J. Korean Math. Soc. **49** (2012), No. 5, pp. 977–991
http://dx.doi.org/10.4134/JKMS.2012.49.5.977

ON TRANSVERSALLY HARMONIC MAPS OF FOLIATED RIEMANNIAN MANIFOLDS

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Abstract. Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliated Riemannian manifolds with M compact. If the transversal Ricci curvature of $\mathcal{F}$ is nonnegative and the transversal sectional curvature of $\mathcal{F}'$ is nonpositive, then any transversally harmonic map $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is transversally totally geodesic. In addition, if the transversal Ricci curvature is positive at some point, then ϕ is transversally constant.

1. Introduction

Transversally harmonic maps of foliated Riemannian manifolds were introduced by Konderak and Wolak [6] in 2003. Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliated Riemannian manifolds and let $\phi : M \to M'$ be a smooth foliated map, i.e., ϕ is a smooth leaf-preserving map. Then ϕ is said to be *transversally harmonic* if the transversal tension field $\tau_b(\phi)$ vanishes. See Section 3 and [6] for details. Equivalently, it is a critical point of the transversal energy functional on any compact domain of M, which is defined in Section 4 (cf. [7]). Also, transversally harmonic maps are considered as harmonic maps between the leaf spaces [6, 7]. So, for the point foliation, transversally harmonic maps are harmonic maps. Therefore transversally harmonic maps are considered as generalizations of harmonic maps. In this paper, we study transversally harmonic maps and give some interesting facts relating to them. The paper is organized as follows. In Section 2, we review the well-known facts on a foliated Riemannian manifold. In Section 3, we review the properties of the transversally harmonic map, which were studied in [7] and give some results. In Section 4, we give a new proof of the first normal variational formula for the transversal energy $E_B(\phi)$ (Theorem 4.1). In the last section, we study the generalized Weitzenböck formula and give some applications (Theorem 5.3 and Theorem 5.4).

Received January 20, 2011; Revised January 31, 2012.
2010 *Mathematics Subject Classification.* 53C12, 58E20.
Key words and phrases. transversal tension field, transversally harmonic map, normal variational formula, generalized Weitzenböck type formula.

2. Preliminaries

Let $(M, g, \mathcal{F})$ be a $(p+q)$-dimensional foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g with respect to $\mathcal{F}$ [8, 13]. A *foliated Riemannian manifold* means a Riemannian manifold with a Riemannian foliation. Let TM be the tangent bundle of M, L the tangent bundle of $\mathcal{F}$, and $Q = TM/L$ the corresponding normal bundle of $\mathcal{F}$. Then we have an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0, \tag{2.1}$$

where $\pi : TM \to Q$ is a projection and $\sigma : Q \to L^{\perp}$ is a bundle map satisfying $\pi \circ \sigma = id$. Let g_Q be the holonomy invariant metric on Q induced by $g = g_L + g_{L^{\perp}}$; that is,

$$g_Q(s,t) = g(\sigma(s), \sigma(t)) \quad \forall\ s, t \in \Gamma Q. \tag{2.2}$$

This means that $\theta(X) g_Q = 0$ for $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative. So we have an identification $L^{\perp}$ with Q via an isometric splitting $(Q, g_Q) \cong (L^{\perp}, g_{L^{\perp}})$. We denote by ∇^Q the transverse Levi-Civita connection on the normal bundle Q [13, 14]. The transversal curvature tensor R^Q of $\nabla^Q \equiv \nabla$ is defined by $R^Q(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for any $X, Y \in \Gamma TM$. It is trivial that $i(X)R^Q = 0$ for any $X \in \Gamma L$, where $i(X)$ is the interior product. Let K^Q, Ric^Q and σ^Q be the transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to ∇, respectively. The foliation $\mathcal{F}$ is said to be *minimal* if $\kappa = 0$, where κ is the mean curvature form of $\mathcal{F}$ [13]. Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e., $\phi \in \Omega^r_B(\mathcal{F})$ if and only if $i(X)\phi = 0$ and $\theta(X)\phi = 0$ for any $X \in \Gamma L$. Then $\Omega^*(M) = \Omega^*_B(\mathcal{F}) \oplus \Omega^*_B(\mathcal{F})^{\perp}$ [1]. Let κ_B be the basic part of κ. Then κ_B is closed, i.e., $d\kappa_B = 0$ [1]. Now, we define the *basic Laplacian* Δ_B acting on $\Omega^*_B(\mathcal{F})$ by

$$\Delta_B = d_B \delta_B + \delta_B d_B, \tag{2.3}$$

where δ_B is the formal adjoint of $d_B = d|_{\Omega^*_B(\mathcal{F})}$ [11]. Let $\{E_a\}_{a=1,\dots,q}$ be a local orthonormal basic frame on Q. We define $\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}} : \Omega^r_B(\mathcal{F}) \to \Omega^r_B(\mathcal{F})$ by

$$\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}} = -\sum_a \nabla^2_{E_a, E_a} + \nabla_{\kappa^{\sharp}_B}, \tag{2.4}$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in \Gamma TM$ and ∇^M denotes the Levi-Civita connection of M. Then the operator $\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}$ is positive definite and formally self adjoint on the space of basic forms [3]. Let $V(\mathcal{F})$ be the space of all transversal infinitesimal automorphisms Y of $\mathcal{F}$, i.e., $[Y, Z] \in \Gamma L$ for all $Z \in \Gamma L$ [5]. Let

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) \mid Y \in V(\mathcal{F})\}. \tag{2.5}$$

Note that $\bar{V}(\mathcal{F}) \cong \Omega^1_B(\mathcal{F})$ [10]. For later use, we recall the transversal divergence theorem [16] on a foliated Riemannian manifold.

