by a linear expansion with respect to a presentation of the quantum representation category of $\mathfrak{sl}(n)$. Moreover, we correct the false conjecture [30] given by Chbili, which addresses the relation between some link polynomials of a periodic link and its factor link such as Alexander polynomial ($n = 0$) and Jones polynomial ($n = 2$) and prove the corrected conjecture not only for HOMFLY polynomial but also for the colored HOMFLY polynomial specialized to a one variable polynomial.

1. Introduction

The discovery of the Jones polynomial [9, 10] brought a Renaissance of knot theory and its generalizations have been studied in many different ways [3, 7, 14, 20, 19, 26, 38, 41]. Using the representation theory of complex simple Lie algebras, Reshetikhin and Turaev found quantized simple Lie algebras invariants of links and 3-manifolds [32, 33] and these invariants have been studied extensively [4, 5, 13, 28, 16, 27, 45].

In the present paper, we study the quantum $\mathfrak{sl}(n)$ representation theory related to the HOMFLY polynomials of periodic links. Murasugi found a strong relation between the Alexander polynomials of a periodic link L and its factor link $\overline{L}$ [24] and a similar relation for the Jones polynomials [25]. There are various results to decide the periodicity of links [29, 35, 11, 42, 43, 44]. A conjecture for the relation between HOMFLY polynomials $P_n(q)$ specialized to a one variable polynomial of a periodic link L and its factor link $\overline{L}$ was found as follows [4].

Conjecture 1.1 ([4]). Let p be a positive integer and L be a p-periodic link in $\mathbb{S}^3$ with its factor link $\overline{L}$. Then,

$$P_n(L) \equiv P_n(\overline{L})^p \qquad \text{modulo } \mathcal{A}_n,$$

where $\mathcal{A}_n$ is the ideal of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $[n]^p - [n]$.

The quantum integers are defined as

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad [n]! = [n][n-1]\cdots[2][1], \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}.$$

For Conjecture 1.1, Chbili provided a proof for $n = 3$ using the representation theory of the quantum $\mathfrak{sl}(3)$ [4]. There were subsequent studies on the conjecture [6]. But, it was shown that *Conjecture 1.1 is **false** for $n \geq 4$* [30]. The counterexamples show that even if the link is colored by the vector representation, the congruence can be involved with other fundamental representations. Focused on the quantum representation category of $\mathfrak{sl}(n, \mathbb{C})$, we may modify the original conjecture as follows.

Conjecture 1.2. Let p be a positive integer and L be a p-periodic link in $\mathbb{S}^3$ with its factor link $\overline{L}$. Then,

$$P_n(L) \equiv P_n(\overline{L})^p \qquad \text{modulo } \mathcal{I}_n,$$

where $\mathcal{I}_n$ is the ideal of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $\begin{bmatrix} n \\ i \end{bmatrix}^p - \begin{bmatrix} n \\ i \end{bmatrix}$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

The study of a presentation of the quantum representation category of $\mathfrak{sl}(n)$ leads us to a powerful computation method of the HOMFLY polynomial $P_n(q)$ specialized to a one variable polynomial, a linear expansion of webs, and its generalization to the colored $\mathfrak{sl}(n)$ HOMFLY polynomial $G_n(L, \mu)$ specialized to a one variable polynomial. Then we not only prove Conjecture 1.2 in Theorem 4.1 but also we show the following theorem that Conjecture 1.2 remains true for $G_n(L, \mu)$.

Theorem 1.3. *Let p be a positive integer and L be a p-periodic link in $\mathbb{S}^3$ with its factor link $\overline{L}$. Let μ be a p-periodic coloring of L and $\overline{\mu}$ be the induced coloring of $\overline{L}$. Then for $n \geq 0$,*

$$G_n(L, \mu) \equiv G_n(\overline{L}, \overline{\mu})^p \qquad \text{modulo } \mathcal{I}_n,$$

where $\mathcal{I}_n$ is the ideal of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $\begin{bmatrix} n \\ i \end{bmatrix}^p - \begin{bmatrix} n \\ i \end{bmatrix}$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Recently, there are significant progresses on the representation theory of the quantum $\mathfrak{sl}(n)$ [23, 34, 40]. In particular, a complete set of relations for the representation theory of the quantum $\mathfrak{sl}(n)$ which contain our relations in Figure 4, Lemma 3.2 and new relation called 'Kekulé relation', which was first found by the second author for $\mathfrak{sl}(4)$ [17], is conjectured in [23]. Furthermore, it was also proven that Remark 3.3 is false [23].

The outline of this paper is as follows. In Section 2, we review the HOMFLY polynomials and the colored HOMFLY polynomials specialized to a one variable polynomial. In Section 3, we develop the representation theory of the quantum $\mathfrak{sl}(n)$. We show that the quantum $\mathfrak{sl}(n)$ skein module of the plane or sphere has dimension 1 using the relations we have found. In Section 4 we prove Conjecture 1.2 and show the conjecture holds for the colored $\mathfrak{sl}(n)$ HOMFLY polynomial specialized to a one variable polynomial. In Section 5, we compare our result with previous works.

2. The HOMFLY polynomials and the colored HOMFLY polynomials specialized to a one variable polynomial

A *link* L is a disjoint union of circles embedded in three dimensional sphere $\mathbb{S}^3$, and a *knot* K is a link with only one component. Here, we assume all links are PL. A link L in $\mathbb{S}^3$ is *p-periodic* if there exists a periodic homeomorphism h of order p such that $fix(h) \cong S^1, h(L) = L$ and $fix(h) \cap L = \emptyset$ where $fix(h)$ is the set of fixed points of h. It is well known that if we consider $\mathbb{S}^3$ as $\mathbb{R}^3 \cup \{\infty\}$, we can assume that h is a rotation by $2\pi/p$ angle around the z-axis. Let $G = \mathbb{Z}/p\mathbb{Z}$ denote the group of homeomorphisms of $\mathbb{S}^3$ generated by h, and let π denote the covering map $\mathbb{S}^3 \to \mathbb{S}^3/G$, branched along z-axis. We call $\overline{L} = \pi(L)$ the factor link of L. For other terms and definitions of knot theory, we refer to [1].

Now we define the HOMFLY polynomial specialized to a one variable polynomial. For the rest of paper, all HOMFLY polynomial and colored HOMFLY polynomial are specialized to a one variable polynomial unless we state differently. For a nonnegative integer n, the HOMFLY polynomial $P_n(q)$ specialized to a one variable polynomial can be calculated uniquely by the following skein relations:

$$P_n(\emptyset) = 1,$$

$$P_n(\bigcirc \cup D) = \left(\frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right) P_n(D),$$

$$q^{\frac{n}{2}} P_n(L_+) - q^{-\frac{n}{2}} P_n(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) P_n(L_0),$$

where $\emptyset$ is the empty diagram, $\bigcirc$ is the trivial knot and L_+, L_- and L_0 are skein triple, three diagrams which are identical except at one crossing as in Figure 1.

The HOMFLY polynomial of links can be recovered from the representation theory of the quantum $\mathfrak{sl}(n)$. For $n = 0$, we use the special linear Lie superalgebra $\mathfrak{gl}(1|1)$ to find that $P_0(q)$ is the Alexander polynomial [15]. For $n = 1$ and for any link, $P_1(q) = 1$. For $n = 2$, $P_2(q)$ is the Jones polynomial [10, 32, 33, 41]. The polynomial $P_n(q)$ can be computed by linearly expanding each crossing into a sum of diagrams of planar trivalent graphs where the edges of these planar graphs are oriented and colored by 1 or 2 as in Figure 2 [27].
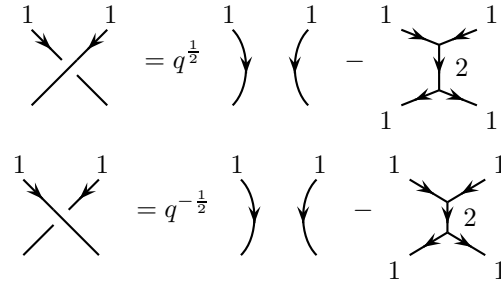
FIGURE 1. The skein triple L_+, L_- and L_0.

