A MEMORY TYPE BOUNDARY STABILIZATION FOR AN EULER-BERNOULLI BEAM UNDER BOUNDARY OUTPUT FEEDBACK CONTROL

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Abstract. In this paper, the memory type boundary stabilization for an Euler-Bernoulli beam with one end fixed and control at the other end is considered. We prove the existence of solutions using the Galerkin method and then investigate the exponential stability of solutions by using multiplier technique.

1. Introduction

In this paper, we consider the following Euler-Bernoulli beam with memory which has one end fixed and control input at the other end:

$$(1.1) \quad y_{tt}(x,t) + y_{xxxx}(x,t) - \int_0^t \kappa(t-\tau)y_{xxxx}(x,\tau)d\tau + g(y_t(x,t)) = 0,$$

$$x \in (0,L), \quad t \geq 0,$$

$$y(0,t) = y_x(0,t) = y_{xx}(L,t) = 0, \quad t \geq 0,$$

$$y_{xxx}(L,t) - \int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau = u(t) - \widetilde{\theta}\sin t, \quad t \geq 0,$$

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,L),$$

$$y_{out}(t) = y_t(L,t), \quad t \geq 0,$$

where κ represents the kernel of memory term, $g : \mathbb{R} \to \mathbb{R}$ is a given function, $\widetilde{\theta}$ is a positive constant, $u : \mathbb{R}^+ \to \mathbb{R}$ is the boundary control force applied at the free end of the beam and $y_{out}(t)$ stands for the measured signal of the system at time t. System (1.1) describes the transverse vibration of an extensible beam clamped at $x = 0$ and supported at $x = L$ by a control force. The advantage of the adaptive stabilization is that stabilization and good control

performance can be automatically achieved even in the presence of various types of uncertainty. In this paper, we consider the stabilization of the system (1.1). To this end, we design the following adaptive output feedback controller:

(1.2)
$$u(t) = h(t)y_t(L,t) + \theta(t)\sin t, \quad t \geq 0,$$
$$h_t(t) = ry_t^2(L,t), \quad h(0) = h_0 > 0, \quad t \geq 0, \quad r > 0,$$
$$\theta_t(t) = y_t(L,t)\sin t, \quad \theta(0) = \theta_0, \quad t \geq 0,$$

where θ_0 is the initial condition of the estimator.

Under this adaptive controller, the closed-loop system (1.1) becomes

(1.3)

$$y_{tt}(x,t) + y_{xxxx}(x,t) - \int_0^t \kappa(t-\tau)y_{xxxx}(x,\tau)d\tau + g(y_t(x,t)) = 0,$$
$$x \in (0,L), \quad t \geq 0,$$
$$y(0,t) = y_x(0,t) = y_{xx}(L,t) = 0, \quad t \geq 0,$$
$$y_{xxx}(L,t) - \int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau = h(t)y_t(L,t) + [\theta(t) - \widetilde{\theta}]\sin t, \quad t \geq 0,$$
$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,L),$$
$$h_t(t) = ry_t^2(L,t), \quad h(0) = h_0 > 0, \quad t \geq 0, \quad r > 0,$$
$$\theta_t(t) = y_t(L,t)\sin t, \quad \theta(0) = \theta_0.$$

The energy of the system (1.3) is given by

$$E(t) = \frac{1}{2}\int_0^t (y_t^2(x,t) + y_{xx}^2(x,t))dx.$$

Ma [12, 13] studied the boundary stabilization for a nonlinear beam on elastic bearings without memory and boundary output feedback control. The existence and uniform decay of solutions for an Euler-Bernoulli beam with memory was consider by Park and Kim [15]. Also, the Euler-Bernoulli beam equation with memory and boundary output feedback control was studied by Park, Kang and Kim [14]; they proved the existence and exponential stability of solutions for the following system:

(1.4) $$y_{tt}(x,t) + y_{xxxx}(x,t) - \int_0^t \kappa(t-\tau)y_{xxxx}(x,\tau)d\tau + g(y_t(x,t)) = 0,$$
$$x \in (0,L), \quad t \geq 0,$$
$$y(0,t) = y_x(0,t) = y_{xx}(L,t) = 0, \quad t \geq 0,$$
$$y_{xxx}(L,t) - \int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau = h(t)y_t(L,t), \quad t \geq 0,$$
$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,L),$$
$$h_t(t) = ry_t^2(L,t), \quad h(0) = h_0 > 0, \quad t \geq 0, \quad r > 0.$$

Conrad and Omër [3] proved the existence and uniform decay for a flexible beam with a tip mass. The boundary stabilization and boundary controllability for the beams were consider by authors [1, 2, 4-6, 9-11, 16, 17]. Guo and Luo [8] studied the stabilization and parameter estimation for a flexible-beam vibration with gain adaptive direct strain feedback control. Recently, Guo and Guo [7] consider the adaptive stabilization for a Kirchhoff-type nonlinear beam under boundary output feedback control. Motivated by [14] and some idea in [7], we can obtain our main results. The objective of this paper is to study the stabilization for a more general Euler-Bernoulli beam with memory under boundary output feedback control. Our choice of boundary feedback is motivated by the fact that boundary controls are easily implemented as the need to act only on the boundary of the spatial domain. This paper is organized as follows. In Section 2, using a constructive Galerkin approximation scheme, we show the existence and uniqueness of the solution for the system (1.3). The exponential decay that is dependent on the initial data is obtained in Section 3 by making use of the multiplier technique.

2. Existence result

In this section we prepare some notation and hypotheses which will be needed in the proof of our result. Let $L^2(0, L)$ be the usual Hilbert space with the inner product $(\cdot, \cdot)$ and the inner product induced norm $\| \cdot \|$. Throughout this paper, we define $V = \{y \in H^2(0, L) \mid y(0) = y_x(0) = 0\}$ equipped with the norm $\|y\|_V = \|y_{xx}\|$, $W = \{y \in V \cap H^4(0, L) \mid y_{xx}(L) = 0\}$ equipped with the norm $\|y\|_W = \|y_{xx}\| + \|y_{xxxx}\|$ and $(u, v) = \int_0^L u(x)v(x)dx$. From the Poincaré inequality, it follows that $\| \cdot \|_V$ and $\| \cdot \|_W$ are equivalent to the standard norms of $H^2(0, L)$ and $H^4(0, L)$, respectively. Now, we state the following hypotheses which will be assumed in this paper.