Theorem 2.1 (Transversal divergence theorem)**.** *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M \operatorname{div}_\nabla \bar{X} = \int_M g_Q(\bar{X}, \kappa_B^\sharp) \tag{2.6}$$

for all $X \in V(\mathcal{F})$, where $\operatorname{div}_\nabla X$ denotes the transversal divergence of X with respect to the connection ∇.

Now we define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in V(\mathcal{F})$ [5] by

$$A_Y \phi = \theta(Y)\phi - \nabla_Y \phi. \tag{2.7}$$

It is well-known [5] that for any $s \in \Gamma Q$

$$A_Y s = -\nabla_{Y_s} \bar{Y}, \tag{2.8}$$

where Y_s is the vector field such that $\pi(Y_s) = s$. So A_Y depends only on $\bar{Y} = \pi(Y)$. Since $\theta(X)\phi = \nabla_X \phi$ for any $X \in \Gamma L$, A_Y preserves the basic forms and depends only on $\bar{Y}$. Now, we recall the generalized Weitzenböck formula on $\Omega_B^*(\mathcal{F})$.

Theorem 2.2 ([3])**.** *On a foliated Riemannian manifold $(M, \mathcal{F})$, we have*

$$\Delta_B \phi = \nabla_{\mathrm{tr}}^* \nabla_{\mathrm{tr}} \phi + F(\phi) + A_{\kappa_B^\sharp} \phi, \quad \phi \in \Omega_B^r(\mathcal{F}), \tag{2.9}$$

where $F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi$. If ϕ is a basic 1-form, then $F(\phi)^\sharp = \operatorname{Ric}^Q(\phi^\sharp)$.

Now we recall a very important lemma for later use. From Proposition 4.1 in [11], it is well-known that $\Delta_B - \kappa_B^\sharp$ on all basic functions is the restriction of $\Delta - \kappa^\sharp$ on all functions. Hence, by maximum and minimum principles, we have the following lemma.

Lemma 2.3 ([4])**.** *Let $(M, g, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g. If $(\Delta_B - \kappa_B^\sharp) f \geq 0$ ($or \leq 0$) for any basic function f, then f is constant.*

3. Transversally harmonic maps

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds. Let ∇^M and $\nabla^{M'}$ be the Levi-Civita connections of M and M', respectively. Let ∇ and ∇' be the transverse Levi-Civita connections on Q and Q', respectively. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth *foliated map*, i.e., $d\phi(L) \subset L'$. Then we define $d_T\phi : Q \to Q'$ by

$$d_T\phi := \pi' \circ d\phi \circ \sigma. \tag{3.1}$$

Then $d_T\phi$ is a section in $Q^* \otimes \phi^{-1}Q'$, where $\phi^{-1}Q'$ is the pull-back bundle on M. Let ∇^ϕ and $\tilde{\nabla}$ be the connections on $\phi^{-1}Q'$ and $Q^* \otimes \phi^{-1}Q'$, respectively. Then

a foliated map $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is called *transversally totally geodesic* if it satisfies

$$\tilde{\nabla}_{\mathrm{tr}} d_T\phi = 0, \tag{3.2}$$

where $(\tilde{\nabla}_{\mathrm{tr}} d_T\phi)(X, Y) = (\tilde{\nabla}_X d_T\phi)(Y)$ for any $X, Y \in \Gamma Q$. Note that if $\phi : M \to M'$ is transversally totally geodesic with $d\phi(Q) \subset Q'$, then, for any transversal geodesic γ in M, $\phi \circ \gamma$ is also transversal geodesic. The *transversal tension field* of ϕ is defined by

$$\tau_b(\phi) = \mathrm{tr}_Q \tilde{\nabla} d_T\phi = \sum_{a=1}^{q} (\tilde{\nabla}_{E_a} d_T\phi)(E_a), \tag{3.3}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q. Trivially, the transversal tension field $\tau_b(\phi)$ is a section of $\phi^{-1}Q'$.

Definition 3.1. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then ϕ is said to be transversally harmonic if the transversal tension field of ϕ vanishes, i.e., $\tau_b(\phi) = 0$.

Now we recall the O'Neill tensors $\mathcal{A}$ and $\mathcal{T}$ [9] on a foliated manifold $(M, \mathcal{F})$, which are defined by

$$\mathcal{A}_X Y = \pi^\perp(\nabla^M_{\pi(X)} \pi(Y)) + \pi(\nabla^M_{\pi(X)} \pi^\perp(Y)), \tag{3.4}$$

$$\mathcal{T}_X Y = \pi^\perp(\nabla^M_{\pi^\perp(X)} \pi(Y)) + \pi(\nabla^M_{\pi^\perp(X)} \pi^\perp(Y)) \tag{3.5}$$

for any $X, Y \in \Gamma TM$, where $\pi^\perp : TM \to L$. It is well-known [9] that

$$\mathcal{A}_{\pi(X)} \pi(Y) = \pi^\perp[\pi(X), \pi(Y)] \tag{3.6}$$

for any vector fields X, Y on M. Then $\mathcal{T} \equiv 0$ is equivalent to the property that all leaves of $\mathcal{F}$ are totally geodesic submanifolds of (M, g) and $\mathcal{A} \equiv 0$ is equivalent to the integrability of Q.

Let $\{E_i\}_{i=1,\dots,p}$ be a local orthonormal basis of L and $\{E_a\}_{a=1,\dots,q}$ be a local orthonormal basic frame on Q. Then we have the following.

