

RICCI CURVATURE AND MONOPOLE CLASSES ON 3-MANIFOLDS

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ABSTRACT. We prove an L^2 -estimate of Ricci curvature in terms of harmonic 1-forms on a closed oriented Riemannian 3-manifold admitting a solution of any rescaled Seiberg-Witten equations. We also give a necessary condition to be a monopole class on some special connected sums.

1. Introduction

A second cohomology class is called a *monopole class* if it arises as the first Chern class of a Spin^c structure for which the Seiberg-Witten equations admit a solution for every choice of a Riemannian metric. It is well-known by LeBrun [9, 10, 12] that the existence of a monopole class gives various curvature estimates of a Riemannian 4-manifold. These immediately give corresponding estimates on 3-manifolds by using the dimensional reduction.

Theorem 1.1 ([19]). *Let (M, g) be a smooth closed oriented Riemannian 3-manifold with $b_1(M) \geq 1$. Suppose that it admits a solution of the Seiberg-Witten equations for a Spin^c structure \mathfrak{s} . Then*

$$\int_M (s_-)_g^2 d\mu_g \geq \frac{16\pi^2 |c_1(\mathfrak{s}) \cup [\omega]|^2}{\int_M |\omega|_g^2 d\mu_g},$$

where $(s_-)_g$ is $\min(s_g, 0)$ at each point and s_g is the scalar curvature of g . Furthermore if the Seiberg-Witten invariant of \mathfrak{s} is nonzero, then for a nonzero element ω in $H_{DR}^1(M)$

$$\int_M |r_g|^2 d\mu_g \geq \frac{8\pi^2 |c_1(\mathfrak{s}) \cup [\omega]|^2}{\int_M |\omega|_g^2 d\mu_g},$$

where r_g is the Ricci curvature of g .

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Here, the Seiberg-Witten invariant in case of $b_1(M) = 1$ means that of the chamber for arbitrarily small perturbations. We conjectured that the above Ricci curvature estimate still holds true when $c_1(\mathfrak{s})$ is a monopole class.

In this article, we show that it holds true if $c_1(\mathfrak{s})$ is a *strong monopole class* meaning that it is the first Chern class of a Spin^c structure of M which admits a solution of the rescaled Seiberg-Witten equations of \mathfrak{s} for any rescaling and any Riemannian metric (See Theorem 3.2).

Following LeBrun [13], we define the f -rescaled Seiberg-Witten equations on a 3-manifold M to be

$$\begin{cases} D_A\Phi = 0 \\ F_A = (\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2}\text{Id})f, \end{cases}$$

where the rescaling factor f is a positive smooth function on M . An obvious but important fact is that if the Seiberg-Witten invariant of \mathfrak{s} is nonzero, then $c_1(\mathfrak{s})$ is a strong monopole class. But it is not known yet whether every monopole class is a strong monopole class.

In general, it is very difficult to find a monopole class which has zero Seiberg-Witten invariant. Connected sums of 4-manifolds with $b_2^+ > 1$ are good candidates. In dimension 4, Bauer and Furuta [2, 1] devised a new refined invariant of Seiberg-Witten moduli space to prove the existence of a monopole class on some connected sums of Kähler surfaces. But it seems that no 3-dimensional example has been found yet. In the final section, we apply our curvature estimates to find a necessary condition to be a monopole class on some special connected sums.

For a brief introduction to the Seiberg-Witten theory, the readers are referred to [16, 20].

2. The Yamabe problem for modified scalar curvature

Let (X, g) be a smooth closed oriented Riemannian 4-manifold and W_+ be the self-dual Weyl curvature. By the modified scalar curvature we mean

$$\mathfrak{S} \equiv s - \sqrt{6}|W_+|.$$

We will denote the set of $C^{2,\alpha}$ metrics for $\alpha \in (0, 1)$ conformal to g by $[g]$. Assume that there exists a metric in $[g]$ with nonpositive $\int_X \mathfrak{S} d\mu$. Then as observed by Gursky [6] and LeBrun [11], the standard proof of the Yamabe problem [14] proves that there exists a $C^{2,\alpha}$ metric in $[g]$ such that \mathfrak{S} is a nonpositive constant. As in the Yamabe problem, it is a “minimizer” realizing

$$\mathfrak{Y}(X, [g]) \equiv \inf_{\tilde{g} \in [g]} \frac{\int_X \mathfrak{S}_{\tilde{g}} d\mu_{\tilde{g}}}{(\text{Vol}_{\tilde{g}})^{\frac{1}{2}}}.$$

We also have:

Lemma 2.1. For $r \in [2, \infty]$,

$$\mathfrak{Y}(X, [g]) = - \inf_{\tilde{g} \in [g]} \left(\int_X |\mathfrak{S}_{\tilde{g}}|^r d\mu_{\tilde{g}} \right)^{\frac{1}{r}} (\text{Vol}_{\tilde{g}})^{\frac{1}{2} - \frac{1}{r}},$$

where the infimum is realized only by the minimizer which is unique up to a constant multiplication.