(H_1) For any $y_0 \in W$, $y_1 \in V$,

$$(2.1) \qquad y_{xxxx}(x, 0) + g(y_1) = 0, \quad x \in (0, L),$$
$$y(0, 0) = y_x(0, 0) = y_{xx}(L, 0) = 0,$$
$$y_{xxx}(L, 0) = h_0 y_t(L, 0).$$

(H_2) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and there exists a positive constant μ such that

$$(2.2) \qquad g(0) = 0, \quad (g(r) - g(s))(r - s) \geq \mu |r - s|^2, \quad r, s \in \mathbb{R}.$$

(H_3) Let the function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive and bounded C^2-function such that

$$(2.3) \qquad \ell = 1 - \int_0^\infty \kappa(r)dr > 0$$

and for some positive $m_i, i = 0, 1, 2$

$$(2.4) \qquad -m_0\kappa(t) \leq \kappa_t(t) \leq -m_1\kappa(t), \quad \forall t \geq 0,$$

$$0 \leq \kappa_{tt}(t) \leq m_2\kappa(t), \quad \forall t \geq 0.$$

Considering the above hypotheses we have the following existence result by Galerkin method.

Theorem 2.1. *Let $y_0 \in W$, $y_1 \in V$. Suppose that (H_1), (H_2) and (H_3) are satisfied. Then the problem (1.3) has a unique solution y in the sense that for any time $T > 0$,*

(2.5)

$$y \in L^\infty(0, T; W), \quad y_t \in L^\infty(0, T; V), \quad y_{tt} \in L^\infty(0, T; L^2(0, L)),$$
$$h \in C^1[0, T], \quad \theta \in C^1[0, T],$$
$$y_{tt}(x, t) + y_{xxxx}(x, t) - \int_0^t \kappa(t - \tau)y_{xxxx}(x, \tau)d\tau + g(y_t(x, t)) = 0$$
$$in \quad L^2(0, T; L^2(0, L)),$$
$$y(0, t) = y_x(0, t) = y_{xx}(L, t) = 0, \quad t \geq 0,$$
$$y_{xxx}(L, t) - \int_0^t \kappa(t - \tau)y_{xxx}(L, t)dx = h(t)y_t(L, t) + [\theta(t) - \widetilde{\theta}]\sin t, \quad t \geq 0,$$
$$y(x, 0) = y_0(x), \quad y_t(x, t) = y_1(x), \quad x \in (0, L),$$
$$h_t(t) = ry_t^2(L, t), \quad t \geq 0, \quad h(0) = h_0 > 0, \quad r > 0,$$
$$\theta_t(t) = y_t(L, t)\sin t, \quad t \geq 0, \quad \theta(0) = \theta_0.$$

By the Sobolev embedding theorem, it follows that $y \in C((0, L) \times [0, T])$.

Proof. Let us solve the variational problem with (2.5) which is given by: find $y(t) \in V$ such that

$$(2.6) \quad (y_{tt}(t), w) + (y_{xx}(t), w_{xx}) - \int_0^t \kappa(t - \tau)(y_{xx}(\tau), w_{xx})d\tau + (g(y_t(t)), w)$$
$$+ \{h(t)y_t(L, t) + (\theta(t) - \widetilde{\theta})\sin t\}w(L) = 0 \quad \text{for all} \quad w \in V.$$

Let $\{w^j\}$ be a complete orthogonal system of V for which $\{y_0, y_1\} \in \text{Span}\{w^1, w^2\}$. For each $m \in \mathbb{N}$, we denote $V_m = \text{Span}\{w^1, w^2, \ldots, w^m\}$. We search for a function

$$y^m(t) = \sum_{j=1}^m k^j(t)w^j$$

such that for any $w \in V_m$, it satisfies the approximate equation

$$(2.7) \quad (y_{tt}^m(t), w) + (y_{xx}^m(t), w_{xx}) - \int_0^t \kappa(t - \tau)(y_{xx}^m(\tau), w_{xx})d\tau + (g(y_t^m(t)), w)$$
$$+ \{h^m(t)y_t^m(L, t) + (\theta^m(t) - \widetilde{\theta})\sin t\}w(L) = 0,$$
$$h_t^m(t) = r\Big[\sum_{j=1}^m k_t^j(t)w^j(L)\Big]^2 = r[y_t^m(L, t)]^2,$$

$$\theta_t^m(t) = \Big[\sum_{j=1}^{m} k_t^j(t) w^j(L) \Big] \sin t = y_t^m(L, t) \sin t,$$

$$h^m(0) = h_0 > 0, \quad \theta^m(0) = \theta_0,$$

$$y^m(0) = y_{0m} \to y_0 \text{ in } W, \quad y_t^m(0) = y_{1m} \to y_1 \text{ in } V.$$

By standard methods in differential equations, we can prove the existence of a solution to (2.7) on some interval, $(0, t_m)$, where $t_m = \infty$ by using the first estimate below. In order to prove Theorem 2.1 it suffices to prove the following a priori estimates.

Estimate I. Replacing w by y_t^m in (2.7), we have

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} \Big(\|y_t^m(t)\|^2 + \|y_{xx}^m(t)\|^2 \Big) + (g(y_t^m(t)), y_t^m(t))$$

$$= \frac{d}{dt} \Big(\int_0^t \kappa(t - \tau)(y_{xx}^m(\tau), y_{xx}^m(t)) d\tau \Big) - \int_0^t \kappa_t(t - \tau)(y_{xx}^m(\tau), y_{xx}^m(t)) d\tau$$

$$- \kappa(0) \|y_{xx}^m(t)\|^2 - \{ h^m(t)[y_t^m(L, t)]^2 + [\theta^m(t) - \widetilde{\theta}] \sin t \cdot y_t^m(L, t) \}.$$

By hypotheses (2.4) and Cauchy-Schwartz inequality, we deduce that

$$(2.9) \quad \Big| \int_0^t \kappa_t(t - \tau)(y_{xx}^m(\tau), y_{xx}^m(t)) d\tau \Big|$$

$$\leq \|y_{xx}^m(t)\| \int_0^t |\kappa_t(t - \tau)| \|y_{xx}^m(\tau)\| d\tau$$

$$\leq \frac{m_0^2}{2} \|y_{xx}^m(t)\|^2 + \frac{1}{2} \Big(\int_0^t \kappa(t - \tau) \|y_{xx}^m(\tau)\| d\tau \Big)^2$$

$$\leq \frac{m_0^2}{2} \|y_{xx}^m(t)\|^2 + \frac{1}{2} \|\kappa\|_{L^1(0,\infty)} \int_0^t \kappa(t - \tau) \|y_{xx}^m(\tau)\|^2 d\tau.$$