Theorem 3.2. *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then*

$$\begin{aligned}\tau(\phi) = {} & \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi) - d_T\phi(\kappa^\sharp) + \mathrm{tr}_g \phi^* \mathcal{T}' + \mathrm{tr}_Q \phi^* \mathcal{A}' \\ & + \sum_a \{\pi^\perp \nabla^{M'}_{\pi d\phi(E_a)} \pi^\perp d\phi(E_a) + \pi^\perp \nabla^{M'}_{\pi^\perp d\phi(E_a)} \pi^\perp d\phi(E_a) - \pi^\perp d\phi(\nabla_{E_a} E_a)\} \\ & + \sum_a \pi \nabla^{M'}_{\pi^\perp d\phi(E_a)} \pi d\phi(E_a),\end{aligned}$$

where $\tau(\phi)$ is the tension field of ϕ and

$$\tau(\phi|_{\mathcal{F}}) = \pi^\perp \sum_i (\tilde{\nabla}_{E_i} d\phi)(E_i). \tag{3.7}$$

Proof. Let $\{E_i, E_a\}_{i=1,\dots,p;a=1,\dots,q}$ be a local orthonormal frame of TM such that $E_i \in \Gamma L$, $E_a \in \Gamma Q$. By the definition of the tension field, we have

$$\tau(\phi) = \sum_{i=1}^{p} (\tilde{\nabla}_{E_i} d\phi)(E_i) + \sum_{a=1}^{q} (\tilde{\nabla}_{E_a} d\phi)(E_a). \tag{3.8}$$

Since ϕ is a foliated map, $\pi d\phi(E_i) = 0$ and $\pi^{\perp} d\phi(E_i) = d\phi(E_i)$. Therefore, we have

$$\sum_{i=1}^{p} (\tilde{\nabla}_{E_i} d\phi)(E_i) = \tau(\phi|_{\mathcal{F}}) + \sum_{i} \{\pi \nabla^{M'}_{d\phi(E_i)} d\phi(E_i) - \pi d\phi(\nabla^{M}_{E_i} E_i)\}$$

and

$$\begin{aligned}
&\sum_{a=1}^{q} (\tilde{\nabla}_{E_a} d\phi)(E_a) \\
&= \tau_b(\phi) + \sum_{a} \{\pi^{\perp} \nabla^{M'}_{\pi d\phi(E_a)} \pi d\phi(E_a) + \nabla^{M'}_{\pi d\phi(E_a)} \pi^{\perp} d\phi(E_a)\} \\
&\quad + \sum_{a} \{\nabla^{M'}_{\pi^{\perp} d\phi(E_a)} \pi d\phi(E_a) + \nabla^{M'}_{\pi^{\perp} d\phi(E_a)} \pi^{\perp} d\phi(E_a) - \pi^{\perp} d\phi(\nabla^{M}_{E_a} E_a)\}.
\end{aligned}$$

From (3.6), we have $\pi^{\perp} \nabla^{M'}_{\pi d\phi(E_a)} \pi d\phi(E_a) = \pi^{\perp} \nabla^{M}_{E_a} E_a = 0$. Hence, from (3.4) and (3.5), we have

$$\begin{aligned}
\tau(\phi) = \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi) - \pi d\phi(\sum_{i} \pi(\nabla^{M}_{E_i} E_i)) + \sum_{i} {\mathcal{T}'}_{d\phi(E_i)} d\phi(E_i) \\
+ \sum_{a} \{{\mathcal{T}'}_{d\phi(E_a)} d\phi(E_a) + {\mathcal{A}'}_{d\phi(E_a)} d\phi(E_a) + \pi \nabla^{M'}_{\pi^{\perp} d\phi(E_a)} \pi d\phi(E_a)\} \\
+ \sum_{a} \pi^{\perp} \{\nabla^{M'}_{\pi^{\perp} d\phi(E_a)} \pi^{\perp} d\phi(E_a) + \nabla^{M'}_{\pi d\phi(E_a)} \pi^{\perp} d\phi(E_a) - d\phi(\pi \nabla^{M}_{E_a} E_a)\}.
\end{aligned}$$

Since $\sum_i \pi(\nabla^{M}_{E_i} E_i) = \kappa^{\sharp}$, the proof is completed. □

Corollary 3.3 ([15]). *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map and $d\phi(Q) \subset Q'$. Then*

$$\tau(\phi) = \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi) - d\phi(\kappa^{\sharp}) + \mathrm{tr}_L \phi^* \mathcal{T}',$$

where $\mathrm{tr}_L \phi^* \mathcal{T}' = \sum_{i=1}^{p} {\mathcal{T}'}_{d\phi(E_i)} d\phi(E_i)$.

Proof. Since $d\phi(Q) \subset Q'$, $\pi^{\perp} d\phi(E_a) = 0$ for all a. Moreover, from (3.5) and (3.6), ${\mathcal{A}'}_X X = 0$ and ${\mathcal{T}'}_X Y = 0$ for any $X, Y \in \Gamma Q'$. From Theorem 3.2, the desired result follows. □

Corollary 3.4 ([7]). *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map and $d\phi(Q) \subset Q'$. Assume that $\mathcal{F}$ is minimal and $\mathcal{F}'$ is totally geodesic. Then ϕ is harmonic if and only if ϕ is transversally harmonic and leaf-wise harmonic, i.e., $\tau(\phi|_{\mathcal{F}}) = 0$.*