Proof. The L^∞ case is an immediate consequence of the other cases because of the inequality

$$\|\mathfrak{S}_{\tilde{g}}\|_{L^\infty} (\text{Vol}_{\tilde{g}})^{\frac{1}{2}} \geq \left(\int_X |\mathfrak{S}_{\tilde{g}}|^r d\mu_{\tilde{g}} \right)^{\frac{1}{r}} (\text{Vol}_{\tilde{g}})^{\frac{1}{2} - \frac{1}{r}},$$

whose equality is attained only when $\mathfrak{S}_{\tilde{g}}$ is constant.

For $r \in [2, \infty)$, we will use the technique of Besson, Courtois, and Gallot [3]. Let g be a minimizer. Let $\tilde{g} = u^2 g$, where $u : X \rightarrow \mathbb{R}^+$ is a C^2 function. Note that u satisfies the modified Yamabe equation

$$\mathfrak{S}_{\tilde{g}} u^3 = \mathfrak{S}_g u + 6\Delta_g u.$$

Therefore

$$\begin{aligned} \left(\int_X |\mathfrak{S}_{\tilde{g}}|^r d\mu_{\tilde{g}} \right)^{\frac{1}{r}} (\text{Vol}_{\tilde{g}})^{\frac{1}{2} - \frac{1}{r}} &= \left(\int_X |\mathfrak{S}_{\tilde{g}}|^r u^4 d\mu_g \right)^{\frac{1}{r}} \left(\int_X u^4 d\mu_g \right)^{\frac{1}{2} - \frac{1}{r}} \\ &\geq \frac{\int_X -\mathfrak{S}_{\tilde{g}} u^2 d\mu_g}{\left(\int_X d\mu_g \right)^{\frac{1}{2}}} \\ &= \frac{\int_X -(\mathfrak{S}_g + 6\frac{1}{u} d^* du) d\mu_g}{(\text{Vol}_g)^{\frac{1}{2}}} \\ &= \frac{\int_X (-\mathfrak{S}_g + 6\frac{|du|^2}{u^2}) d\mu_g}{(\text{Vol}_g)^{\frac{1}{2}}} \\ &\geq \frac{\int_X -\mathfrak{S}_g d\mu_g}{(\text{Vol}_g)^{\frac{1}{2}}}, \end{aligned}$$

where the first inequality is an application of the Hölder inequality, and the equality holds if and only if u is a positive constant. It also follows that any minimizer is a constant multiple of g . \square

3. Ricci curvature estimate

Let us start with the following lemma:

Lemma 3.1. Let (M, g) be a smooth closed oriented Riemannian 3-manifold and \mathfrak{s} be a $Spin^c$ structure on it. If it admits a solution for a rescaled Seiberg-Witten equations, then any $C^{2,\alpha}$ -metric $\tilde{g} \in [g]$ also has a solution of the rescaled Seiberg-Witten equations for \mathfrak{s} .

Proof. We claim that if (A, Ψ) is a solution with respect to g , then $(A, e^{-\varphi}\Psi)$ is a solution with respect to $\tilde{g} = e^{2\varphi}g$. Mapping an orthonormal frame $\{e_1, e_2, e_3\}$ of g to an orthonormal frame $\{e^{-\varphi}e_1, e^{-\varphi}e_2, e^{-\varphi}e_3\}$ of \tilde{g} gives a global isomorphism of two orthonormal frame bundles and hence a global isometry of the Clifford bundles. Then the identity map between the spinor bundles is an isometry.