Integrating (2.8) over $(0, t)$ and using (2.2) and (2.9), we obtain

$$(2.10) \quad \|y_t^m(t)\|^2 + \|y_{xx}^m(t)\|^2 + \frac{1}{r}[h^m(t)]^2 + [\theta^m(t) - \widetilde{\theta}]^2 + 2\mu \int_0^t \|y_t^m(\tau)\|^2 d\tau$$

$$\leq \|y_t^m(0)\|^2 + \|y_{xx}^m(0)\|^2 + \frac{1}{r}[h^m(0)]^2 + [\theta_0 - \widetilde{\theta}]^2$$

$$+ 2 \int_0^t \kappa(t - \tau)(y_{xx}^m(\tau), y_{xx}^m(t)) d\tau$$

$$+ \Big(m_0^2 + \|\kappa\|_{L^1(0,\infty)}^2 - 2\kappa(0) \Big) \int_0^t \|y_{xx}(\tau)\|^2 d\tau.$$

Using Schwartz inequality and Young's inequality, we get

$$(2.11) \quad \Big| \int_0^t \kappa(t - \tau)(y_{xx}^m(\tau), y_{xx}^m(t)) d\tau \Big|$$

$$\leq \|y_{xx}^m(t)\| \|\kappa\|_{L^1(0,\infty)}^{\frac{1}{2}} \Big(\int_0^t \kappa(t - \tau) \|y_{xx}^m(\tau)\|^2 d\tau \Big)^{\frac{1}{2}}$$

$$\leq \frac{1}{8}\|y_{xx}^m(t)\|^2 + 2\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}\int_0^t\|y_{xx}^m(\tau)\|^2 d\tau.$$

Combining the inequalities (2.10), (2.11) and applying Gronwall's Lemma, we see the first estimate:

$$(2.12) \qquad \|y_t^m(t)\|^2 + \|y_{xx}^m(t)\|^2 + \frac{1}{r}[h^m(t)]^2 + [\theta^m(t) - \widetilde{\theta}]^2 \leq M_1,$$

where $M_1 > 0$ depends on the initial data y_0, y_1, h_0, θ_0.

From this, we obtain $y_t^m(L, t) \in L^\infty(0, \infty)$. Therefore, the approximate solution can be extended to the whole interval $[0, T]$, where $T = \infty$.

Estimate II. First of all, we estimate the L^2-norm of $y_{tt}^m(0)$. Considering $t = 0$ and $w = y_{tt}^m(0)$ in (2.7), then we get

$$\|y_{tt}^m(0)\|^2 + (y_{xx}^m(0), y_{xxtt}^m(0)) + (g(y_t^m(0)), y_{tt}^m(0)) + h_0 y_t^m(L, 0)y_{tt}^m(L, 0) = 0.$$

Since the compatibility condition (H_1), we obtain

$$(y_{xx}^m(0), y_{xxtt}^m(0)) = y_{xx}^m(L, 0)y_{xtt}^m(L, 0) - y_{xx}^m(0, 0)y_{xtt}^m(0, 0) - y_{xxx}^m(L, 0)y_{tt}^m(L, 0)$$
$$+ y_{tt}^m(0, 0)y_{xxx}^m(0, 0) + (y_{xxxx}^m(0), y_{tt}^m(0))$$
$$= -h_0 y_t^m(L, 0)y_{tt}^m(L, 0) - (g(y_t^m(0)), y_{tt}^m(0)).$$

Therefore, from the above equality, there exists the following result

$$(2.13) \qquad\qquad\qquad \|y_{tt}^m(0)\| = 0.$$

Finally, differentiating (2.7) and writing the equation with $w = y_{tt}^m(t)$, we have

(2.14)

$$\frac{1}{2}\frac{d}{dt}(\|y_{tt}^m(t)\|^2 + \|y_{xxt}^m(t)\|^2) + \kappa(0)\|y_{xxt}^m(t)\|^2 + (g'(y_t^m(t))y_{tt}^m(t), y_{tt}^m(t))$$
$$+ \Big(r[y_t^m(L, t)]^3 + h^m(t)y_{tt}^m(L, t)\Big)y_{tt}^m(L, t)$$
$$+ \Big(y_t^m(L, t)sin^2 t + (\theta^m(t) - \widetilde{\theta})\cos t\Big)y_{tt}^m(L, t)$$
$$= \frac{d}{dt}\Big(\int_0^t \kappa_t(t - \tau)(y_{xx}^m(\tau), y_{xxt}^m(t))d\tau\Big) - \int_0^t \kappa_{tt}(t - \tau)(y_{xx}^m(\tau), y_{xxt}^m(t))d\tau$$
$$- \kappa_t(0)(y_{xx}^m(t), y_{xxt}^m(t)) + \kappa(0)\frac{d}{dt}(y_{xx}^m(t), y_{xxt}^m(t)).$$

From hypotheses (2.4), we deduce that

$$(2.15) \qquad \Big|\int_0^t \kappa_{tt}(t - \tau)(y_{xx}^m(\tau), y_{xxt}^m(t))d\tau\Big|$$
$$\leq \frac{m_2^2}{2}\|y_{xxt}(t)\|^2 + \frac{1}{2}\|\kappa\|_{L^1(0,\infty)}\int_0^t \kappa(t - \tau)\|y_{xx}^m(\tau)\|^2 d\tau.$$

Since $g \in C^1(\mathbb{R})$ and $y_t^m(t)$ is bounded by (2.12), there exists $C_1 > 0$ depends on the initial datas y_0, y_1, h_0, θ_0 such that

$$(2.16) \qquad\qquad |(g'(y_t^m(t))y_{tt}^m(t), y_{tt}^m(t))| \leq C_1\|y_{tt}^m(t)\|^2.$$

Using (2.13), (2.15) and (2.16), we integrate (2.14) over $(0, t)$ to obtain
(2.17)

$$\frac{1}{2}\|y_{tt}^m(t)\|^2 + \frac{1}{2}\|y_{xxt}^m(t)\|^2 + \kappa(0)\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau + \frac{r}{4}[y_t^m(L,t)]^4$$

$$\leq C_2 + \frac{m_2^2}{2}\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau + \frac{1}{2}\|\kappa\|_{L^1(0,\infty)}^2 \int_0^t \|y_{xx}^m(\tau)\|^2 d\tau$$

$$+ C_1 \int_0^t \|y_{tt}^m(\tau)\|^2 d\tau + \kappa(0)(y_{xx}^m(t), y_{xxt}^m(t)) - \kappa_t(0)\int_0^t (y_{xx}^m(\tau), y_{xxt}^m(\tau)) d\tau$$

$$+ \int_0^t \kappa_t(t-\tau)(y_{xx}^m(\tau), y_{xxt}^m(t)) d\tau + \frac{r}{4}[y_t^m(L,0)]^4 - \int_0^t h^m(\tau)[y_{tt}^m(L,\tau)]^2 d\tau$$

$$- \frac{1}{2}[y_t^m(L,t)]^2 \sin^2 t + \frac{1}{2}\int_0^t [y_t^m(L,\tau)]^2 \sin 2\tau d\tau - y_t^m(L,t)[\theta^m(t) - \widetilde{\theta}]\cos t$$

$$+ y_t^m(L,0)[\theta_0 - \widetilde{\theta}] + \int_0^t y_t^m(L,\tau)\{y_t^m(L,\tau)\sin\tau\cos\tau - [\theta^m(\tau) - \widetilde{\theta}]\sin\tau\} d\tau,$$

where $C_2 > 0$ depends on the initial data y_0, y_1, h_0.