Proof. Since $\mathcal{F}$ is minimal and $\mathcal{F}'$ is totally geodesic, i.e, $\kappa = 0$ and $\mathcal{T}' = 0$, by Corollary 3.3, we have

$$\tau(\phi) = \tau(\phi|_{\mathcal{F}}) + \tau_b(\phi),$$

which yields the results. □

Corollary 3.5. *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map and $d\phi(Q) \subset Q'$. Then ϕ is a transversally harmonic map if and only if*

$$\pi(\tau(\phi)) = \mathrm{tr}_L \phi^* \mathcal{T}' - d\phi(\kappa^\sharp).$$

Now, let $\mathcal{F}$ be a Riemannian flow defined by a unit vector field V on a Riemannian manifold (M^{n+1}, g). Then

$$\kappa^\sharp = \pi(\nabla^M_V V) = \nabla^M_V V. \tag{3.9}$$

In fact, $\nabla^M_V V$ is already orthogonal to the leaves since $g(\nabla^M_V V, V) = 0$. Moreover, it is trivial that $\mathcal{F}$ is totally geodesic if and only if $\mathcal{F}$ is minimal, i.e., $\mathcal{T} = 0$ if and only if $\kappa^\sharp = 0$. Let $\mathcal{F}$ and $\mathcal{F}'$ be two Riemannian flows defined by unit vector fields V and V' on Riemannian manifolds (M, g) and (M', g'), respectively. Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a smooth foliated map. Then

$$\tau(\phi|_{\mathcal{F}}) = V(\lambda)V' - \pi^\perp d\phi(\kappa^\sharp), \quad \lambda = (\phi^*\omega')(V), \tag{3.10}$$

where ω' is a dual 1-form of V'. Hence if $d\phi(Q) \subset Q'$, then ϕ is leaf-wise harmonic if and only if λ is basic, i.e., $V(\lambda) = 0$. Hence we have the following corollary.

Corollary 3.6. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two Riemannian flows defined by a unit vector fields V and V' on a Riemannian manifolds M and M', respectively. Assume that $\mathcal{F}$ and $\mathcal{F}'$ are minimal. Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map and $d\phi(Q) \subset Q'$. Then ϕ is harmonic if and only if ϕ is transversally harmonic and $\phi^*(\omega')(V)$ is basic.*

Proof. Since $\mathcal{F}$ is minimal, from (3.10)

$$\tau(\phi|_{\mathcal{F}}) = V(\lambda)V', \quad \lambda = (\phi^*\omega')(V).$$

Hence the proof follows from Corollary 3.4. □

Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ and $\psi : (M', \mathcal{F}') \to (M'', \mathcal{F}'')$ be smooth foliated maps. Then the composition $\psi \circ \phi : (M, \mathcal{F}) \to (M'', \mathcal{F}'')$ is a smooth foliated map. Moreover, we have

$$d_T(\psi \circ \phi) = d_T\psi \circ d_T\phi. \tag{3.11}$$

Hence we have the following proposition.

Proposition 3.7. *Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ and $\psi : (M', \mathcal{F}') \to (M'', \mathcal{F}'')$ be smooth foliated maps. Then*

$$\tilde{\nabla}_{\mathrm{tr}} d_T(\psi \circ \phi) = d_T\psi(\tilde{\nabla}_{\mathrm{tr}} d_T\phi) + \phi^* \tilde{\nabla}_{\mathrm{tr}} d_T\psi, \tag{3.12}$$

where

$$(\phi^*\tilde{\nabla}_{\mathrm{tr}}d_T\psi)(X,Y) = (\tilde{\nabla}_{d_T\phi(X)}d_T\psi)(d_T\phi(Y))$$

for any $X, Y \in \Gamma Q$.

Proof. From (3.11), we have that, for any $X, Y \in \Gamma Q$,

$$\begin{aligned}(\tilde{\nabla}_{\mathrm{tr}}d_T(\psi\circ\phi))(X,Y) &= \nabla^{\psi\circ\phi}_X d_T(\psi\circ\phi)(Y) - d_T(\psi\circ\phi)(\nabla_X Y)\\ &= (\tilde{\nabla}_{d_T\phi(X)}d_T\psi)(d_T\phi(Y)) + d_T\psi((\tilde{\nabla}_X d_T\phi)(Y))\\ &= (\phi^*\tilde{\nabla}_{\mathrm{tr}}d_T\psi)(X,Y) + d_T\psi(\tilde{\nabla}_{\mathrm{tr}}d_T\phi)(X,Y),\end{aligned}$$

which proves (3.12). □

Corollary 3.8. *Let* $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ *and* $\psi : (M',\mathcal{F}') \to (M'',\mathcal{F}'')$ *be smooth foliated maps. Then the transversal tension field of the composition is given by*

$$\tau_b(\psi\circ\phi) = d_T\psi(\tau_b(\phi)) + \mathrm{tr}_Q\phi^*\tilde{\nabla}_{\mathrm{tr}}d_T\psi, \tag{3.13}$$

where

$$\mathrm{tr}_Q\phi^*\tilde{\nabla}_{\mathrm{tr}}d_T\psi = \sum_{a=1}^{q}(\tilde{\nabla}_{d_T\phi(E_a)}d_T\psi)(d_T\phi(E_a)).$$

Corollary 3.9. *Let* $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ *be a transversally harmonic map and let* $\psi : (M',\mathcal{F}') \to (M'',\mathcal{F}'')$ *be a transversally totally geodesic map. Then* $\psi\circ\phi : (M,\mathcal{F}) \to (M'',\mathcal{F}'')$ *is a transversally harmonic map.*

4. The first normal variational formula

Let $\phi : (M,g,\mathcal{F}) \to (M',g',\mathcal{F}')$ be a smooth foliated map. Let Ω be a compact domain of M. Then the *transversal energy* of ϕ on $\Omega \subset M$ is defined by

$$E_B(\phi;\Omega) = \frac{1}{2}\int_\Omega |d_T\phi|^2\mu_M, \tag{4.1}$$

where $|d_T\phi|^2 = \sum_a g_{Q'}(d_T\phi(E_a), d_T\phi(E_a))$ and μ_M is the volume element of M.