FIGURE 2. Expansions of crossings for $P_n(L)$.

To define the colored HOMFLY polynomial, we review the quantum representation category of $\mathfrak{sl}(n, \mathbb{C})$ [20, 16]. The color 1 of an edge in the definition of HOMFLY polynomial presents the vector representation V of the quantum $\mathfrak{sl}(n)$, and the color 2 for its exterior power $\wedge^2 V$. The trivalent vertex is the unique (up to scaling) intertwiner of $V^{\otimes 2} \otimes \wedge^2 V$. This setup works for arbitrary exterior powers of V [27]. Oriented edges of graphs in their calculus carry colors from 1 to $n - 1$ that denote the fundamental representations of the quantum $\mathfrak{sl}(n)$. Kuperberg generalized Temperley-Lieb algebras to $\mathfrak{sl}(3)$ web spaces [20]. In Section 3 we develop the quantum $\mathfrak{sl}(n)$ representation theory by extending the idea of webs in [27]. Recently, Westbury found a web space for spin representations of $\mathfrak{so}(7)$ [40]. A precise and algebraic overview of the quantum $\mathfrak{sl}(n)$ representation theory can be found in [23]. By expanding all crossings as in Figure 17, Murakami, Ohtsuki and Yamada found a regular isotopy invariant $[D]_n$ [27]. In Section 4, we modify writhes suitably to define an isotopy invariant $K_n(L, \mu)$, the $\mathfrak{sl}(n)$ HOMFLY polynomial, where μ is a coloring of L by a fundamental representation of the quantum $\mathfrak{sl}(n)$. Using the quantum $\mathfrak{sl}(n)$ representation theory, we show $K_n(L, \mu)$ can be computed by a linear expansion with respect to the relations of web spaces in Theorem 3.4.

For $n = 2$, we can decorate L by any other irreducible representations V_n using the highest-weight projection

$$f_n : V_1^{\otimes n} \to V_1^{\otimes n}$$

FIGURE 3. Generators of the quantum $\mathfrak{sl}(n)$ web space.

whose image is V_n where V_1 is the vector representation of $\mathfrak{sl}(2)$. This projection is called a *Jones-Wenzl projector* [39]. It does exist for $n \geq 3$ and called a *clasp* [20]. Using these clasps, Lickorish first found a quantum $\mathfrak{sl}(2)$ invariants of 3-manifolds [22]. Ohtsuki and Yamada generalized it for the quantum $\mathfrak{sl}(3)$ [28] and Yokota did for the quantum $\mathfrak{sl}(n)$ [45]. A benefit of using the quantum $\mathfrak{sl}(n)$ representation theory for link invariants is that some nontrivial facts from the original work of [10, 32, 33] do follow easily such as the integrality [21].

By decorating each component by μ, we can define the colored $\mathfrak{sl}(n)$ HOM-FLY polynomial, denoted by $G_n(L, \mu)$ as follows. For a given colored link L of l components say, $L_1, L_2, \ldots, L_l$, where each component L_i is colored by an irreducible representation $V_{a(i)_1\lambda_1 + a(i)_2\lambda_2 + \cdots + a(i)_{n-1}\lambda_{n-1}}$ of $\mathfrak{sl}(n)$ and λ_1, λ_2, ..., λ_{n-1} are the fundamental weights of $\mathfrak{sl}(n)$. The coloring is denoted by $\mu = (a(1)_1\lambda_1 + a(1)_2\lambda_2 + \cdots + a(1)_{n-1}\lambda_{n-1}, a(2)_1\lambda_1 + a(2)_2\lambda_2 + \cdots + a(2)_{n-1}\lambda_{n-1}, \ldots, a(l)_1\lambda_1 + a(l)_2\lambda_2 + \cdots + a(l)_{n-1}\lambda_{n-1})$. First we replace each component L_i by $a(i)_1 + a(i)_2 + \cdots + a(i)_{n-1}$ copies of parallel lines and each $a(i)_j$ line is colored by the weight λ_j. Then we put a clasp of weight $(a(i)_1\lambda_1 + a(i)_2\lambda_2 + \cdots + a(i)_{n-1}\lambda_{n-1})$ for L_i. If we assume the clasps are far away from crossings, we expand each crossing as in Figure 17, then clasps [45]. The value we obtained after removing all faces by using the relations is *the colored $\mathfrak{sl}(n)$ HOMFLY polynomial $G_n(L, \mu)$ of L*.

3. The quantum $\mathfrak{sl}(n)$ representation theory

We refer to [8, 20, 12, 16] for the general representation theory. The webs are generated by the two types of shapes in Figure 3 where $i + j + k = n$. At the second web in Figure 3, we can change the directions of the edges to inward by using the duality, $(V_{\lambda_i})^* \cong V_{\lambda_{n-i}}$, then the colors of the resulting edges are $n-i, n-j$ and $n-k$. Then we say a vertex has a *sign type* $+$ if the sum of the colors is n, $-$ if the sum is $2n$. The generators in Figure 3 have sign type $+, -$ from left. Naturally we can assign a sign type for each face. Using last two relations in Figure 4, we only need to look at the face of a *valid sign type* in which $+, -$ are appearing alternatively. One can easily see that none of faces with odd numbers of edges can have a valid sign type.

A few computations of tensor products of fundamental representations show the relations in Figure 4 (up to scaling): the quantum dimension of the fundamental representation V_{λ_i} is $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$, the dimension of the invariant space of $V_{\lambda_i} \otimes$
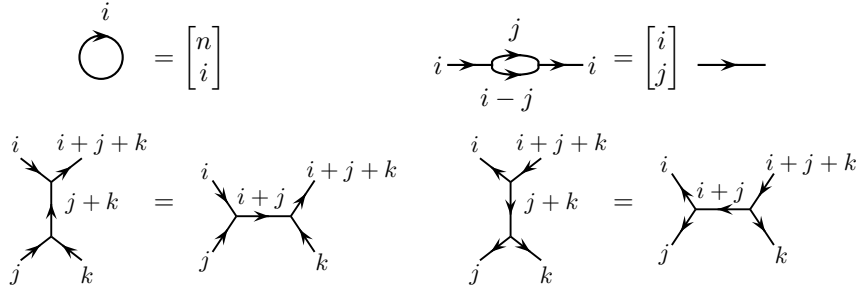
FIGURE 4. Relations obtained by the quantum dimension and scaling.

$V_{\lambda_i}^*$ is 1 and the dimension of the invariant space of $V_{\lambda_i} \otimes V_{\lambda_j} \otimes V_{\lambda_k} \otimes V_{\lambda_i+\lambda_j+\lambda_k}^*$ is 1.

3.1. Rectangular relations

To discuss rectangular relations, we first prove the following lemma.