Now, since (2.4), for any $\eta > 0$ we see that

(2.18) $$|\kappa(0)(y_{xx}^m(t), y_{xxt}^m(t))| \leq \frac{(\kappa(0))^2}{4\eta}\|y_{xx}^m(t)\|^2 + \eta\|y_{xxt}^m(t)\|^2,$$

(2.19)
$$\left|\kappa_t(0)\int_0^t (y_{xx}^m(\tau), y_{xxt}^m(\tau)) d\tau\right| \leq \frac{m_0^2(\kappa(0))^2}{4\eta}\int_0^t \|y_{xx}^m(\tau)\|^2 d\tau + \eta\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau$$

and

(2.20) $$\left|\int_0^t \kappa_t(t-\tau)(y_{xx}^m(\tau), y_{xxt}^m(t)) d\tau\right|$$

$$\leq \frac{m_0^2}{4\eta}\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}\int_0^t \|y_{xx}^m(\tau)\|^2 d\tau + \eta\|y_{xxt}^m(t)\|^2.$$

From the inequality $ab \leq \delta_1 a^2 + \frac{1}{4\delta_1}b^2$, $\delta_1 > 0$ and the Sobolev embedding theorem $\|y_t(L,t)\|^2 \leq \delta_2\|y_{xxt}(t)\|^2$, $\delta_2 > 0$, we deduce

(2.21) $$\left|-\frac{1}{2}[y_t^m(L,t)]^2 \sin^2 t - y_t^m(L,t)[\theta^m(t) - \widetilde{\theta}]\cos t\right|$$

$$\leq \delta_1[y_t^m(L,t)]^2 \sin^2 t + \delta_1[y_t^m(L,t)]^2 \cos^2 t + \frac{1}{4\delta_1}[\theta^m(t) - \widetilde{\theta}]^2$$

$$\leq \widetilde{\delta_1}\|y_{xxt}^m(t)\|^2 + \frac{1}{4\delta_1}[\theta^m(t) - \widetilde{\theta}]^2, \quad \text{where} \quad \widetilde{\delta_1} = \delta_1\delta_2$$

and using Young's inequality we obtain

(2.22) $$\left|\int_0^t y_t^m(L,\tau)\{y_t^m(L,\tau)\sin\tau\cos\tau - [\theta^m(\tau) - \widetilde{\theta}]\sin\tau\} d\tau\right|$$

$$\leq \int_0^t [y_t^m(L,\tau)]^2 d\tau + \frac{1}{2}\int_0^t [\theta^m(\tau) - \widetilde{\theta}]^2 d\tau$$

$$\leq \frac{h^m(t)}{r} - \frac{h_0}{r} + \frac{1}{2}\int_0^t [\theta^m(\tau) - \widetilde{\theta}]^2 d\tau.$$

Thus from (2.17)-(2.22), we have

(2.23)

$$\frac{1}{2}\|y_{tt}^m(t)\|^2 + \frac{1}{2}\|y_{xxt}^m(t)\|^2 + \kappa(0)\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau + \frac{r}{4}[y_t^m(L,t)]^4$$

$$\leq C_2 + C_1\int_0^t \|y_{tt}^m(\tau)\|^2 d\tau + \frac{(\kappa(0))^2}{4\eta}\|y_{xx}^m(t)\|^2$$

$$+ \left(\frac{m_2^2}{2} + \eta\right)\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau + 2\eta\|y_{xxt}^m(t)\|^2$$

$$+ \left(\frac{1}{2}\|\kappa\|_{L^1(0,\infty)}^2 + \frac{m_0^2(\kappa(0))^2}{4\eta} + \frac{m_0^2}{4\eta}\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}\right)\int_0^t \|y_{xx}^m(\tau)\|^2 d\tau$$

$$+ \frac{r}{4}[y_t^m(L,0)]^4 - \int_0^t h^m(\tau)[y_{tt}^m(L,\tau)]^2 d\tau - \frac{1}{2}[y_t^m(L,t)]^2 \sin^2 t$$

$$+ \frac{1}{2}\int_0^t [y_t^m(L,\tau)]^2 \sin 2\tau d\tau - y_t^m(L,t)[\theta^m(t) - \widetilde{\theta}]\cos t + y_t^m(L,0)[\theta_0 - \widetilde{\theta}]$$

$$+ \int_0^t y_t^m(L,\tau)\{y_t^m(L,\tau)\sin\tau\cos\tau - [\theta^m(\tau) - \widetilde{\theta}]\sin\tau\}d\tau$$

$$\leq C_2 + C_1\int_0^t \|y_{tt}^m(\tau)\|^2 d\tau + \frac{(\kappa(0))^2}{4\eta}\|y_{xx}^m(t)\|^2$$

$$+ \left(\frac{m_2^2}{2} + \eta\right)\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau + 2\eta\|y_{xxt}^m(t)\|^2$$

$$+ \left(\frac{1}{2}\|\kappa\|_{L^1(0,\infty)}^2 + \frac{m_0^2(\kappa(0))^2}{4\eta} + \frac{m_0^2}{4\eta}\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}\right)\int_0^t \|y_{xx}^m(\tau)\|^2 d\tau$$

$$+ \frac{r}{4}[y_t^m(L,0)]^4 + \frac{3h^m(t)}{2r} + \widetilde{\delta_1}\|y_{xxt}^m(t)\|^2 + \frac{1}{4\delta_1}[\theta^m(t) - \widetilde{\theta}]^2 + \frac{1}{2}[y_t^m(L,0)]^2$$

$$+ \frac{1}{2}[\theta_0 - \widetilde{\theta}]^2 + \frac{1}{2}\int_0^t [\theta^m(\tau) - \widetilde{\theta}]^2 d\tau - \frac{3h_0}{2r}.$$

From the inequalities (2.12) and (2.23) and choosing $\eta > 0$ sufficiently small, we get

$$\frac{1}{2}\|y_{tt}^m(t)\|^2 + \left(\frac{1}{2} - \widetilde{\delta_1}\right)\|y_{xxt}^m(t)\|^2 + \kappa(0)\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau + \frac{r}{4}[y_t^m(L,t)]^4$$

$$\leq C_3 + C_1\int_0^t \|y_{tt}^m(\tau)\|^2 d\tau + \left(\frac{m_2^2}{2} + \eta\right)\int_0^t \|y_{xxt}^m(\tau)\|^2 d\tau,$$

where $C_3 > 0$ depends on the initial data y_0, y_1, h_0, θ_0.