Let $V \in \phi^{-1}Q'$. Obviously, V may be considered as a vector field on Q' along ϕ. Then there is a 1-parameter family of foliated maps ϕ_t with $\phi_0 = \phi$ and $\frac{d\phi_t}{dt}|_{t=0} = V$. The family $\{\phi_t\}$ is said to be a *foliated variation* of ϕ with the *normal variation vector field* V. Then we have the first normal variational formula(cf. [7]).

Theorem 4.1. (The first normal variational formula) *Let* $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ *be a smooth foliated map. Let* $\{\phi_t\}$ *be a smooth foliated variation of* ϕ *supported in a compact domain* Ω. *Then*

$$\frac{d}{dt}E_B(\phi_t,\Omega)|_{t=0} = -\int_\Omega \langle V, \tau_b(\phi) - d_T\phi(\kappa_B^\sharp)\rangle\mu_M, \tag{4.2}$$

where $V = \frac{d\phi_t}{dt}|_{t=0}$ *is the normal variation vector field of* $\{\phi_t\}$ *and* $\langle\cdot,\cdot\rangle$ *is the pull-back metric on* $\phi^{-1}Q'$.

Proof. Let Ω be a compact domain of M and let $\{\phi_t\}$ be a foliated variation of ϕ supported in Ω with the normal variation vector field $V \in \phi^{-1}Q'$. Choose a local orthonormal basic frame $\{E_a\}$ on Q such that $(\nabla E_a)(x) = 0$. Define $\Phi : M \times (-\epsilon, \epsilon) \to M'$ by $\Phi(x,t) = \phi_t(x)$ and set $E = \Phi^{-1}Q'$. Let ∇^Φ denote the pull-back connection on E. Obviously, $d_T\Phi(E_a) = d_T\phi_t(E_a)$ and $d\Phi(\frac{\partial}{\partial t}) = \frac{d\phi_t}{dt}$. Moreover, we have $\nabla^\Phi_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \nabla^\Phi_{\frac{\partial}{\partial t}}E_a = \nabla^\Phi_{E_a}\frac{\partial}{\partial t} = 0$. Hence we have

$$\begin{aligned}
\frac{d}{dt}E_B(\phi_t,\Omega) &= \int_\Omega \sum_a \langle \nabla^\Phi_{\frac{\partial}{\partial t}} d_T\Phi(E_a), d_T\Phi(E_a)\rangle \mu_M \\
&= \int_\Omega \sum_a \langle \nabla^\Phi_{E_a} d\Phi(\frac{\partial}{\partial t}), d_T\Phi(E_a)\rangle \mu_M \\
&= \int_\Omega \sum_a \{E_a\langle \frac{d\phi_t}{dt}, d_T\phi_t(E_a)\rangle - \langle \frac{d\phi_t}{dt}, \nabla^{\phi_t}_{E_a} d_T\phi_t(E_a)\rangle\}\mu_M \\
&= \int_\Omega \sum_a E_a\langle \frac{d\phi_t}{dt}, d_T\phi_t(E_a)\rangle \mu_M - \int_\Omega \langle \frac{d\phi_t}{dt}, \tau_b(\phi_t)\rangle \mu_M
\end{aligned}$$

Now we define a normal vector field W_t by

$$W_t = \sum_a \langle \frac{d\phi_t}{dt}, d_T\phi_t(E_a)\rangle E_a.$$

Then we have

$$\mathrm{div}_\nabla W_t = \sum_a E_a\langle \frac{d\phi_t}{dt}, d_T\phi_t(E_a)\rangle.$$

By the transversal divergence theorem (Theorem 2.1), we have

$$\begin{aligned}
\frac{d}{dt}E_B(\phi_t,\Omega) &= \int_\Omega \mathrm{div}_\nabla W_t \mu_M - \int_\Omega \langle \frac{d\phi_t}{dt}, \tau_b(\phi_t)\rangle \mu_M \\
&= \int_\Omega \langle W_t, \kappa_B^\sharp\rangle \mu_M - \int_\Omega \langle \frac{d\phi_t}{dt}, \tau_b(\phi_t)\rangle \mu_M \\
&= -\int_\Omega \langle \frac{d\phi_t}{dt}, \tau_b(\phi_t) - d_T\phi(\kappa_B^\sharp)\rangle \mu_M,
\end{aligned}$$

which proves (4.2). □

Corollary 4.2. *Let* $\phi : (M,\mathcal{F}) \to (M',\mathcal{F}')$ *be a smooth foliated map. Assume that* $\mathcal{F}$ *is minimal. Then* ϕ *is transversally harmonic if and only if* ϕ *is a critical point of the trasnversal energy of* ϕ *supported in a compact domain.*

5. A generalized Weitzenböck type formula and its applications

Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds and let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a smooth foliated map. Note that $|d_T\phi|^2 \in \Omega_B^0(\mathcal{F})$ [6]. Let $\Omega_B^r(E) = \Omega_B^r(\mathcal{F}) \otimes E$ be the space of E-valued basic r-forms, where $E = \phi^{-1}Q'$. We define $d_\nabla : \Omega_B^r(E) \to \Omega_B^{r+1}(E)$ by

$$d_\nabla(\omega \otimes s) = (-1)^r \omega \wedge \nabla^\phi s + d_B\omega \otimes s \tag{5.1}$$

for any $s \in \Gamma E$ and $\omega \in \Omega_B^r(\mathcal{F})$. Let δ_∇ be a formal adjoint of d_∇. Then we have the following.