Lemma 3.1.   (1) *If $i \leq j \leq n - j - 1$, then*
$$\dim(\mathrm{inv}(V_{\lambda_i} \otimes V_{\lambda_j} \otimes V_{\lambda_i}^* \otimes V_{\lambda_j}^*)) = i + 1.$$

(2) *If $j \geq i \geq k \geq 1, n - j - 1 \geq i$ and $n - i - j - 1 \geq l \geq 1$, then*
$$\dim(\mathrm{inv}(V_{\lambda_i} \otimes V_{\lambda_{j+l}} \otimes V_{\lambda_{i+l}}^* \otimes V_{\lambda_j}^*)) = i + 1.$$

Proof. If $i \leq j \leq n - j - 1$, we obtain the following isomorphism by the Clebsch-Gordan formula.
$$V_{\lambda_i} \otimes V_{\lambda_j} \cong V_{\lambda_{j-i}} \oplus V_{\lambda_{j-i+2}} \oplus \cdots \oplus V_{\lambda_{j+i}}.$$

For irreducible representations V, W of a simple Lie algebra, by a simple application of Schur's lemma we find
$$\dim(\mathrm{inv}(V \otimes W^*)) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \not\cong W. \end{cases}$$

These two facts imply that $\dim(\mathrm{inv}(V_{\lambda_i} \otimes V_{\lambda_j} \otimes V_{\lambda_i}^* \otimes V_{\lambda_j}^*)) = i+1$. Similarly one can prove the other.   □

From Lemma 3.1, we know the number of basis webs that we need for each expansion. For $n - i - 1 \geq j \geq i \geq 0$, we can have two sets of basis webs and each has the same sign type as in Figure 5. Throughout the section we will use the basis in the left hand side of Figure 5. There are only two possible types of rectangular relations as in Lemma 3.2, all other can be taken care of by relations in Figure 4. The equation (1) in Lemma 3.2 was first appeared in [27] without a proof.
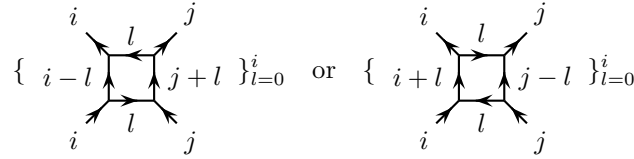
$$\{\ \underset{i}{\overset{i\quad\quad j}{}}\ i-l\ \underset{l}{\overset{l}{\square}}\ j+l\ \}_{l=0}^{i}\quad\text{or}\quad\{\ i+l\ \underset{l}{\overset{l}{\square}}\ j-l\ \}_{l=0}^{i}$$

FIGURE 5. Two basis webs with the boundary $V_{\lambda_i}\otimes V_{\lambda_j}\otimes V_{\lambda_i}^*\otimes V_{\lambda_j}^*$.

Lemma 3.2. *For $n-i-1\geq j\geq i\geq k\geq 0$ and $n-i-j-1\geq l\geq 0$, we find*

$$(1)\qquad \begin{array}{c}\text{(left web)}\end{array}\ =\ \sum_{m=0}^{i}\begin{bmatrix}l\\k-m\end{bmatrix}\begin{array}{c}\text{(right web)}\end{array}$$

For $n-j-1\geq i\geq j\geq k\geq 1$ and $n-i-j-1\geq l\geq 1$, we have

$$(2)\qquad \begin{array}{c}\text{(left web)}\end{array}\ =\ \sum_{m=0}^{j}\begin{bmatrix}l\\k-m\end{bmatrix}\begin{array}{c}\text{(right web)}\end{array}$$

Just for these two equations, we use a different convention of quantum integers that $\begin{bmatrix}0\\0\end{bmatrix}=1$ but $\begin{bmatrix}0\\s\end{bmatrix}=0$ if $s\neq 0$.

Proof. Let $a(k,m),b(k,m)$ be the coefficients in the righthand side of the equations (1) and (2). We induct on $(\min\{i,j\},k)$ in lexicographic order. The key idea is to prove both equations simultaneously. If $k=0$ and $j\geq i$, we find the equation in Figure 6. For the case $k=0$ and $i\geq j$, it is identical except the weight on horizontal arrow is replaced by the weight $i-j$ of the opposite direction. For $i=0$, we find the equations in Figure 7. One can do for the case $j=0$ similarly.

Now we are set to proceed to the induction step. Let us look at the first case $n-i-1\geq j\geq i\geq k\geq 0$, $n-i-j-1\geq l\geq 0$. On the top of each web in the equation (1), we can attach i different H's, given in Figure 8 where $i\geq s\geq 1$. If $s=i$, we can easily get

$$\begin{bmatrix}j+k\\i\end{bmatrix}\begin{bmatrix}i+l\\k\end{bmatrix}=\sum_{m=0}^{i}a(i,m)\begin{bmatrix}j+l+m\\m\end{bmatrix}\begin{bmatrix}j\\i-m\end{bmatrix}.$$

From the case $s=1$, first we apply the bottom two relations in Figure 4 at the upper rectangle of the web in the left hand side of the first equality in Figure 9 then the second relation in Figure 4 to obtain the first equality.
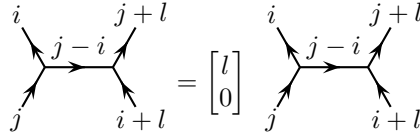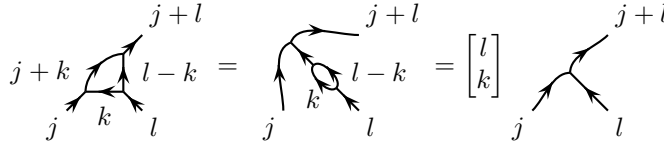
FIGURE 6. Initial induction step $k = 0$ and $j \geq i$.
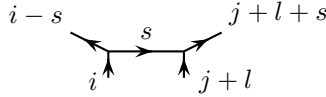

FIGURE 7. Initial induction step $i = 0$.

FIGURE 8. H's attached on top of webs in Lemma 3.2.

The second equality in Figure 9 follows from the induction hypothesis of the equation (1).

Next we look at each term in right hand side of the equation (1) as in the first web on in Figure 10. Now we can use the equation (2) for the upper rectangle of the first web because of the induction hypothesis, where the indices of the boundary are $i-1$, $i-m$, $(i-1)+(j+l+m-i+1)$ and $(i-m)+(j+l-i+m+1)$ from the northwest corner counter-clockwisely. Since $k = 1$ there are only two nonzero terms as in the right hand side of the first equality in Figure 10 where $\alpha = [-i+j+l+m+1]$, $\beta = 1$. For the second web in the right hand side of the first equality in Figure 10, one can see that a similar step of relations, which was used in Figure 9 can be applied for the lower rectangle to get the next equality.

At last, by comparing coefficients of each basis element, we get the following i equations

$$[-i+j+k+1]\begin{bmatrix} l+1 \\ k-t \end{bmatrix} = a(i,t)[-i+j+t+1] + a(i,t+1)[-i+j+l+t+2],$$

where $i-1 \geq t \geq 0$. Since these $i+1$ equations are independent, we plug in the answer to equations to check $a(i,m) = \begin{bmatrix} l \\ k-m \end{bmatrix}$ is correct. One can follow the proof for the second case, $n-j-1 \geq i \geq j \geq k \geq 1$, $n-i-j-1 \geq l \geq 1$. $\square$

3.2. The quantum $\mathfrak{sl}(n)$ skein modules

Skein modules were introduced independently by V. Turaev [36] and J. Przytycki [31] as a $\mathbb{C}[A^{\pm 1}]$-module associated to a 3-manifold M generated

FIGURE 9. The reduction on the left side of equation (1).

FIGURE 10. The reduction on the right side of equation (1).

by framed links inside M with local relations known as Kauffman relations. In the case of $M = \mathbb{S}^3$ this construction reduces to the Jones polynomial and in the general case, the evaluation of the skein module at the root of unity is known to fir with the Topological Quantum Field Theory constructed in [2]. It can be generalized for arbitrary Lie algebra, remind that the Jones polynomial came from the representation category of the quantum $\mathfrak{sl}(2)$. By replacing M by $F \times [0,1]$, framed links by framed links with a color and Kauffman relation by relations given Figure 1 and Lemma 3.2 one can obtain a $\mathbb{C}[A^{\pm 1}]$-module.