Taking $\widetilde{\delta}_1$ with $\frac{1}{2} - \widetilde{\delta}_1 > 0$ and using of Gronwall's Lemma, we have the second estimate:

$$(2.24) \qquad \|y_{tt}^m(t)\|^2 + \|y_{xxt}^m(t)\|^2 + r[y_t^m(L,t)]^4 \le M_2,$$

where M_2 depends on the initial data y_0, y_1, h_0, θ_0.

Analysis of the nonlinear terms. By estimates (2.12) and (2.24), we deduce that

$$(2.25) \qquad \{y^m\} \text{ is bounded in } L^\infty(0,T;V),$$
$$\{y_t^m\} \text{ is bounded in } L^\infty(0,T;V),$$
$$\{y_{tt}^m\} \text{ is bounded in } L^\infty(0,T;L^2(0,L)),$$
$$\{y_t^m(L,t)\} \text{ is bounded in } L^2(0,T),$$
$$\{h^m\} \text{ is bounded in } L^\infty(0,T),$$
$$\{h_t^m\} = \{r(y_t^m(L,t))^2\} \text{ is bounded in } L^\infty(0,T),$$
$$\{\theta^m\} \text{ is bounded in } L^\infty(0,T),$$
$$\{\theta_t^m\} = \{y_t^m(L,t)\sin t\} \text{ is bounded in } L^\infty(0,T).$$

Therefore, there exists a subsequence of $\{y^m\}$, still denoted by $\{y^m\}$ such that

$$(2.26) \qquad y^m \to y \text{ weakly star in } L^\infty(0,T;V),$$
$$y_t^m \to y_t \text{ weakly star in } L^\infty(0,T;V),$$
$$y_{tt}^m \to y_{tt} \text{ weakly star in } L^\infty(0,T;L^2(0,L)),$$
$$y_t^m(L,t) \to y_t(L,t) \text{ weakly in } L^2(0,T),$$
$$h^m \to h \text{ weakly star in } L^\infty(0,T),$$
$$h_t^m \to h_t \text{ weakly star in } L^\infty(0,T),$$
$$\theta^m \to \theta \text{ weakly star in } L^\infty(0,T),$$
$$\theta_t^m \to \theta_t \text{ weakly star in } L^\infty(0,T).$$

Due to the compact embedding $V \hookrightarrow L^2(0,L)$, we can get a subsequence such that

$$(2.27) \qquad y_t^m \to y_t \text{ strongly in } L^2(0,T;L^2(0,L)).$$

From (H_2) and (2.27), we get

$$(2.28) \qquad g(y_t^m) \to g(y_t) \text{ a.e. in } x \in (0,L), \;\; t > 0.$$

From the above convergence and due to the boundedness of the sequence $\{g(y_t^m)\}$ in $L^2(0,T;L^2(0,L))$, we conclude by Lion's Lemma that

$$g(y_t^m) \to g(y_t) \text{ weakly in } L^2(0,T;L^2(0,L)).$$

Moreover, by the Sobolev embedding theorem and (2.26), we see that $h \in C^1[0,T]$ and $h^m(t)y_t^m(L,t) \to h(t)y_t(L,t)$ weakly in $L^2(0,T)$.

We also derive that

$$\theta \in C^1[0,T] \text{ and } \theta^m(t)\sin t \to \theta(t)\sin t \text{ weakly in } L^2(0,T).$$

The above convergence results are sufficient to pass to the limit in the nonlinear terms of (2.7). Then it is a matter of routine to deduce the existence of global solutions in $[0, T]$. The uniqueness can be proved by the straightforward methods and Gronwall's inequality. $\square$

3. Exponential stability

Having established global existence of solution to (1.3), we focus our attention on exponential decay that can be obtained for the energy function.

We define the energy $E(t)$ of problem (1.3) by $E(t) = \frac{1}{2}\|y_t(t)\|^2 + \frac{1}{2}\|y_{xx}(t)\|^2$. Then the derivative of the energy is given by

$$E_t(t) = -(g(y_t(t)), y_t(t)) + \int_0^t \kappa(t-\tau)(y_{xx}(\tau), y_{xxt}(\tau))d\tau - h(t)(y_t(L,t))^2$$
$$- [\theta(t) - \widetilde{\theta}]\sin t \cdot y_t(L,t).$$

Defining

$$(3.1) \qquad (\kappa \square y_{xx})(t) = \int_0^t \kappa(t-\tau)\|y_{xx}(\tau) - y_{xx}(t)\|^2 d\tau.$$

By simple computation, we obtain

$$(3.2) \qquad (\kappa \square y_{xx})_t(t) = (\kappa_t \square y_{xx})(t) - 2\int_0^t \kappa(t-\tau)(y_{xx}(\tau), y_{xxt}(t))d\tau$$
$$+ \frac{d}{dt}\left(\|y_{xx}(t)\|^2 \int_0^t \kappa(\tau)d\tau\right) - \kappa(t)\|y_{xx}(t)\|^2.$$

From (3.2), we see that

$$(3.3) \qquad \int_0^t \kappa(t-\tau)(y_{xx}(\tau), y_{xxt}(t))d\tau$$
$$= -\frac{1}{2}(\kappa \square y_{xx})_t(t) + \frac{1}{2}(\kappa_t \square y_{xx})(t) + \frac{1}{2}\frac{d}{dt}\left(\|y_{xx}(t)\|^2 \int_0^t \kappa(\tau)d\tau\right)$$
$$- \frac{1}{2}\kappa(t)\|y_{xx}(t)\|^2.$$

Define the modified energy by

$$(3.4) \qquad e(t) = \frac{1}{2}\|y_t(t)\|^2 + \frac{1}{2}(\kappa \square y_{xx})(t) + \frac{1}{2}\left(1 - \int_0^t \kappa(\tau)d\tau\right)\|y_{xx}(t)\|^2$$
$$+ \frac{1}{2}[\theta(t) - \widetilde{\theta}]^2.$$