$$d_\nabla = \sum_a \theta^a \wedge \tilde{\nabla}_{E_a}, \quad \delta_\nabla = -\sum_a i(E_a)\tilde{\nabla}_{E_a} + i(\kappa_B^\sharp), \tag{5.2}$$

where $i(X)(\omega \otimes s) = i(X)\omega \otimes s$ for any $X \in \Gamma TM$. Then the Laplacian Δ on $\Omega_B^*(E)$ is defined by

$$\Delta = d_\nabla\delta_\nabla + \delta_\nabla d_\nabla. \tag{5.3}$$

Moreover, the operators A_X and $\theta(X)$ are extended to $\Omega_B^r(E)$ as follows:

$$A_X(\omega \otimes s) = A_X\omega \otimes s, \tag{5.4}$$

$$\theta(X)(\omega \otimes s) = \theta(X)\omega \otimes s + \omega \otimes \nabla_X^\phi s \tag{5.5}$$

for any $\omega \otimes s \in \Omega_B^r(E)$ and $X \in \Gamma TM$. Then $\theta(X) = d_\nabla i(X) + i(X)d_\nabla$ for any $X \in \Gamma TM$. Hence $\Phi \in \Omega_B^*(E)$ if and only if $i(X)\Phi = 0$ and $\theta(X)\Phi = 0$ for all $X \in \Gamma L$. Then the generalized Weitzenböck type formula (2.9) is extended to $\Omega_B^*(E)$ as follows:

$$\Delta\Phi = \tilde{\nabla}_{\mathrm{tr}}^*\tilde{\nabla}_{\mathrm{tr}}\Phi + F(\Phi) + A_{\kappa_B^\sharp}\Phi, \quad \forall\Phi \in \Omega_B^r(E), \tag{5.6}$$

where $F(\Phi) = \sum_{a,b=1}^q \theta^a \wedge i(E_b)\tilde{R}(E_b, E_a)\Phi$. Note that $d_T\phi \in \Omega_B^1(E)$. Then we have the following.

Theorem 5.1. *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Then the generalized Weitzenböck type formula is given by*

$$\frac{1}{2}\Delta_B|d_T\phi|^2 = \langle\Delta d_T\phi, d_T\phi\rangle - |\tilde{\nabla}_{\mathrm{tr}}d_T\phi|^2 - \langle A_{\kappa_B^\sharp}d_T\phi, d_T\phi\rangle - \langle F(d_T\phi), d_T\phi\rangle,$$

where

$$\begin{aligned}\langle F(d_T\phi), d_T\phi\rangle &= \sum_a g_{Q'}(d_T\phi(\mathrm{Ric}^Q(E_a)), d_T\phi(E_a)) \\ &\quad - \sum_{a,b} g_{Q'}(R^{Q'}(d_T\phi(E_b), d_T\phi(E_a))d_T\phi(E_a), d_T\phi(E_b)).\end{aligned} \tag{5.7}$$

Proof. Let $\{E_a\}(a=1,\dots,q)$ be a local orthonormal basic frame such that at $x\in M$, $(\nabla E_a)_x=0$. Then, at x, we have from (2.4)

$$\frac{1}{2}\Delta_B|d_T\phi|^2=\langle\tilde{\nabla}^*_{\rm tr}\tilde{\nabla}_{\rm tr}d_T\phi,d_T\phi\rangle-|\tilde{\nabla}_{\rm tr}d_T\phi|^2. \tag{5.8}$$

From (5.6) and (5.8), we have

$$\frac{1}{2}\Delta_B|d_T\phi|^2=\langle\Delta d_T\phi,d_T\phi\rangle-|\tilde{\nabla}_{\rm tr}d_T\phi|^2-\langle A_{\kappa_B^\sharp}d_T\phi,d_T\phi\rangle-\langle F(d_T\phi),d_T\phi\rangle.$$

Now, we compute $\langle F(d_T\phi),d_T\phi\rangle$. Let $\{V_\alpha\}(\alpha=1,\dots,q')$ be a local orthonormal basic frame of Q' and ω^α be its dual coframe field. Let $f^\alpha=\phi^*\omega^\alpha$. Then $d_T\phi$ is expressed by

$$d_T\phi=\sum_{\alpha=1}^{q'}f^\alpha\otimes V_\alpha, \tag{5.9}$$

where $V_\alpha(x)\equiv V_\alpha(\phi(x))$. By a direct calculation, we have

$$\tilde{R}(E_a,E_b)d_T\phi=\sum_\alpha R^Q(E_a,E_b)f^\alpha\otimes V_\alpha+\sum_\alpha f^\alpha\otimes R^E(E_a,E_b)V_\alpha, \tag{5.10}$$

where $R^E(E_a,E_b)V_\alpha=R^{Q'}(d_T\phi(E_a),d_T\phi(E_b))V_\alpha$. From (5.10), we have

$$\begin{aligned}\langle F(d_T\phi),d_T\phi\rangle&=\langle\sum_{a,b}\theta^a\wedge i(E_b)\tilde{R}(E_b,E_a)d_T\phi,d_T\phi\rangle\\&=\sum_{a,b,\alpha,\beta}\langle\theta^a\wedge i(E_b)R^Q(E_b,E_a)f^\alpha\otimes V_\alpha,f^\beta\otimes V_\beta\rangle\\&\quad+\sum_{a,b,\alpha,\beta}g_Q(\theta^a\wedge i(E_b)f^\alpha,f^\beta)g_Q(R^E(E_a,E_b)V_\alpha,V_\beta).\end{aligned}$$

Note that $d_T\phi(E_a)=\sum_\alpha f^\alpha(E_a)V_\alpha$. Then we have

$$\begin{aligned}&\sum_{a,b,\alpha}g_Q(\theta^a\wedge i(E_b)R^Q(E_b,E_a)f^\alpha,f^\alpha)\\&=\sum_a g_{Q'}(d_T(\mathrm{Ric}^Q(E_a)),d_T\phi(E_a)).\end{aligned} \tag{5.11}$$