For a proof of the Spin^c Dirac equation, one is referred to [8], and the curvature equation is immediate from the fact that

$$|F_A|_{\tilde{g}} = e^{-2\varphi}|F_A|_g = e^{-2\varphi}f|\Psi|_g^2 = f|e^{-\varphi}\Psi|_{\tilde{g}}^2,$$

where f is the rescaling factor. □

Theorem 3.2. *Let (M, g) be a smooth closed oriented Riemannian 3-manifold with $b_1(M) \geq 1$ and \mathfrak{s} be a Spin^c structure on it. Suppose that it admits a solution for the rescaled Seiberg-Witten equations for any rescaling. Then for any smooth metric \tilde{g} conformal to g and any nonzero $\omega \in H_{DR}^1(M)$,*

$$\int_M |r_{\tilde{g}}|^2 d\mu_{\tilde{g}} \geq \frac{8\pi^2 |c_1(\mathfrak{s}) \cup [\omega]|^2}{\int_M |\omega|_{\tilde{g}}^2 d\mu_{\tilde{g}}},$$

and the equality holds if and only if (M, \tilde{g}) is a Riemannian submersion onto S^1 with totally geodesic fiber isometric to a compact oriented surface of genus ≥ 1 with a non-positive constant curvature metric whose volume form is a multiple of ω for \tilde{g} -harmonic ω , and $[c_1(\mathfrak{s})]$ is a multiple of $[*\omega]$ in $H_{DR}^2(M)$, where $*$ denotes the Hodge star with respect to \tilde{g} .*

Proof. In order to prove the inequality, we may assume $c_1(\mathfrak{s}) \neq 0 \in H^2(M, \mathbb{R})$. Then the Seiberg-Witten equations have an irreducible solution for any metric in $[g]$, implying that there cannot exist a metric in $[g]$ with nonnegative scalar curvature, and hence there exists a smooth metric in $[g]$ with negative scalar curvature.

For notational convenience, let g be any smooth metric in $[g]$. Consider the product metric $g + dt^2$ on $M \times S^1$, where $t \in [0, 1]$ is a global coordinate of S^1 . By the previous section, there exists a $C^{2,\alpha}$ metric $\hat{g} \in [g + dt^2]$ which minimizes $\mathfrak{M}(M \times S^1, [g + dt^2])$ satisfying that $s - \sqrt{6}|W_+|$ is a negative constant.

Lemma 3.3. *\hat{g} is invariant under the translation along S^1 -direction.*

Proof. Let $\hat{g} = (g + dt^2)\psi$ for a positive smooth function ψ on $M \times S^1$, and we will show $\psi(x, t) = \psi(x, t + c)$ for $(x, t) \in M \times S^1$ for any c .

Since $(g + dt^2)\psi(x, t + c)$ is also a minimizer, by Lemma 2.1 there exists a smooth positive function φ on S^1 such that

$$\psi(x, t + c) = \varphi(c)\psi(x, t)$$

for any (x, t) . For any c ,

$$\int_{M \times S^1} \psi(x, t) d\mu_{g+dt^2} = \int_{M \times S^1} \psi(x, t + c) d\mu_{g+dt^2}$$

$$= \varphi(c) \int_{M \times S^1} \psi(x, t) \, d\mu_{g+dt^2},$$

where the first equality is due to the translation invariance of dt^2 . Since $\psi > 0$, we conclude that $\varphi(c) = 1$ for any c . \square

We write the metric \hat{g} as the warped form $h + f^2 dt^2$ for $f : M \rightarrow \mathbb{R}^+$ where h is the metric $f^2 g$ on M . Let $\{e_1, e_2, e_3, e_4 = \frac{\partial}{\partial t}\}$ be a local orthonormal frame $M \times S^1$ with respect to $h + dt^2$, and $\{\omega^i \mid i = 1, \dots, 4\}$ its dual coframe. Recall the first Cartan's structure equations:

$$d\omega^i = -\omega_j^i \wedge \omega^j,$$

where ω_j^i are the connection 1-forms of $h + dt^2$. Obviously ω_4^i are all zero for all i . Take an orthonormal coframe of \hat{g} as $\{\omega^1, \omega^2, \omega^3, f\omega^4\}$ and apply the first Cartan's structure equations to $\{\omega^1, \omega^2, \omega^3, f\omega^4\}$, then one can see that the connection 1-forms $\hat{\omega}_j^i$ of \hat{g} are given by

$$\begin{aligned} \hat{\omega}_j^i &= \omega_j^i \quad \text{for } i, j = 1, 2, 3, \\ \hat{\omega}_j^4 &= \frac{\partial f}{\partial e_j} \omega^4 \quad \text{for } j = 1, 2, 3. \end{aligned}$$

Let (A, Φ) be a solution of the $\frac{1}{f}$ -rescaled Seiberg-Witten equations for \mathfrak{s} on (M, h) , whose existence is guaranteed by Lemma 3.1. Then it is a translation-invariant solution of the Seiberg-Witten equations for \mathfrak{s} on $(M \times S^1, h + dt^2)$.