We call it the *quantum* $\mathfrak{sl}(n)$ *skein modules*. In this section, we concentrate on the quantum $\mathfrak{sl}(n)$ skein modules and prove Theorem 3.4.

The authors have been trying to find a complete relation of the quantum $\mathfrak{sl}(n)$ representation theory, but we find that the size of a polygon we have to find a suitable relation is increasing as n increases: there a rectangular relation for the quantum $\mathfrak{sl}(3)$, a hexagonal relation for the quantum $\mathfrak{sl}(4)$ and an octagonal relation for the quantum $\mathfrak{sl}(6)$. In particular, a complete set of relations for the quantum $\mathfrak{sl}(4)$ representation theory is conjectured [17].

Remark 3.3. For a given n and sufficiently large m, *i.e.*, $n \ll m$, we conjecture that any $2n$ polygon of the sign type $(+, -, +, \ldots, -)$ can be expanded to a sum of webs of polygons of smaller sizes and $2n$ polygons of the sign type $(-, +, -, \ldots, +)$ by relations of the quantum $\mathfrak{sl}(m)$ representation theory.

As mentioned before, a complete set of relations for the representation theory of the quantum $\mathfrak{sl}(n)$ which contain our relations in Figure 4, Lemma 3.2 and new relation called 'Kekulé relation', which was first found by the second author for $\mathfrak{sl}(4)$ [17], is conjectured in [23]. Furthermore, it was also shown that Remark 3.3 is false [23]. Without using extra relations found in [23], we can prove the following theorem.

Theorem 3.4. *The quantum* $\mathfrak{sl}(n)$ *skein module of the plane or sphere has dimension* 1.

Proof. If we look at these webs without decorations, they are directed, weighted, trivalent and planar graphs. We will consider these graphs on $\mathbb{S}^2$ instead of $\mathbb{R}^2$ and assume that all webs are without boundary for the rest of proof, *i.e.*, no vertex of valence 1.

We will claim that the dimension of the web space without boundary is 1. Suppose it is not true, then there exists a web for which we can not take a value in $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by repeatedly applying relations for the representation theory of the quantum $\mathfrak{sl}(n)$, say D. We assume that D has the smallest number of faces among all counterexamples. Since a face of size less than 4 can be removed by using relations found in Figure 4, we assume that all faces in D have at least 4 edges. Since it can be shown easily that there are finitely many trivalent graphs with a fixed number of faces whose sizes are bigger than 3, we further assume that D has the maximum number of rectangular faces among counterexamples of the minimal number of faces. Let V, E and F be the number of vertices, edges and faces in D on $\mathbb{S}^2$, respectively. Let F_i be the number of faces with i edges. We can easily find the following equations:

$$(3) \qquad\qquad 3V = 2E,$$
$$(4) \qquad\qquad 2 = V - E + F,$$
$$(5) \qquad\qquad 2E = \sum_{4 \le i} iF_i.$$

The outline of the proof is as follow. First, we prove the existence of rectangular face in D in Lemma 3.7. Then we divide cases depending on the neighborhood of the rectangular faces. Lemma 3.6 will show that if a rectangular face is adjacent to a pentagon, then the adjacent pentagon is unique, we call it an *isolated* rectangular face. If a rectangular face is not isolated, we call it a *non-isolated* rectangular face. For non-isolated rectangular faces, we further divide them as follow; if the sizes of all adjacent faces are bigger than 6, then we call it a *non-moveable* rectangular face. If a rectangular face is non-isolated and it is adjacent to a hexagon, then we call it a *moveable* rectangular face. Then we observe the neighborhood of rectangular faces, (i) isolated rectangular faces in Lemma 3.8, (ii) moveable rectangular faces in Lemma 3.9. By looking at all polygons whose sizes are bigger than 6, we find a contradictory inequality to prove the theorem.

Lemma 3.5. *Let D be a counterexample with the hypothesis, the minimality of the number of faces and the maximality of the number of rectangular faces among counterexamples. Then, there can not be two adjacent rectangles of a valid sign type in D.*

Proof. Suppose not, then there exist two adjacent rectangles of a valid sign type in D. We apply an equation in Lemma 3.2 to change one of adjacent rectangles to a linear combination of webs with the rectangular faces with the opposite sign type. Then, two opposite edges of the other rectangle of adjacent rectangles have the same signs at each ends. By applying the last two relations in Figure 4, a web with two adjacent rectangles are now changed to a linear combination of webs with a hexagon and a bigon as depicted in Figure 11, consequently, just a hexagon. These processes result that the original web D is a linear combination of webs of less numbers of faces, however, since D is a counterexample, at least one of webs in this linear combination has to be a counterexample. But this can not be happened because of the minimality of the number of faces in D. $\qquad\square$

Let I be the number of non-isolated rectangular faces in D. The following lemma shows that each pentagon is adjacent to a unique isolated rectangular face, *i.e.*, $F_4 - I = F_5$.

Lemma 3.6. *Let D be a counterexample with the hypothesis, the minimality of the number of faces and the maximality of the number of rectangular faces among counterexamples. If there exists a pentagonal face in D, the pentagonal face is adjacent to a unique rectangular face.*

Proof. Suppose the pentagon is not adjacent to a rectangle, *i.e.*, the sizes of all adjacent faces of the pentagon are bigger than 4. But by applying one of the last two relations as shown in Figure 4 to the pentagonal face, it can be changed to a rectangle without changing the total number of faces but increasing the number of the rectangles by 1 in D. This contradicts the maximality
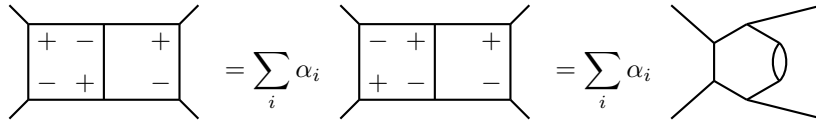
FIGURE 11. Two adjacent rectangles with a valid sign type and its linear expansion of webs with less number of faces.

of the number of the rectangular faces in D. Therefore, it must be adjacent to a rectangle. To show the uniqueness of the adjacent rectangle we consider the pentagon relations. There are five possible shapes by rotating the web in Figure 12 by $\frac{2\pi}{5}$. By applying the equations in Lemma 3.2, one can see that any two out of these five shapes are related as the equation as illustrated in Figure 13, which is called a *pentagon relation*. If two rectangles are adjacent to a pentagon, we use a pentagon relation to make a linear combinations of webs with two adjacent rectangles and webs of one less number of faces. Since D is a counterexample, one of these webs in the linear combination has to be a counterexample. However, a web with two adjacent rectangles can not be a counterexample as described before. A web with one less number of faces than that of D can not be a counterexample neither because of the minimality of the number of faces in D. Therefore, the adjacent rectangle of the pentagon has to be unique. □

The rectangular faces play the key role in the proof as mentioned before. We will show the existence of a rectangular face in the following lemma.