From (5.11), we have

$$\begin{aligned}\langle F(d_T\phi),d_T\phi\rangle=&\sum_a g_{Q'}(d_T\phi(\mathrm{Ric}^Q(E_a)),d_T\phi(E_a))\\&-\sum_{a,b}g_{Q'}(R^{Q'}(d_T\phi(E_a),d_T\phi(E_b))d_T\phi(E_b),d_T\phi(E_a)),\end{aligned}$$

which completes the proof. □

Remark. (1) Let $\phi:(M,\mathcal{F})\to(M',\mathcal{F}')$ be a smooth foliated map. Then

$$d_\nabla(d_T\phi)=0,\quad \delta_\nabla d_T\phi=-\tau_b(\phi)+i(\kappa_B^\sharp)d_T\phi. \tag{5.12}$$

(2) If a foliated map $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is transversally harmonic, then

$$\Delta d_T\phi = d_\nabla i(\kappa_B^\sharp) d_T\phi. \tag{5.13}$$

Corollary 5.2. *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a transversally harmonic map. Then*

$$\frac{1}{2}\Delta_B |d_T\phi|^2 = -|\tilde{\nabla}_{\mathrm{tr}} d_T\phi|^2 - \langle F(d_T\phi), d_T\phi\rangle + \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2). \tag{5.14}$$

Proof. Since $d_\nabla(d_T\phi) = 0$, we have

$$A_X d_T\phi = -\tilde{\nabla}_X d_T\phi + d_\nabla i(X) d_T\phi, \quad \forall X \in \Gamma Q. \tag{5.15}$$

Hence (5.14) follows from (5.13) and (5.15). □

As applications of the generalized Weitzenböck formula, we have the following theorems.

Theorem 5.3. *Let $(M, g, \mathcal{F})$ be a compact foliated Riemannian manifold of nonnegative transversal Ricci curvature, and let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold of nonpositive transversal sectional curvature. If $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is transversally harmonic, then ϕ is transversally totally geodesic, i.e., $\nabla_{\mathrm{tr}} d_T\phi = 0$. Furthermore,*

1. *If the transversal Ricci curvature Ric^Q of $\mathcal{F}$ is positive somewhere, then ϕ is transversally constant, i.e., the induced map between leaf spaces is constant.*
2. *If the transversal sectional curvature $K^{Q'}$ of $\mathcal{F}'$ is negative, then ϕ is either transversally constant or $\phi(M)$ is a transversally geodesic closed curve.*

Proof. Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a transversally harmonic map. Then, from (5.14), we have

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|d_T\phi|^2 = -|\tilde{\nabla}_{\mathrm{tr}} d_T\phi|^2 - \langle F(d_T\phi), d_T\phi\rangle. \tag{5.16}$$

Since $\mathrm{Ric}^Q \geq 0$ and $K^{Q'} \leq 0$, from (5.7) we have

$$\langle F(d_T\phi), d_T\phi\rangle \geq 0. \tag{5.17}$$

Hence $(\Delta_B - \kappa_B^\sharp)|d_T\phi|^2 \leq 0$. From Lemma 2.3, $|d_T\phi|$ is constant. Hence again, we have from (5.16)

$$|\tilde{\nabla}_{\mathrm{tr}} d_T\phi|^2 + \langle F(d_T\phi), d_T\phi\rangle = 0. \tag{5.18}$$

Hence $\tilde{\nabla}_{\mathrm{tr}} d_T\phi = 0$ and by assumptions

$$g_{Q'}(d_T\phi(\mathrm{Ric}^Q(E_a), d_T\phi(E_a)) = 0, \tag{5.19}$$

$$g_{Q'}(R^{Q'}(d_T\phi(E_a), d_T\phi(E_b))d_T\phi(E_a), d_T\phi(E_b)) = 0 \tag{5.20}$$

for any indices a and b. Therefore ϕ is transversally totally geodesic. Moreover, from (5.19), if Ric^Q is positive at some point, then $d_T\phi = 0$, i.e., ϕ is transversally constant, which proves (1). For the proof of (2), if there exists a point $x \in M$ such that at least two vectors in $\{d_T\phi(E_a)\}$ are linearly independent at $\phi(x)$, say, $d_T\phi(E_1)$ and $d_T\phi(E_2)$, then from the hypothesis,

$$g_{Q'}(R^{Q'}(d_T\phi(E_1), d_T\phi(E_2))d_T\phi(E_2), d_T\phi(E_1)) < 0,$$

which contradicts (5.20). Hence the rank of $d_T\phi < 2$, that is, the rank of $d_T\phi$ is zero or one everywhere. If the rank of $d_T\phi$ is zero, then ϕ is transversally constant. If the rank of $d_T\phi$ is one, then $\phi(M)$ is closed transversally geodesic. □

Next, we extend Theorem 5.3 under the weaker transversal sectional curvature of $\mathcal{F}'$. Let $\mathrm{rank}_T(\phi)$ be the rank of $d_T\phi$.