We claim that $(A, \frac{\Phi}{\sqrt{f}})$ is a solution of the unrescaled Seiberg-Witten equations for \mathfrak{s} on $(M \times S^1, h + f^2 dt^2)$. Let's denote the objects of $\hat{g} = h + f^2 dt^2$ corresponding to that of $h + dt^2$ by $\hat{\cdot}$. The Spin^c Dirac equation reads

$$\begin{aligned} & \hat{D}_A(f^{-\frac{1}{2}}\Phi) \\ &= \sum_{i=1}^4 \hat{e}_i \hat{\nabla}_{\hat{e}_i} (f^{-\frac{1}{2}}\Phi) \\ &= \sum_{i=1}^4 \hat{e}_i \left(\frac{\partial}{\partial \hat{e}_i} (f^{-\frac{1}{2}}\Phi) + \frac{1}{2} \left(\sum_{j < k} \hat{\omega}_j^k(\hat{e}_i) \hat{e}_j \hat{e}_k + A(\hat{e}_i) \right) f^{-\frac{1}{2}}\Phi \right) \\ &= \sum_{i=1}^3 e_i \left(\frac{\partial}{\partial e_i} (f^{-\frac{1}{2}}\Phi) + \frac{1}{2} \left(\sum_{j < k \leq 3} \omega_j^k(e_i) e_j e_k + A(e_i) \right) f^{-\frac{1}{2}}\Phi \right) \\ & \quad + \frac{\hat{e}_4}{2} \left(\sum_{j=1}^3 \frac{\partial f}{\partial e_j} \omega^4(\hat{e}_4) e_j \hat{e}_4 \right) f^{-\frac{1}{2}}\Phi \\ &= \sum_{i=1}^3 e_i \left(-\frac{f^{-\frac{3}{2}}}{2} \frac{\partial f}{\partial e_i} \Phi + f^{-\frac{1}{2}} \frac{\partial \Phi}{\partial e_i} + \frac{1}{2} \left(\sum_{j < k \leq 3} \omega_j^k(e_i) e_j e_k + A(e_i) \right) f^{-\frac{1}{2}}\Phi \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \left(\sum_{j=1}^3 \frac{\partial f}{\partial e_j} \frac{1}{f} e_j \right) f^{-\frac{1}{2}} \Phi \\
 & = f^{-\frac{1}{2}} D_A \Phi \\
 & = 0,
 \end{aligned}$$

and the curvature equation reads

$$\begin{aligned}
 F_A^\dagger & = \frac{1}{2} (F_A + (*_h F_A) \wedge \hat{\omega}^4) \\
 & \simeq \frac{1}{2} (F_A + (*_h F_A) \wedge \omega^4) \\
 & = \frac{1}{f} (\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} \text{Id}) \\
 & = (f^{-\frac{1}{2}} \Phi) \otimes (f^{-\frac{1}{2}} \Phi)^* - \frac{|f^{-\frac{1}{2}} \Phi|^2}{2} \text{Id},
 \end{aligned}$$

where the equivalence in the second line means the identification as an endomorphism of the plus spinor bundle (Mapping an orthonormal frame $\{e_1, \dots, e_4\}$ of $h + dt^2$ to an orthonormal frame $\{e_1, \dots, e_3, \hat{e}_4\}$ of $h + f^2 dt^2$ gives a global isomorphism of two orthonormal frame bundles, inducing a global isometry of the Clifford bundles. Then the identity map between the spinor bundles is an isometry).

Lemma 3.4.

$$\int_{M \times S^1} \left(\frac{2}{3} s_{g+dt^2} - 2\sqrt{\frac{2}{3}} |W_+|_{g+dt^2} \right)^2 d\mu_{g+dt^2} \geq 32\pi^2 ((\pi^* c_1)^+)^2,$$

where $(\pi^* c_1)^+$ is the self-dual harmonic part of $\pi^* c_1$ with respect to $g + dt^2$, and $\pi : M \times S^1 \rightarrow M$ is the projection map.