Then, from (3.3) and (3.4) we obtain

(3.5)

$$e_t(t) = -(g(y_t(t)), y_t(t)) + \frac{1}{2}(\kappa_t \square y_{xx})(t) - \frac{1}{2}\kappa(t)\|y_{xx}(t)\|^2 - h(t)(y_t(L,t))^2$$
$$\leq -(g(y_t(t)), y_t(t)) - \frac{m_1}{2}(\kappa \square y_{xx})(t) - \frac{1}{2}\kappa(t)\|y_{xx}(t)\|^2$$

$$- h(t)(y_t(L,t))^2 \leq 0, \quad \forall t \geq 0.$$

We observe that in view of hypotheses (2.3) and (2.4), we see that $e(t) \geq 0$ and $E(t) \leq \ell^{-1}e(t)$. Therefore it is enough to obtain the desired exponential stability for the modified energy $e(t)$ which will be done below. In order to carry the proof of Theorem 3.2, we need the following Theorem 3.1.

Theorem 3.1. *Let y be the solution given by Theorem 2.1 and $e(t)$ be defined by (3.4). Then we have*

$$\lim_{t \to \infty} e(t) = 0, \quad \lim_{t \to \infty} h(t) \leq \sqrt{2re(0) + (h(0))^2} \ ,$$
$$h(t) \leq \sqrt{2re(0) + (h(0))^2}, \quad \forall t \geq 0.$$

Now, we define the perturbed energy by

(3.6)

$$e_\varepsilon(t) = e(t) + \varepsilon\psi(t), \quad \text{where} \ \ \psi(t) = \delta \int_0^t y(x,t)y_t(x,t)dx \ \ \text{with} \ \ 0 < \delta < \frac{1}{2}.$$

Then, we obtain the following two propositions.

Proposition 3.1. *There exists $C_1' > 0$ such that*

(3.7) $$|e_\varepsilon(t) - e(t)| \leq \varepsilon C_1' e(t), \quad \forall t \geq 0 \ \ and \ \ \forall \varepsilon > 0.$$

Proof. The proof of this proposition is the same the proof of [15, Proposition 3.1], so we will omit this proof. □

Proposition 3.2. *There exist positive constants C_2', C_3' and C_4' such that*

$$\frac{d}{dt}e_\varepsilon(t) \leq \ -\varepsilon C_2'e(t) + \varepsilon C_3'(1 + h(t))(y(L,t))^2 + \varepsilon C_4'[\theta(t) - \widetilde{\theta}]^2,$$
$$\forall t \geq 0 \ \ and \ \ \forall \varepsilon \in [0, \varepsilon_1].$$

Proof. Using (1.1), we deduce that

(3.8) $$\frac{d}{dt}\psi(t) = \delta \int_0^L y_t^2(t)dx + \delta \int_0^L y(t)y_{tt}(t)dx$$

$$= \delta \int_0^L (y_t(t))^2 dx + \delta \int_0^t y(t)(-y_{xxxx}(t))dx$$

$$+ \delta \int_0^L y(t)\Big(\int_0^t \kappa(t - \tau)y_{xxxx}(\tau)d\tau \Big) dx$$

$$- \delta \int_0^L y(t)g(y_t(t))dx.$$

By $y(0,t) = y_x(0,t) = y_{xx}(L,t) = 0$ and $y \in L^\infty(0,T;W)$, we have the following two inequalities

$$\int_0^L y(t)(-y_{xxxx}(t))dx$$

$$= -y(L,t)y_{xxx}(L,t) + \int_0^L y_x(t)y_{xxx}(t)dx$$

$$= -y(L,t)y_{xxx}(L,t) + y_x(L,t)y_{xx}(L,t) - \int_0^L (y_{xx}(t))^2 dx$$

$$= -y(L,t)y_{xxx}(L,t) - \int_0^L (y_{xx}(t))^2 dx$$

and

$$\int_0^L y(t)\Big(\int_0^t \kappa(t-\tau)y_{xxxx}(\tau)d\tau \Big)dx$$

$$= y(L,t)\Big(\int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau \Big) - \int_0^L y_x(t)\Big(\int_0^t \kappa(t-\tau)y_{xxx}(\tau)d\tau \Big)dx$$

$$= y(L,t)\Big(\int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau \Big) - y_x(L,t)\Big(\int_0^t \kappa(t-\tau)y_{xx}(L,\tau)d\tau \Big)$$

$$+ \int_0^L y_{xx}(t)\Big(\int_0^t \kappa(t-\tau)y_{xx}(\tau)d\tau \Big)dx$$

$$= y(L,t)\Big(\int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau \Big) + \int_0^L y_{xx}(t)\Big(\int_0^t \kappa(t-\tau)y_{xx}(\tau)d\tau \Big)dx.$$

Thus, from (3.8) we get

(3.9)
$$\frac{d}{dt}\psi(t) = \delta \int_0^t (y_t(t))^2 dx - \delta y(L,t)\Big[y_{xxx}(L,t) - \int_0^t \kappa(t-\tau)y_{xxx}(L,\tau)d\tau \Big]$$

$$- \delta \int_0^L (y_{xx}(t))^2 dx + \delta \int_0^L y_{xx}(t)\Big(\int_0^t \kappa(t-\tau)y_{xx}(\tau)d\tau \Big)dx$$

$$- \delta \int_0^L y(t)g(y_t(t))dx.$$

From the assumptions on g, Young's inequality and Sobolev imbedding theorem, we see that

(3.10)
$$\delta \Big| \int_0^t y(t)g(y_t(t))dx \Big| \le M_1'\|y_t(t)\|^2 + M_2'\|y_{xx}(t)\|^2,$$

where M_1' and M_2' are some positive constants.