Lemma 3.7. *Let D be a counterexample with the hypothesis, the minimality of the number of faces and the maximality of the number of rectangular faces among counterexamples. Then, there exists a rectangular face in D.*

Proof. By Lemma 3.6, if there does not exist a rectangular face in D, then the size of all faces are bigger than 5, *i.e.*, $2E \geq 6F$. By combining with equations (3) and (4), we easily find a contradiction as follows,

$$2 = V - E + F \leq \frac{2}{3}E - E + \frac{1}{3}E = 0.$$ □

To proceed the proof of Theorem 3.4, we introduce a way modifying the web which has a moveable rectangular face, recall that a moveable rectangular face is a rectangular face adjacent to a hexagon. If a rectangular face is moveable, the relations in Lemma 3.2 allow us to change the web D to a linear combination of webs with a rectangular face of an opposite sign type and webs of less number of faces. For the webs with a rectangular face of an opposite sign type, one can see that two edges in the hexagonal face have the same sign at the ends. By applying relations shown in Figure 4, the positions of the rectangle
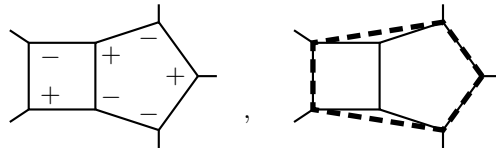
FIGURE 12. A valid sign type for an isolated rectangle and its adjacent pentagon and their corresponding new pentagon presented by dashed lines considered in Lemma 3.8.

FIGURE 13. A pentagonal relation.

FIGURE 14. A general swapping move for a moveable rectangular face.

and the hexagon are now interchanged as illustrated in Figure 14 which is called a *swapping move*. Because of the hypothesis of D, one of these new webs with adjacent rectangle and hexagon has to be a counterexample, say D'. Slightly abusing notation, we will say that the original counterexample D can be changed to D' by a swapping move.

In the following lemma, we look at the neighborhood of each isolated rectangular face. Since every isolated rectangular face is adjacent to a pentagon, we will treat these two adjacent polygons as a new pentagon as depicted in the right side of Figure 12.

Lemma 3.8. *Let D be a counterexample with the hypothesis, the minimality of the number of faces and the maximality of the number of rectangular faces among counterexamples. Suppose D has an isolated rectangular face. Then, the size of all adjacent polygons of a new pentagon drawn for an isolated rectangular face and its adjacent pentagon as in the right side of Figure 12 must be bigger than 7.*

Proof. First we already know that these polygons adjacent to the pentagon can not be a rectangle nor a pentagon. Suppose one of them is either a hexagon or a
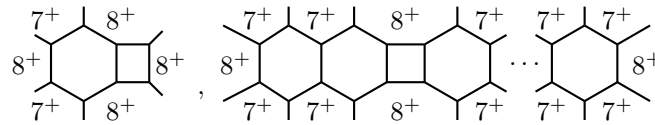
FIGURE 15. Two possible neighborhoods of the moveable rectangular face.

heptagon. Either case, we can rotate the pentagon using the pentagon relations as shown in Figure 13 such that the rectangular face is adjacent to the hexagon or the heptagon (since the rectangle must have a valid sign type, otherwise, it can be removed resulting a contradiction of the minimality of the number of faces in D). But, we can change the heptagon to a hexagon using the relations as illustrated in Figure 4. Then by a swapping move as depicted in Figure 14, the rectangle in D can be separated from the pentagon. Then every web in the linear combinations has either one less number of faces or a pentagon which is not adjacent to a rectangle, thus this pentagon can be changed to a rectangle using the relations shown in Figure 4, consequently, it increases the number of rectangular faces. Since D is a counterexample, at least one webs in the linear combinations has to be a counterexample but neither cases is possible because of the hypothesis of D.   □

Now we proceed to the next lemma, we look at the neighborhood of a moveable rectangular face. One can easily see that all adjacent hexagonal faces must have a valid sign type (otherwise, they become two adjacent rectangular faces using the relations shown in Figure 4, which is impossible).

Lemma 3.9. *Let D be a counterexample with the hypothesis, the minimality of the number of faces and the maximality of the number of rectangular faces. The only possible neighborhood of a moveable rectangular face must be either one of webs in Figure 15 or a circular web obtained from the right one in Figure 15 by attaching the rightmost and leftmost hexagons.*

Proof. We will divide cases by the number of adjacent hexagonal faces of the moveable rectangular face. If there is only one hexagon in the neighborhood of the rectangle, we do swap the rectangle with the hexagon. After a swapping move, we repeat the process from the new rectangle. If there is still only one hexagon, then we find the desired result as shown in the left side of Figure 15. If there are more than one hexagon, it will be dealt with in the next cases.

If there are two adjacent hexagonal faces, there are also two possibilities. If these two hexagons of valid sign types are adjacent to each other, we use the relations in Lemma 3.2 to change the rectangular face to a linear combination of webs of less number of faces or webs with hexagons which can be changed into two rectangles as illustrated in Figure 16. Since D is a counterexample,
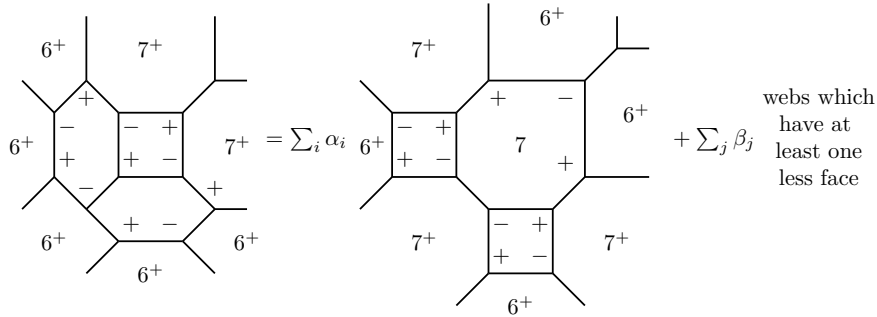
FIGURE 16. Two adjacent hexagons in the neighborhood of the rectangle can not exist in D.

one of webs in the linear combination has to be a counterexample but neither cases can be a counterexample because of the hypothesis of D. If these two hexagons are not adjacent each other, then after a swapping move toward to both directions, we can repeat the process at the new rectangles. Since there are only finitely many faces, this process either stops in finite steps which gives a web in the right side of Figure 15 or repeats infinitely which gives a circular web obtained from the right one in Figure 15 by attaching the rightmost and leftmost hexagons. For three or four adjacent hexagons, we must have at least two adjacent hexagons which is not possible as described previously.

Furthermore, the traces of swapping moves of different moveable rectangular faces are disjoint. Otherwise, we can repeat swapping moves to make these two moveable rectangular faces adjacent. But as we mentioned before, it is not possible to have adjacent rectangles.   □

Now, we are set to find a contradictory inequality. Instead of rectangular faces, we are going to look at all polygons whose sizes are bigger than 6. From Lemma 3.8, three edges from isolated rectangular face and four edges from its adjacent pentagon, total $4F_5 + 3(F_4 - I)$ edges, can be common edges of faces which have more than seven edges. From Lemma 3.9, for edges in polygons whose size is bigger than 6 which is not a common edge with isolated rectangles and their adjacent pentagon, there exists at most one edge of a moveable rectangular face can travel to become the common edge of the given edge by a sequence of swapping moves. Furthermore, since all webs are trivalent and no two rectangles are adjacent, for each face of size $n \geq 9$, there are at most $\lfloor \frac{2n}{3} \rfloor$ rectangular faces or pentagonal faces which are originally adjacent to the given face or adjacent to the given face by a finite sequence of swapping moves. For a heptagon, it is not adjacent to an isolated rectangle by Lemma 3.8 and we can see that three rectangles can not be adjacent to the heptagon because one can easily see that two of these three rectangles can be adjacent by a swapping move, and having two adjacent rectangles is impossible as mentioned before.