Proof. This immediately follows from LeBrun’s method of Theorem 2.2 in [11]. First by using Lemma 2.1,

$$\begin{aligned}
 & \int_{M \times S^1} \left(\frac{2}{3} s_{g+dt^2} - 2\sqrt{\frac{2}{3}} |W_+|_{g+dt^2} \right)^2 d\mu_{g+dt^2} \\
 & \geq \int_{M \times S^1} \left(\frac{2}{3} s_{\hat{g}} - 2\sqrt{\frac{2}{3}} |W_+|_{\hat{g}} \right)^2 d\mu_{\hat{g}},
 \end{aligned}$$

and the right hand side is equal to

$$(3.1) \quad \left(\int_{M \times S^1} d\mu_{\hat{g}} \right)^{\frac{1}{3}} \left(\int_{M \times S^1} \left| \frac{2}{3} s_{\hat{g}} - 2\sqrt{\frac{2}{3}} |W_+|_{\hat{g}} \right|^3 d\mu_{\hat{g}} \right)^{\frac{2}{3}},$$

because \hat{g} has constant $\frac{2}{3} s - 2\sqrt{\frac{2}{3}} |W_+|$. Now we use the fact that $(M \times S^1, \hat{g})$ admits a solution of the unrescaled Seiberg-Witten equations for \mathfrak{s} . Combining its Weitzenböck formula with the Weitzenböck formula for the self-dual harmonic 2-forms, we conclude that (3.1) is greater than or equal to $32\pi^2 ((\pi^* c_1)^+)^2$. \square

Now using the above lemma, we get

$$\begin{aligned}
 \int_M |r_g|^2 d\mu_g &= \int_{M \times S^1} |r_{g+dt^2}|^2 d\mu_{g+dt^2} \\
 &= 8 \int_{M \times S^1} \left(\frac{s_{g+dt^2}^2}{24} + \frac{1}{2} |W_+|_{g+dt^2}^2 \right) d\mu_{g+dt^2} \\
 &\quad - 8\pi^2(2\chi + 3\tau)(M \times S^1) \\
 (3.2) \quad &\geq \frac{1}{2} \int_{M \times S^1} \left(\frac{2}{3} s_{g+dt^2} - 2\sqrt{\frac{2}{3}} |W_+|_{g+dt^2} \right)^2 d\mu_{g+dt^2} - 0 \\
 (3.3) \quad &\geq 16\pi^2((\pi^*c_1)^+)^2 \\
 (3.4) \quad &\geq \frac{8\pi^2|c_1 \cup [\omega]|^2}{\int_M |\omega|_g^2 d\mu_g},
 \end{aligned}$$

where the second equality is due to the 4-dimensional Chern-Gauss-Bonnet theorem, and the first inequality is simple applications of Hölder inequality which was proved in LeBrun [12].

Lemma 3.5. *The equality of the theorem statement holds if and only if $(M \times S^1, g + dt^2)$ is a Kähler manifold of non-positive constant scalar curvature with the Kähler form a multiple of $*\omega + \omega \wedge dt$ for harmonic ω , and $[c_1(\mathfrak{s})]$ is a multiple of $[\omega]$ in $H_{DR}^2(M)$.*

Proof. Let's first consider the case when $[c_1] \neq 0 \in H_{DR}^2(M)$. It is shown in [12] that both equalities in (3.2) and (3.3) hold if and only if $g + dt^2$ is a Kähler metric of negative constant scalar curvature with the Kähler form a multiple of $(\pi^*c_1)^+$. The equality in (3.4) holds if and only if

$$\omega = \omega^h = *c_1^h,$$

where $(\cdot)^h$ denotes the g -harmonic part.

When $[c_1] = 0 \in H_{DR}^2(M)$, the equality implies that the metric is Ricci-flat (In fact, it's a flat manifold T^3/Γ , because the dimension is 3). By the Weitzenböck formula for 1-forms, ω^h is a nonzero parallel 1-form. Then $*\omega^h + \omega^h \wedge dt$ is a nonzero parallel 2-form on $(M \times S^1, g + dt^2)$, and hence a Kähler form with the obvious complex structure compatible with the orientation. Conversely suppose that $(M \times S^1, g + dt^2)$ is scalar-flat Kähler. Since a Kähler curvature is a (symmetric) section of $\wedge^{1,1} \otimes \wedge^{1,1}$, on any scalar-flat Kähler surface the Riemann curvature restricted to self-dual two forms must be zero, and hence so is W_+ . Then by the 4-dimensional Chern-Gauss-Bonnet theorem,