Moreover, for any $\eta > 0$, from Fubini's theorem, we have

(3.11)
$$\int_0^L y_{xx}(t)\Big(\int_0^t \kappa(t-\tau)y_{xx}(\tau)d\tau \Big)dx$$

$$= \int_0^t \kappa(t-\tau)\Big(\int_0^L y_{xx}(\tau)y_{xx}(t)dx \Big)d\tau$$

$$= \int_0^t \kappa(t-\tau)\Big(\int_0^L (y_{xx}(\tau) - y_{xx}(t))y_{xx}(t)dx \Big)d\tau$$

$$+ \int_0^t \kappa(t-\tau) \Big(\int_0^L (y_{xx}(t))^2 dx \Big) d\tau$$

$$\leq \eta \|y_{xx}(t)\|^2 + \frac{1}{4\eta} \int_0^t \int_0^L (\kappa(t-\tau))^2 (y_{xx}(\tau) - y_{xx}(t))^2 dx d\tau$$

$$+ \int_0^t \kappa(\tau) \Big(\int_0^L (y_{xx}(t))^2 dx \Big) d\tau$$

$$\leq \eta \|y_{xx}(t)\|^2 + \frac{1}{4\eta} \|\kappa\|_{L^\infty(0,\infty)} (\kappa \square y_{xx})(t) + \int_0^t \kappa(\tau) d\tau \|y_{xx}(t)\|^2.$$

Combining (3.9)-(3.11), we obtain

(3.12)

$$\frac{d}{dt} \psi(t) \leq \delta \int_0^L (y_t(t))^2 dx - \delta y(L,t) \Big[y_{xxx}(L,t) - \int_0^t \kappa(t-\tau) y_{xxx}(L,\tau) d\tau \Big]$$

$$- \delta \int_0^t (y_{xx}(t))^2 dx + M_1' \|y_t(t)\|^2 + M_2' \|y_{xx}(t)\|^2 + \eta\delta \|y_{xx}(t)\|^2$$

$$+ \frac{\delta}{4\eta} \|\kappa\|_{L^\infty(0,\infty)} (\kappa \square y_{xx})(t) + \delta \int_0^t \kappa(\tau) d\tau \|y_{xx}(t)\|^2$$

$$\leq (\delta + M_1') \|y_t(t)\|^2 - \delta y(L,t) [h(t) y_t(L,t) + (\theta(t) - \widetilde{\theta}) \sin t]$$

$$+ \Big(M_2' + \eta\delta + \delta \int_0^t \kappa(\tau) d\tau - \delta \Big) \|y_{xx}(t)\|^2$$

$$+ \frac{\delta}{4\eta} \|\kappa\|_{L^\infty(0,\infty)} (\kappa \square y_{xx})(t)$$

$$= - \delta e(t) + \Big(M_1' + \frac{3}{2}\delta \Big) \|y_t(t)\|^2$$

$$+ \Big(M_2' + \eta\delta + \frac{\delta}{2} \int_0^t \kappa(\tau) d\tau - \frac{\delta}{2} \Big) \|y_{xx}(t)\|^2$$

$$+ \Big(\frac{\delta}{2} + \frac{\delta}{4\eta} \|\kappa\|_{L^\infty(0,\infty)} \Big) (\kappa \square y_{xx})(t)$$

$$- \delta y(L,t) [h(t) y_t(L,t) + (\theta(t) - \widetilde{\theta}) \sin t] + \frac{\delta}{2} [\theta(t) - \widetilde{\theta}]^2.$$

On the other hand, from (2.2), (3.5), (3.6) and (3.12), we get

(3.13) $$\frac{d}{dt} e_\varepsilon(t) = \frac{d}{dt} e(t) + \varepsilon \frac{d}{dt} \psi(t)$$

$$= - (g(y_t(t)), y_t(t)) - \frac{m_1}{2} (\kappa \square y_{xx})(t) - \frac{1}{2} \kappa(t) \|y_{xx}(t)\|^2$$

$$- h(t)(y_t(L,t))^2 - \varepsilon \delta e(t) + \varepsilon \Big(M_1' + \frac{3}{2}\delta \Big) \|y_t(t)\|^2$$

$$+ \varepsilon \Big(M_2' + \eta\delta + \frac{\delta}{2} \int_0^t \kappa(\tau) d\tau - \frac{\delta}{2} \Big) \|y_{xx}(t)\|^2$$

$$+ \varepsilon \Big(\frac{\delta}{2} + \frac{\delta}{4\eta} \|\kappa\|_{L^\infty(0,\infty)}\Big)(\kappa \square y_{xx})(t)$$

$$- \varepsilon \delta y(L,t)[h(t)y_t(L,t) + (\theta(t) - \widetilde{\theta})\sin t] + \frac{\varepsilon\delta}{2}[\theta(t) - \widetilde{\theta}]^2$$

$$\leq - \varepsilon \delta e(t) - \Big(\mu - \varepsilon\Big(M_1' + \frac{3}{2}\delta\Big)\Big)\|y_t(t)\|^2$$

$$- \Big(\frac{m_1}{2} - \varepsilon\Big(\frac{\delta}{2} + \frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)}\Big)\Big)(\kappa \square y_{xx})(t)$$

$$- \Big(\frac{1}{2}\kappa(t) - \varepsilon\Big(M_2' + \eta\delta + \frac{\delta}{2}\int_0^t \kappa(\tau)d\tau - \frac{\delta}{2}\Big)\Big)\|y_{xx}(t)\|^2$$

$$- h(t)\Big(1 - \frac{\delta\varepsilon}{2}\Big)(y_t(L,t))^2$$

$$+ \frac{\varepsilon\delta}{2}(1 + h(t))(y(L,t))^2 + \varepsilon\delta[\theta(t) - \widetilde{\theta}]^2.$$

Now, we define

(3.14)
$$\varepsilon_1 = \min\Big\{\frac{2\mu}{2M_1' + 3\delta}, \frac{2m_1\eta}{\delta(2\eta + \|\kappa\|_{L^\infty(0,\infty)})}, \frac{\kappa(t)}{2M_2' + 2\eta\delta + \delta\int_0^t \kappa(\tau)d\tau - \delta}, \frac{2}{\delta}\Big\}.$$

Considering $\varepsilon \in [0, \varepsilon_1]$ and choosing $\delta > 0$ such that $(2\eta - \ell)\delta + 2M_2' > \delta$, then from (3.13) and (3.14), we obtain

(3.15)
$$\frac{d}{dt}e_\varepsilon(t) \leq -\varepsilon\delta e(t) + \frac{\varepsilon\delta}{2}(1 + h(t))(y(L,t))^2 + \varepsilon\delta[\theta(t) - \widetilde{\theta}]^2.$$

This completes the proof of Proposition 3.2. □

Proof of Theorem 3.1. From Proposition 3.1, we see that

(3.16)
$$(1 - \varepsilon C_1')e(t) \leq e_\varepsilon(t) \leq (1 + \varepsilon C_1')e(t), \quad \forall t \geq 0.$$

From (3.15) and (3.16), we have

(3.17)
$$\frac{d}{dt}e_\varepsilon(t) \leq \frac{-\varepsilon\delta}{1 + \varepsilon C_1'}e_\varepsilon(t) + \frac{\varepsilon\delta}{2}(1 + h(t))(y(L,t))^2 + \varepsilon\delta[\theta(t) - \widetilde{\theta}]^2.$$