For an octagon, it is not adjacent to an isolated rectangle by Lemma 3.8 and it can be adjacent to up to 4 rectangular faces. These can be summarized as the following inequality,

$$(6) \quad 4F_5 + 3(F_4 - I) + 4I \leq 2F_7 + 4F_8 + 6F_9 + 6F_{10} + \cdots + \lfloor \frac{2n}{3} \rfloor F_n + \cdots.$$

Since $I = F_4 - F_5$, we have $4F_5 + 3(F_4 - I) + 4I = 4F_5 + 3F_4 + I = 3F_5 + 4F_4$. Using the fact $(n-6) \geq \lceil \frac{1}{2} \lfloor \frac{2n}{3} \rfloor \rceil$ for all $n \geq 9$, we get the following inequalities,

$$(7) \quad 4F_4 + 2F_5 \leq 4F_4 + 3F_5 \leq 2F_7 + 4F_8 + 6F_9 + 6F_{10} + \cdots + \lfloor \frac{2n}{3} \rfloor F_n + \cdots,$$

$$(8) \quad 2F_4 + F_5 \leq 1F_7 + 2F_8 + 3F_9 + 4F_{10} + \cdots + (n-6)F_n + \cdots,$$

$$(9) \quad 0 \leq -2F_4 - F_5 + 1F_7 + 2F_8 + 3F_9 + 4F_{10} + \cdots + (n-6)F_n + \cdots.$$

By adding the last inequality (9) to the equality (10), we obtain the desired inequality between the number of faces and edges as follows,

$$(10) \quad 6F \quad = \quad 6F_4 + 6F_5 + 6F_6 + 6F_7 + 6F_8 + \cdots + 6F_n + \cdots,$$

$$(11) \quad 6F \quad \leq \quad 4F_4 + 5F_5 + 6F_6 + 7F_7 + 8F_8 + \cdots + nF_n + \cdots = 2E.$$

If we substitute the inequality (11), $F \leq \frac{1}{3}E$, and the equation (3) into the equation (4), then we find a contradiction as

$$2 = V - E + F \leq \frac{2}{3}E - E + \frac{1}{3}E = 0.$$

Therefore, this completes the proof. □

4. Proofs of main results

If links are decorated by the fundamental representations V_{λ_i} of the quantum $\mathfrak{sl}(n)$, denoted by i, Murakami, Ohtsuki and Yamada [27] found a quantum invariant for framed links by resolving each crossing in a link diagram D of L as shown in Figures 12 and 13.

For negative crossings, we replace q with q^{-1}. $P_n(q)$ is the special case of $q^{-\omega(D)\frac{n}{2}}[D]_n$, when all components are colored by the fundamental representation V_{λ_1}. They showed that $P_n(q)$ is an isotopy invariant and that $[D]_n$ is a regular isotopy invariant for other colorings [27]. However, one can make it a link invariant by using a suitable writhe but we have to be careful since there are more than one colors. For a coloring μ of a diagram D of a link L, we first consider a *colored writhe* $\omega_i(D)$ as the sum of writhes of components colored by i. Then we set

$$K_n(L, \mu) = \prod_i q^{-\omega_i(D)\frac{i(n-i+1)}{2}}[D]_n,$$

where the product runs over all colors i.

By Theorem 3.4, we know $K_n(L, \mu)$ is well defined. It was shown that $[D]_n$ is a regular isotopy invariant [27], so is $K_n(L, \mu)$. One can find the equations in Figure 18 by the second relation in Figure 4 and a routine induction. Then

$$(12) \quad \left[\begin{matrix} i & & j \\ & \times & \\ j & & i \end{matrix} \right]_n = \sum_{k=0}^{i} (-1)^{k+(j+1)i} q^{\frac{(i-k)}{2}} \left[\begin{matrix} i & \overset{j+k-i}{\longrightarrow} & j \\ j+k & & i-k \\ j & \underset{k}{\longrightarrow} & i \end{matrix} \right]_n$$

$$(13) \quad \left[\begin{matrix} i & & j \\ & \times & \\ j & & i \end{matrix} \right]_n = \sum_{k=0}^{j} (-1)^{k+(i+1)j} q^{\frac{(j-k)}{2}} \left[\begin{matrix} i & \overset{i+k-j}{\longrightarrow} & j \\ j-k & & i+k \\ j & \underset{k}{\longrightarrow} & i \end{matrix} \right]_n$$

FIGURE 17. Skein expansions of a crossing.

$$\left[\begin{matrix} \end{matrix} \right]_n = q^{\frac{i(n-i+1)}{2}} \left[\begin{matrix} \end{matrix} \right]_n , \quad \left[\begin{matrix} \end{matrix} \right]_n = q^{-\frac{i(n-i+1)}{2}} \left[\begin{matrix} \end{matrix} \right]_n$$

FIGURE 18. Resolving Reidemeister move I.

$K_n(L,\mu)$ is invariant under Reidemeister move I. Therefore, $K_n(L,\mu)$ is an isotopy invariant of a link L. A coloring μ on a p-periodic link L is a p-periodic coloring of L if the periodic homeomorphism h of order p used for the periodicity of L also preserves the coloring $i.e.$, $h(L,\mu) = (L,\mu)$. For such a coloring μ on a periodic link L, we also denote the factor link $\overline{L} = \pi(L)$ and natural coloring on $\overline{L}$ by $\overline{\mu}$. Now we discuss the relation between $K_n(L,\mu)$ and $K_n(\overline{L},\overline{\mu})$ for a p-periodic link L in the following theorems.

Theorem 4.1. *Let p be a positive integer and L be a p-periodic link in $\mathbb{S}^3$ with its factor link $\overline{L}$. Let μ be a p-periodic coloring of L and $\overline{\mu}$ be the induced coloring of $\overline{L}$. Then for $n \geq 0$,*

$$K_n(L,\mu) \equiv K_n(\overline{L},\overline{\mu})^p \qquad modulo \ \mathcal{I}_n,$$

where $\mathcal{I}_n$ is the ideal of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]^p - \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$ for $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$.

Proof. Let μ be a p-periodic coloring of L. Let R be the fundamental region by the action h. Let $\mathcal{C}$ be the set of all crossings of L and let $\overline{\mathcal{C}}$ be the set of all crossings in the region R. Let $i(c)$ and $j(c)$ be the weights of two components at the crossing c as in Figure 17. Let $J(c)$ be the minimum of $i(c)$ and $j(c)$ for the crossing c. Let D be the diagram after the expansion by the equations

in Figures 17. Let D' be the diagram obtained by identical expanding of the crossings which are the same by the action h and D'' be the diagram obtained from D' by identical applications of relations at the faces which are in the same orbit by the action h. Let $\overline{D''} = D''/\mathbb{Z}_p$ and $\mathbb{Z}_p$ is generated by the action h. Let $\mathcal{I}_n$ be the ideal of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $\left[\begin{smallmatrix}n\\i\end{smallmatrix}\right]^p - \left[\begin{smallmatrix}n\\i\end{smallmatrix}\right]$ for $i = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$.