$$\begin{aligned}
 \int_{M \times S^1} |r_{g+dt^2}|^2 d\mu_{g+dt^2} &= \int_{M \times S^1} \left(\frac{1}{3} (s_{g+dt^2})^2 + 4|W_+|_{g+dt^2}^2 \right) d\mu_{g+dt^2} \\
 &\quad - 8\pi^2(2\chi + 3\tau)(M \times S^1) \\
 &= 0,
 \end{aligned}$$

giving the equality. □

If the equality holds, we have a parallel splitting of TM by ω and $*\omega$, each of which gives a manifold of constant scalar curvature of dimension 1 and 2 respectively by the above lemma. Thus the universal cover of (M, g) is isometric to $\mathbb{H}^2 \times \mathbb{R}^1$ or \mathbb{R}^3 . In the first case, M is a quotient by a discrete subgroup of $PSL(2, \mathbb{R}) \times \mathbb{Z}$, and in the second case, $M \times S^1$ is a complex torus or a bielliptic surface by the Enriques-Kodaira classification. Therefore (M, g) is obtained by identifying two boundaries of $\Sigma \times [0, 1]$ by an orientation-preserving isometry of a compact Riemann surface (Σ, g_c) of genus ≥ 1 with a constant curvature metric g_c . Then (M, g) is locally a Riemannian product of (Σ, g_c) and S^1 , i.e., a Riemannian submersion onto S^1 with totally geodesic fiber (Σ, g_c) .

Conversely, suppose that (M, g) is such an oriented Riemannian submersion $\pi : M \rightarrow S^1$. The metric being locally a product, $(M \times S^1, g + dt^2)$ is a Kähler manifold with an obvious complex structure and a Kähler form $d\Omega + \pi^* ds \wedge dt$ where $d\Omega$ is the volume form of (Σ, g_c) and ds is the volume form of the base. The scalar curvature of g is just the constant scalar curvature of g_c and the first Chern class of Σ is a multiple of $d\Omega$ which is equal to $*\pi^* ds$. By the above lemma, (M, g) attains the equality, thereby completing the proof. □

Remark. As seen in the proof, the condition that (M, g) admits a solution for the rescaled Seiberg-Witten equations for any rescaling is superfluous. It is enough to suppose that $(M \times S^1, \hat{g})$ has a solution for the unrescaled Seiberg-Witten equations.

In particular, if (M, g) has an isometric G -action which can be lifted to \mathfrak{s} for a compact Lie group G , then it is enough for M to have solutions of the Seiberg-Witten equations for any G -invariant metric. In this case, $c_1(\mathfrak{s})$ is called a G -monopole class, and a G -monopole class sometimes exists even when the ordinary Seiberg-Witten invariant vanishes. This will be dealt with in our forthcoming paper ([21]).

Remark. It is well-known that on a compact Kähler surface (X, g, ϖ) with $b_2^+(X) = 1$, the Seiberg-Witten invariant of its canonical Spin^c structure for a small perturbation and a Riemannian metric $\tilde{g} \in [g]$ is equal to ± 1 , if $\text{deg}(c_1(X)) := [c_1(X)] \cdot [\varpi]$ is negative (See [16]), and also true for any perturbation and Riemannian metric if $c_1(X)$ is a torsion by using the celebrated Taubes's theorem [22] and the wall crossing formula [4, 15].

Now let's discuss some immediate implications of the above theorem. First, by taking ω to be $*c_1^h(\mathfrak{s})$ where $(\cdot)^h$ denotes the harmonic part, we have

$$\int_M |r_{\tilde{g}}|^2 d\mu_{\tilde{g}} \geq 8\pi^2 \int_M |c_1^h(\mathfrak{s})|_{\tilde{g}}^2 d\mu_{\tilde{g}}.$$

More interestingly we can get a lower bound of L^2 -norm of a harmonic 1-form on M :

Corollary 3.6. *Under the same hypothesis as Theorem 3.2, if \bar{g} is not flat,*

$$\left(\int_M |\omega|_{\bar{g}}^2 d\mu_{\bar{g}}\right)^{\frac{1}{2}} \geq \frac{2\sqrt{2}\pi|\alpha \cup [\omega]|}{\left(\int_M |r_{\bar{g}}|^2 d\mu_{\bar{g}}\right)^{\frac{1}{2}}},$$

where α is a convex combination of any two such $c_1(\mathfrak{s})$'s.