Theorem 5.4. *Let $(M, g, \mathcal{F})$ be a compact foliated Riemannian manifold and let $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold. Assume that $\mathrm{Ric}^Q \geq \lambda$ id. and $K^{Q'} \leq \mu$ for any positive constants λ and μ. Let $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a transversally harmonic map with $\max\{\mathrm{rank}_T(\phi)\} \leq C$, where $C \geq 2$ is constant. If $|d_T\phi|^2 \leq \frac{\lambda C}{\mu(C-1)}$, then ϕ is transversally constant or ϕ is transversally totally geodesic. In particular, if $|d_T\phi|^2 \leq \frac{\lambda}{\mu}$, then ϕ is transversally constant.*

Proof. Let $\{E_a\}$ be a local orthonormal basic frame of Q. From (5.7), we have

$$\begin{aligned} &\langle F(d_T\phi), d_T\phi\rangle \\ &= \sum_a g_{Q'}(d_T\phi(\mathrm{Ric}^Q(E_a)), d_T\phi(E_a)) \\ &\quad - \sum_{a,b}\{|d_T\phi(E_a)|^2|d_T\phi(E_b)|^2 - g_{Q'}(d_T\phi(E_a), d_T\phi(E_b))^2\}K^{Q'}_{ab}, \end{aligned} \tag{5.21}$$

where

$$K^{Q'}_{ab} = g_{Q'}(R^{Q'}(d_T\phi(E_a), d_T\phi(E_b))d_T\phi(E_b), d_T\phi(E_a))$$

is the transversal sectional curvature spanned by $d_T\phi(E_a)$ and $d_T\phi(E_b)$. Let $\mathrm{rank}_T(\phi) = r \leq C$. Now, we choose a local orthonormal basic frame $\{E_a\}$ such that $g_{Q'}(d_T(E_a), d_T(E_b))|_x = \lambda_a\delta_{ab}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. Then, from (5.14) and (5.21), we have

$$\begin{aligned} &\frac{1}{2}\Delta_B|d_T\phi|^2 \\ &= -|\tilde{\nabla}_{\mathrm{tr}}d_T\phi|^2 + \frac{1}{2}\kappa^\sharp_B(|d_T\phi|^2) - \sum_a g_{Q'}(d_T\phi(\mathrm{Ric}^Q(E_a)), d_T\phi(E_a)) \\ &\quad + \sum_{a,b}\{|d_T\phi(E_a)|^2|d_T\phi(E_b)|^2 - g_{Q'}(d_T\phi(E_a), d_T\phi(E_b))^2\}K^{Q'}_{ab} \end{aligned}$$

$$\leq -|\tilde{\nabla}_{\mathrm{tr}} d_T\phi|^2 + \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2) - \lambda|d_T\phi|^2 + \mu\{|d_T\phi|^4 - \sum_{a=1}^{r}\lambda_a^2\}.$$

Using the Schwarz's inequality, we have

$$|d_T\phi|^4 = \sum_{a,b}^{r}\lambda_a\lambda_b \leq \frac{1}{2}\sum_{a,b=1}^{r}(\lambda_a^2+\lambda_b^2) = r\sum_{a=1}^{r}\lambda_a^2 \leq C\sum_{a=1}^{r}\lambda_a^2. \tag{5.22}$$

From (5.22), we have

$$|d_T\phi|^4 - \sum_{a=1}^{r}\lambda_a^2 \leq \frac{C-1}{C}|d_T\phi|^4. \tag{5.23}$$

From (5.23), we have

$$\begin{aligned} &\frac{1}{2}\Delta_B|d_T\phi|^2 \\ &\leq -|\tilde{\nabla}_{\mathrm{tr}} d_T\phi|^2 + \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2) - |d_T\phi|^2\{\lambda - \frac{(C-1)\mu}{C}|d_T\phi|^2\} \\ &\leq \frac{1}{2}\kappa_B^\sharp(|d_T\phi|^2). \end{aligned} \tag{5.24}$$

Hence, from Lemma 2.3, $|d_T\phi|$ is constant and then

$$|\tilde{\nabla}_{\mathrm{tr}} d_T\phi|^2 + |d_T\phi|^2\{\lambda - \frac{(C-1)\mu}{C}|d_T\phi|^2\} = 0. \tag{5.25}$$

Therefore $\tilde{\nabla}_{\mathrm{tr}} d_T\phi = 0$ and $|d_T\phi|^2\{\lambda - \frac{(C-1)\mu}{C}|d_T\phi|^2\} = 0$. Hence ϕ is transversally totally geodesic. If $d_T\phi = 0$, then ϕ is transversally constant. If $d_T\phi \neq 0$, then $|d_T\phi|^2 = \frac{\lambda C}{\mu(C-1)}$ and ϕ is transversally totally geodesic. In particular, if $|d_T\phi|^2 \leq \frac{\lambda}{\mu}$, then ϕ is transversally constant. □

Remark. For the point foliation, Theorem 5.3 and Theorem 5.4 are found in [2, page 124] and [12], respectively.

Example. Let T^2 be the flat 2-torus parametrized by the angles (u, v) with $0 \leq u, v < 2\pi$. Let $\bar{\phi} : T^2 \to S^3$ be defined by

$$\bar{\phi}(u, v) = (\cos u, \sin u, \cos v, \sin v)/\sqrt{2},$$

considered as a point in $\mathbb{R}^4$. Then $\bar{\phi}$ is harmonic but not totally geodesic [2, page 132]. Now let (F, h) and (F', h') be Riemannian manifolds. Consider the foliations on $T^2 \times F$ and $S^3 \times F'$ given by the projections on the first component $\pi_1 : T^2 \times F \to T^2$, $\pi_2 : S^3 \times F' \to S^3$, respectively. Then the projections $\pi_i (i = 1, 2)$ are Riemannian fibrations, and so the foliations are

Riemannian. Let $\phi : T^2 \times F \to S^3 \times F'$ be a foliated smooth map, which is given by

$$\phi((u,v),x) = (\bar{\phi}(u,v), f(u,v,x))$$

for any $x \in F$, where $f : T^2 \times F \to F'$ is smooth. Then ϕ is transversally harmonic because $\bar{\phi}$ is harmonic. But ϕ is not totally geodesic because $\bar{\phi}$ is not totally geodesic [6, Theorem 3.1].

Acknowledgements. The authors would like to thank the referee for his or her significant corrections and kind comments. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0021005).

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