$$(14) \quad K_n(L, \mu) = \prod_{c \in \mathcal{C}} \sum_{k=0}^{J(c)} (-1)^{k+(i(c)+1)j(c)} q^{\frac{(j(c)-k)}{2}} [D]_n$$

$$\equiv \prod_{\overline{c} \in \overline{\mathcal{C}}} \sum_{k=0}^{J(\overline{c})} ((-1)^{k+(i(\overline{c})+1)j(\overline{c})} q^{\frac{(j(\overline{c})-k)}{2}})^p [D']_n \qquad (\mathrm{mod} \quad p)$$

$$\equiv \prod_{\overline{c} \in \overline{\mathcal{C}}} \sum_{k=0}^{J(\overline{c})} ((-1)^{k+(i(\overline{c})+1)j(\overline{c})} q^{\frac{(j(\overline{c})-k)}{2}})^p [D'']_n \qquad (\mathrm{mod} \quad p)$$

$$\equiv \prod_{\overline{c} \in \overline{\mathcal{C}}} \sum_{k=0}^{J(\overline{c})} ((-1)^{k+(i(\overline{c})+1)j(\overline{c})} q^{\frac{(j(\overline{c})-k)}{2}})^p ([\overline{D''}]_n)^p \qquad (\mathrm{mod} \quad \mathcal{I}_n)$$

$$\equiv \prod_{\overline{c} \in \overline{\mathcal{C}}} \sum_{k=0}^{J(\overline{c})} ((-1)^{k+(i(\overline{c})+1)j(\overline{c})} q^{\frac{(j(\overline{c})-k)}{2}} [\overline{D}]_n)^p \qquad (\mathrm{mod} \quad p)$$

$$\equiv (\prod_{\overline{c} \in \overline{\mathcal{C}}} \sum_{k=0}^{J(\overline{c})} (-1)^{k+(i(\overline{c})+1)j(\overline{c})} q^{\frac{(j(\overline{c})-k)}{2}} [\overline{D}]_n)^p \qquad (\mathrm{mod} \quad p)$$

$$\equiv (K_n(\overline{L}, \overline{\mu}))^p \qquad (\mathrm{mod} \quad p).$$

If any expansion of crossings occurs in R, it must be used identically for all other $p - 1$ copies of R. Otherwise there will be p identical shapes by the rotation of order p, then the term in the expansion is congruent to zero modulo p. This implies the first congruence in equation (14). By the same philosophy, if any expansion of relations occurs in R, it must be used identically for all other $p - 1$ copies of R. Otherwise it is congruent to zero modulo p and this implies the second congruence. Let us remark that we have not used relations that might occur in the faces which are not entirely contained in R. If there is an unknot in $\overline{D''}$ which was in D'', there are p copies in D''. Thus raising p-power gives us the equality. But if there is an unknot in $\overline{D''}$ which was not in D'', it should count once in D'' and $\overline{D''}$, thus we have to use $\mathcal{I}_n$ congruence. $\square$

Remark 4.2. In the proof, one can easily generalize the congruence relation between the quantum invariants $[\ ]_n$ of a web diagram W of symmetry of order p and its quotient $\overline{W}$ modulo $\mathcal{I}_n$ as

$$[W]_n \equiv ([\overline{W}]_n)^p \qquad (\mathrm{mod} \quad \mathcal{I}_n).$$

But in general, this congruence is not true if $\overline{W}$ is a base web diagram [18].

Next, we find that Theorem 4.1 remains true for the colored $\mathfrak{sl}(n)$ HOMFLY polynomial of a periodic link whose components are colored by irreducible representations of quantum $\mathfrak{sl}(n)$.

Theorem 4.3. *Let p be a positive integer and L be a p-periodic link in $\mathbb{S}^3$ with its factor link $\overline{L}$. Let μ be a p-periodic coloring of L and $\overline{\mu}$ be the induced coloring of $\overline{L}$. Then for $n \geq 0$,*

$$G_n(L,\mu) \equiv G_n(\overline{L},\overline{\mu})^p \qquad modulo \ \ \mathcal{I}_n,$$

where $\mathcal{I}_n$ is the ideal of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]^p - \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Proof. Since all clasps are idempotents, we put p-copies of clasps to each copy of the fundamental region by the action h of the periodicity of L. Thus without expanding the clasps, we obtain the theorem by the same idea of the proof of the Theorem 4.1. $\qquad \square$

We also give a criterion for periodic links by using the invariant $K_n(L,\mu)$ and mirror image of knots in the following theorem.

Theorem 4.4. *Let L be a p-periodic link for a prime p and let L^* be the link obtained from the mirror image of a diagram of L. Let μ be a coloring of L and μ^* be the coloring of L^* induced from the coloring μ of L. Then we have*

$$K_n(L,\mu) \equiv K_n(L^*,\mu^*) \bmod (p, q^p - 1).$$

Proof. For a colored link diagram D we denote its mirror image by D^* which is the colored link diagram obtained from D by changing all of the crossings. We study another necessary condition for a colored link to be periodic by using the invariant $K_n(L,q)$. Let L be a colored link diagram with a crossing x. Let L_+, L_-, and L_0 be the link diagrams obtained by resolving the crossing x as shown in Figure 1 respectively. Then from Figure 17, we see that there exist web diagrams $L_1, \ldots, L_m$ and polynomials $f_1, \ldots, f_m \in Z[q^{\pm\frac{1}{2}}]$ for some positive integer m such that

$$[L_+]_n - [L_-]_n = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(f_1[L_1]_n + \cdots + f_m[L_m]_n).$$

In particular, by applying this relation repeatedly for a colored periodic link we obtain the following lemma.

Lemma 4.5. *Let D be a colored p-periodic link diagram and π be the quotient map of the periodic homeomorphism h of order p and $\overline{D}$ be the factor link of D so that $\pi^{-1}(\overline{D}) = D$. Let x be a crossing of a nontrivial diagram $\overline{D}$, and let $\overline{D}_+$ and $\overline{D}_-$ be the colored link diagram obtained by changing the crossing x to a positive crossing and negative crossing respectively. Then we have*

$$[\pi^{-1}(\overline{D}_+)]_n \equiv [\pi^{-1}(\overline{D}_-)]_n \bmod (p, q^p - 1).$$

Proof. Let $\overline{D}_1, \ldots, \overline{D}_m$ be web diagrams and let $f_1, \ldots, f_m \in Z[q^{\pm\frac{1}{2}}]$ be polynomials for some positive integer m such that

$$[\overline{D}_+]_n - [\overline{D}_-]_n = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(f_1[\overline{D}_1]_n + \cdots + f_m[\overline{D}_m]_n).$$

Then by considering the periodic action induced from the map h, we get

$$[\pi^{-1}(\overline{D}_+)]_n - [\pi^{-1}(\overline{D}_-)]_n$$
$$\equiv (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^p(f_1^p[\pi^{-1}(\overline{D}_1)]_n + \cdots + f_m^p[\pi^{-1}(\overline{D}_m)]_n) \bmod (p)$$
$$\equiv 0 \bmod (p, q^p - 1). \qquad \square$$

Now, we are ready to prove theorem. For a tangle T, we denote its closure by $Cl(T)$ if it is well defined. Let L be a p–periodic link and T be a tangle such that L is the closure $Cl(T^p)$ of the tangle T^p which is p times self-product of T. Let D be a diagram of L and x be a crossing of the diagram D. Let T_+ and T_- be the diagram obtained from the diagram of T by changing the crossing x to a positive crossing and negative crossing respectively. If the two colorings of the strands near the crossing x are different then by using Lemma 4.5, we see that

$$K_n(Cl((T_+)^p), \mu) - K_n(Cl((T_-)^p), \mu)$$
$$= \prod_i q^{-w_i(D)\frac{i(n-i+1)}{2}}([Cl((T_+)^p)]_n - [Cl((T_-)^p)]_n)$$
$$\equiv 0 \bmod (p, q^p - 1).$$

If two colorings of the strands near the crossing x are equal, say i, then $w_i(Cl((T_+)^p)) = w_i(Cl((T_+)^p)) + 2p$. Then we see that there exists a polynomial $g \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ such that

$$K_n(Cl((T_+)^p), \mu) - K_n(Cl((T_-)^p), \mu)$$
$$= g([Cl((T_+)^p)]_n - q^{pi(n-i+1)}[Cl((T_-)^p)]_n)$$
$$\equiv 0 \bmod (p, q^p - 1).$$