4. Monopole classes on connected sums

Our curvature estimates provide an easy toolkit in the study of a closed 3-manifold M with a non-torsion monopole. In [19], we derived the inevitability of collapsing when such a manifold has zero Yamabe invariant which implies the existence of a sequence of unit-volume Riemannian metrics $\{g_i\}$ on M satisfying $\inf_i \int_M s_{g_i}^2 d\mu_{g_i} = 0$. We also found a necessary condition to be a monopole class in a specific example as follows:

Proposition 4.1 ([19]). *Let M be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and N be a closed oriented 3-manifold with $b_1(N) = 0$. Then the rational part of a monopole class, if any, of $M\#N$ is of the form $m[F]$ for an integer m satisfying $|m| \leq |\chi(F)|$, where $\chi(F)$ is the Euler characteristic of the fiber F .*

(Warning: In the statement of Theorem 1.4 of [19], $b_1(N) = 0$ is missing by mistake.)

We give a generalization of this to connected sums:

Proposition 4.2. *Let M_i for $i = 1, \dots, n$ be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and N be any closed oriented 3-manifold. Then the rational part of a monopole class, if any, of $M_1\#\dots\#M_n\#N$ is of the form*

$$\beta + \sum_{i=1}^n m_i[F_i]$$

for $\beta \in H^2(N, \mathbb{Z})$ and an integer m_i satisfying $|m_i| \leq -\chi(F_i)$, where $\chi(F_i)$ is the Euler characteristic of the fiber F_i in M_i .

Proof. Let α be a monopole class of $X = M_1\#\dots\#M_n\#N$.

First if any F_i is a 2-sphere, then the only possibility for M_i is $S^1 \times S^2$. Letting $[\omega]$ be the Poincaré-dual of F_i , we only have to show that it pairs zero with α . Let $0 < \varepsilon \ll 1$.

Take a metric of positive scalar curvature on the M_i . For the connected sum, we take a small ball on M_i , and a representative ω of $[\omega]$ to be supported outside of that ball. Then perform the Gromov-Lawson type surgery [5, 17, 18] on it keeping the positivity of scalar curvature to get a compact manifold M'_i with a cylindrical end. And then contract it small enough so that

$$\int_{M'_i} |\omega|^2 d\mu \leq \varepsilon.$$

On the other part of X , we put any metric such that it satisfies

$$(4.1) \quad \int (s_-)^2 d\mu < 1,$$

and perform the Gromov-Lawson surgery such that the cylindrical end matches with that of the above-made M'_i while still satisfying (4.1).

After gluing these two pieces, we have that

$$\int_X |\omega|^2 d\mu \leq \varepsilon \quad \text{and} \quad \int_X (s_-)^2 d\mu < 1.$$

Applying Theorem 1.1, we get

$$|4\pi\alpha \cup [\omega]|^2 < \varepsilon,$$

which proves $\alpha \cup [\omega] = 0$.

Secondly, let's consider the case of M_i with $\chi(F_i) \leq 0$. Let $[\omega] \in H^1(M_i, \mathbb{R})$. By the Mayer-Vietoris principle, $H^1(M_i, \mathbb{R})$ is generated by

$$\pi_i^* dt, \quad \text{and} \quad \{[\sigma] \in H^1(F_i, \mathbb{R}) \mid f_i^*[\sigma] = [\sigma]\},$$

where $\pi_i : M_i \rightarrow S^1$ is the projection map and f_i is the monodromy diffeomorphism.

When $[\omega]$ is one of the latter ones, we have to show that it pairs zero with α . We can express ω as

$$\frac{1}{d_i} \sum_{n=1}^{d_i} (f_i^n)^* \sigma$$

for such σ satisfying $[f_i^* \sigma] = [\sigma]$, where d_i is the order of f_i .

By taking a f_i -invariant metric on F_i , we can put a locally-product metric on M_i such that π_i is a Riemannian submersion with totally geodesic fibers onto a circle of radius ε and

$$(4.2) \quad \int (s_-)^2 d\mu < \varepsilon.$$

We can take a small simply-connected open set B in F_i , which is invariant under f_i , and take a representative σ of the above $[\sigma] \in H^1(F_i, \mathbb{R})$ to be supported outside of B .