The last congruence relation in the above formulae can be obtained by using Lemma 4.5. Thus we see that

$$K_n(Cl((T_+)^p), \mu) \equiv K_n(Cl((T_-)^p), \mu) \mod (p, q^p - 1).$$

Since T can be obtained from its mirror image T^* by a finite sequence of crossing changes, we get

$$K_n(L, \mu) \equiv K_n(L^*, \mu^*) \mod (p, q^p - 1).$$

It completes the proof the theorem. $\qquad \square$

5. Discussion

To compare our result with previous results, one has to compare the size of ideal used in Theorem 4.1 with ideals in [4, 6, 30, 25]. As we have mentioned, we have corrected the false conjecture given by Chbili [4] and Przytycki and Sikora [30]. To compare with the ideal in [25], we observe that the ideal $\mathcal{I}_n$ of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $\begin{bmatrix} n \\ i \end{bmatrix}^p - \begin{bmatrix} n \\ i \end{bmatrix}$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ is a **subset** of the ideal generated by p and $[2]^p - [2]$ by the strong integrality of the quantum link invariant [21]. Furthermore, if n is odd, the ideal $\mathcal{I}_n$ is a subset of the ideal generated by p and $[3]^p - [3]$. To compare the ideal generated by p and $[2]^p - [2]$ with the ideal of Murasugi's [25] generated by p and

$$\xi_p(t) = \sum_{j=0}^{p-1} (-t)^j - t^{\frac{p-1}{2}},$$

we observe

$$(t+1)\xi_p(t) \equiv q^{-\frac{p}{2}}([2]^p - [2])\big|_{\sqrt{q}=-\frac{1}{\sqrt{t}}} \pmod{p}.$$

To compare with the ideal in [6], we use only fundamental representations of the quantum Lie algebras $\mathfrak{sl}(n)$ which are finite but Chen and Le used all representations of the quantum $\mathfrak{sl}(n)$ which are obviously infinite. Thus, our criterion is sharper than other previous results. However, we were not able to find a new periodicity of knots using our criteria.

References

[1] C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, New York, W. H. Freeman, 1994.

[2] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), no. 4, 883–927.

[3] S. Cautis and J. Kamnitzer, *Knot homology via derived categories of coherent sheaves I. The SL(2) case*, Duke Math. J. **142** (2008), no. 3, 511–588.

[4] N. Chbili, *The quantum SU(3) invariant of links and Murasugi's congruence*, Topology Appl. **122** (2002), no. 3, 479–485.

[5] ———, *Quantum invariants and finite group actions on three-manifolds*, Topology Appl. **136** (2004), no. 1-3, 219–231.

[6] Q. Chen and T. Le, *Quantum invariants of periodic links and periodic 3-manifolds*, Fund. Math. **184** (2004), 55–71.

[7] I. Frenkel and M. Khovanov, *Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$*, Duke Math. J. **87** (1997), no. 3, 409–480.

[8] W. Fulton and J. Harris, *Representation Theory*, Graduate Texts in Mathematics, 129, Springer-Verlag, New York-Heidelberg-Berlin, 1991.

[9] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25.

[10] ———, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), no. 2, 335–388.

[11] M.-J. Jeong and C.-Y. Park, *Lens knots, periodic links and Vassiliev invariants*, J. Knot Theory Ramifications **13** (2004), no. 8, 1041–1056.

[12] C. Kassel, M. Rosso, and V. Turaev, *Quantum Groups and Knot Invariants*, Panoramas et Syntheses, 5, Societe Mathematique de France, 1997.

[13] M. Khovanov, *sl(3) link homology*, Algebr. Geom. Topol. **4** (2004), 1045–1081.

[14] ———, *Categorifications of the colored Jones polynomial*, J. Knot Theory Ramifications **14** (2005), no. 1, 111–130.

[15] ———, private communication.

[16] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology*, Fund. Math. **199** (2008), no. 1, 1–91.

[17] D. Kim, *Graphical Calculus on Representations of Quantum Lie Algebras*, Thesis, UC-Davis, 2003, arXiv:math.QA/0310143.

[18] D. Kim and J. Lee, *The quantum sl(3) invariants of cubic bipartite planar graphs*, J. Knot Theory Ramifications **17** (2008), no. 3, 361–375.

[19] R. Kirby and P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin-Turaev for $\mathfrak{sl}(2)$*, Invent. Math. **105** (1991), no. 3, 473–545.

[20] G. Kuperberg, *Spiders for rank 2 Lie algebras*, Comm. Math. Phys. **180** (1996), no. 1, 109–151.

[21] T. Le, *Integrality and symmetry of quantum link invariants*, Duke Math. J. **102** (2000), no. 2, 273–306.

[22] W. Lickorish, *Distinct 3-manifolds with all $SU(2)_q$ invariants the same*, Proc. Amer. Math. Soc. **117** (1993), no. 1, 285–292.

[23] S. Morrison, *A Diagrammatic Category for the Representation Theory of $U_q(sl_n)$*, UC Berkeley Ph.D. thesis, arXiv:0704.1503.

[24] K. Murasugi, *On periodic knots*, Comment. Math. Helv. **46** (1971), 162–174.

[25] ———, *Jones polynomials of periodic links*, Pacific J. Math. **131** (1988), no. 2, 319–329.

[26] H. Murakami, *Asymptotic Behaviors of the colored Jones polynomials of a torus knot*, Internat. J. Math. **15** (2004), no. 6, 547–555.

[27] H. Murakami, T. Ohtsuki, and S. Yamada, *Homfly polynomial via an invariant of colored plane graphs*, Enseign. Math. (2) **44** (1998), no. 3-4, 325–360.

[28] T. Ohtsuki and S. Yamada, *Quantum SU(3) invariant of 3-manifolds via linear skein theory*, J. Knot Theory Ramifications **6** (1997), no. 3, 373–404.

[29] J. H. Przytycki, *On Murasugi's and Traczyk's criteria for periodic links*, Math. Ann. **283** (1989), no. 3, 465–478.

[30] J. Przytycki and A. Sikora, *SU_n-quantum invariants for periodic links*, Diagrammatic morphisms and applications (San Francisco, CA, 2000), 199–205, Contemp. Math., 318, Amer. Math. Soc., Providence, RI, 2003.

[31] ———, *On skein algebras and $Sl_2(C)$-character varieties*, Topology **39** (2000), no. 1, 115–148.

[32] N. Yu. Reshetikhin and V. G. Turaev, *Ribbob graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26.

[33] ———, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), no. 3, 547–597.

[34] A. Sikora and B. Westbury, *Confluence theory for graphs*, Algebr. Geom. Topol. **7** (2007), 439–478.

[35] P. Traczyk, *A criterion for knots of period 3*, Topology Appl. **36** (1990), no. 3, 275–281.

[36] V. G. Turaev, *The Conway and Kauffman modules of a solid torusa*, (translation) J. Soviet Math. **52** (1990), no. 1, 2799–2805.

[37] T. Van Zandt, *PSTricks: PostScript macros for generic T$_E$X*, Available at `ftp://ftp.princeton.edu/pub/tvz/`.

[38] M. Vybornov, *Solutions of the Yang-Baxter equation and quantum $\mathfrak{sl}(2)$*, J. Knot Theory Ramifications **8** (1999), no. 7, 953–961.

[39] H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada **9** (1987), no. 1, 5–9.

[40] B. Westbury, *Invariant tensors for the spin representation of $\mathfrak{so}(7)$*, Math. Proc. Cambridge Philos. Soc. **144** (2008), no. 1, 217–240.

[41] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399.

[42] Y. Yokota, *The skein polynomial of periodic knots*, Math. Ann. **291** (1991), no. 2, 281–291.

[43] ———, *The Jones polynomial of periodic knots*, Proc. Amer. Math. Soc. **113** (1991), no. 3, 889–894.

[44] ———, *The Kauffman polynomial of periodic knots*, Topology **32** (1993), no. 2, 309–324.

[45] ———, *Skein and quantum $SU(N)$ invariants of 3-manifolds*, Math. Ann. **307** (1997), no. 1, 109–138.

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