For the connected sum, we perform the Gromov-Lawson surgery on $B \times I(\frac{\varepsilon}{10}) \subset M_i$ where $I(\frac{\varepsilon}{10})$ is the interval of length $\frac{\varepsilon}{10}$ to get M'_i with a cylindrical end while still satisfying (4.2). On the other part of X , as before we put a metric with a cylindrical end isometric to that of this M'_i while satisfying (4.1).

After gluing two pieces, we have

$$\int_X |\omega|^2 d\mu \leq C\varepsilon \quad \text{and} \quad \int_X (s_-)^2 d\mu < 1 + \varepsilon$$

for a constant $C > 0$. Hence by Theorem 1.1

$$|4\pi\alpha \cup [\omega]|^2 < C\varepsilon(1 + \varepsilon),$$

proving $\alpha \cup [\omega] = 0$.

Finally when $\omega = \pi_i^* dt$, the adjunction inequality on a 3-manifold X , which can be proved in the same way as 4-manifolds ([7]) by considering $X \times S^1$ gives

$$|\langle \alpha, [F_i] \rangle| \leq -\chi(F_i),$$

completing the proof. \square

Remark. As noted, $M_i \times S^1$ for M_i as above admits a Kähler metric of constant scalar curvature, and each M_i admits a d_i -fold covering space which is $F_i \times S^1$.

For M_i with $F_i = S^2$, one can use the argument of gluing of moduli spaces of Seiberg-Witten equations along cylindrical ends to prove $4\pi\alpha \cup dt = 0$.

It seems plausible to conjecture:

Conjecture 4.3. *Let N_i for $i = 1, \dots, n$ be a closed oriented 3-manifold. Then (the rational part of) any monopole class of $N_1 \# \dots \# N_n$ is expressed as $\sum_i^n \alpha_i$ where α_i is a monopole class of N_i .*

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References

- [1] S. Bauer, *A stable cohomotopy refinement of Seiberg-Witten invariants. II*, Invent. Math. **155** (2004), no. 1, 21–40.
- [2] S. Bauer and M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, Invent. Math. **155** (2004), no. 1, 1–19.
- [3] G. Besson, G. Courtois, and S. Gallot, *Volume et entropie minimale des espaces localement symétriques*, Invent. Math. **103** (1991), no. 2, 417–445.
- [4] H.-D. Cao and J. Zhou, *Equivariant cohomology and wall crossing formula in Seiberg-Witten theory*, Math. Res. Lett. **5** (1998), no. 6, 711–721.
- [5] M. Gromov and H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. **111** (1980), no. 3, 423–434.
- [6] M. Gursky, *Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere*, Math. Ann. **318** (2000), no. 3, 417–431.
- [7] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), no. 6, 797–808.
- [8] H. B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, 1989.
- [9] C. LeBrun, *Four-manifolds without Einstein metric*, Math. Res. Lett. **3** (1996), no. 2, 133–147.
- [10] ———, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom. **5** (1997), no. 3, 535–553.
- [11] ———, *Curvature and smooth topology in dimension four*, Global analysis and harmonic analysis (Marseille-Luminy, 1999), 179–200, Semin. Congr., 4, Soc. Math. France, Paris, 2000.
- [12] ———, *Ricci curvature, minimal volumes, and Seiberg-Witten theory*, Invent. Math. **145** (2001), no. 2, 279–316.
- [13] ———, *Einstein metrics, four-manifolds, and differential topology*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), 235–255, Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003.

- [14] J. Lee and T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), no. 1, 37–91.
- [15] T. J. Li and A. Liu, *General wall crossing formula*, Math. Res. Lett. **2** (1995), no. 6, 797–810.
- [16] J. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifold*, Princeton University Press, 1996.
- [17] C. Sung, *Surgery, curvature, and minimal volume*, Ann. Global Anal. Geom. **26** (2004), no. 3, 209–229.
- [18] ———, *Surgery and equivariant Yamabe invariant*, Differential Geom. Appl. **24** (2006), no. 3, 271–287.
- [19] ———, *Collapsing and monopole classes of 3-manifolds*, J. Geom. Phys. **57** (2007), no. 2, 549–559.
- [20] ———, *Surgery, Yamabe invariant, and Seiberg-Witten theory*, J. Geom. Phys. **59** (2009), no. 2, 246–255.
- [21] ———, *Finite group actions and G-monopole classes on smooth 4-manifolds*, arXiv:1108.3875.
- [22] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), no. 6, 809–822.

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