Let $C_\varepsilon = \varepsilon\delta/(1 + \varepsilon C_1')$ and apply Gronwall's inequality to (3.17), we get

(3.18)
$$e_\varepsilon(t) \leq e^{-C_\varepsilon t}e_\varepsilon(0) + \frac{\varepsilon\delta}{2}\int_0^t e^{-C_\varepsilon(t-\tau)}(1 + h(\tau))(y(L,\tau))^2 d\tau$$

$$+ \varepsilon\delta\int_0^t e^{-C_\varepsilon(t-\tau)}[\theta(t) - \widetilde{\theta}]^2 d\tau$$

$$\leq e^{-C_\varepsilon t}e_\varepsilon(0) + \frac{\varepsilon\delta}{2}\sup_{t \geq 0}[1 + |h(t)|]\int_0^t e^{-C_\varepsilon(t-\tau)}(y(L,\tau))^2 d\tau$$

$$+ \varepsilon\delta\int_0^t e^{-C_\varepsilon(t-\tau)}[\theta(t) - \widetilde{\theta}]^2 d\tau.$$

Referring to the paper [7], we deduce that

$$\int_0^t e^{-C_\varepsilon(t-\tau)}(y(L,\tau))^2 d\tau$$

$$\leq \int_0^{\frac{t}{2}} e^{-C_\varepsilon(t-\tau)}(y(L,\tau))^2 d\tau + \int_{\frac{t}{2}}^t e^{-C_\varepsilon(t-\tau)}(y(L,\tau))^2 d\tau$$

$$\leq \max_{0\leq\tau\leq\frac{t}{2}} e^{-C_\varepsilon(t-\tau)} \int_0^{\frac{t}{2}}(y(L,\tau))^2 d\tau + \max_{\frac{t}{2}\leq\tau\leq t} e^{-C_\varepsilon(t-\tau)} \int_{\frac{t}{2}}^t (y(L,\tau))^2 d\tau$$

$$\leq e^{-\frac{C_\varepsilon}{2}t} \int_0^{\frac{t}{2}}(y(L,\tau))^2 d\tau + \int_{\frac{t}{2}}^\infty (y(L,\tau))^2 d\tau.$$

By Theorem 2.1 and $W \hookrightarrow L^2(0,T;L^2(0,L))$, $y(L,t) \in L^2(0,\infty)$, we obtain

$$(3.19) \qquad \int_0^t e^{-C_\varepsilon(t-\tau)}(y(L,\tau))^2 d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

Similarly, we see that

$$(3.20) \qquad \int_0^t e^{-C_\varepsilon(t-\tau)}[\theta(\tau) - \widetilde{\theta}]^2 d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

From (3.18), (3.19) and (3.20), we deduce

$$(3.21) \qquad \lim_{t\to\infty} e_\varepsilon(t) = 0.$$

Let $\varepsilon_0 = \min\{\varepsilon_1, 1/(2C_1')\}$, where C_1' is given in Proposition 3.1.

Since $\varepsilon \leq 1/(2C_1')$ ($\varepsilon \in (0,\varepsilon_0]$) and from (3.16), we get

$$(3.22) \qquad \frac{1}{2}e(t) \leq e_\varepsilon(t) \leq \frac{3}{2}e(t).$$

Therefore, from (3.21) and (3.22), we have

$$(3.23) \qquad \lim_{t\to\infty} e(t) = 0.$$

Now, we consider the Lyapunov functional $U(t)$ for the system (1.3) as follows:

$$U(t) = e(t) + \frac{(h(t))^2}{2r}.$$

Then from (3.5), we see that

$$(3.24) \qquad U_t(t) \leq 0.$$

Since $y_t(L,\cdot) \in L^2(0,\infty)$, we obtain

$$\sup_{t\geq 0}\left\{e(t) + \frac{(h(t))^2}{2r}\right\} \leq M_3',$$

where $M_3' > 0$ is a constant depending on the initial data.

From $U(t)$ is decreasing, we get

$$(3.25) \qquad e(\infty) + \frac{1}{2r}(h(\infty))^2 \leq e(0) + \frac{1}{2r}(h(0))^2.$$

Thus, from (3.23) and (3.25), we have

$$h(\infty) \leq \sqrt{2re(0) + (h(0))^2}.$$

Since $h(t)$ is nondecreasing, we deduce that

(3.26) $$h(t) \leq \sqrt{2re(0) + (h(0))^2}.$$

Thus, the proof of Theorem 3.1 is completed. □

We can now proceed to state our main exponential stability result.

Theorem 3.2. *Let y be the solution of Theorem 2.1 and let $e(t)$ be defined by (3.4). Then there exist constants $K > 0$ and $\nu > 0$ depending on the initial data such that*

$$E(t) \leq Ke^{-\nu t}, \quad \forall t \geq 0.$$

Proof. From (3.17), we have

(3.27) $$\frac{d}{dt}e_\varepsilon(t) \leq -C_\varepsilon e_\varepsilon(t) + \frac{\varepsilon\delta}{2}(1 + h(t))(y(L,t))^2 + \varepsilon\delta[\theta(t) - \widetilde{\theta}]^2,$$

where $C_\varepsilon = \delta\varepsilon/(1 + \varepsilon C_1')$.

Using (3.26) and by integrating over $(0, t)$ in (3.27) we obtain

$$e_\varepsilon(t) \leq e_\varepsilon(0) - C_\varepsilon \int_0^t e_\varepsilon(\tau)d\tau + \frac{\varepsilon\delta}{2}\int_0^t (1 + h(\tau))(y(L,\tau))^2 d\tau$$

$$+ \varepsilon\delta \int_0^t [\theta(t) - \widetilde{\theta}]^2 d\tau$$

$$\leq e_\varepsilon(0) - C_\varepsilon \int_0^t e_\varepsilon(\tau)d\tau + \frac{\varepsilon\delta}{2}(1 + \|h\|_{L^\infty(0,\infty)})\int_0^t (y(L,\tau))^2 d\tau$$

$$+ \varepsilon\delta \int_0^t [\theta(t) - \widetilde{\theta}]^2 d\tau.$$

Since $\int_0^\infty (y(L,\tau))^2 d\tau$ and $\int_0^\infty [\theta(t) - \widetilde{\theta}]^2 d\tau$ are bounded, by Gronwall's inequality, we deduce that

$$e_\varepsilon(t) \leq (K_1 + e_\varepsilon(0))\exp(-C_\varepsilon t),$$

where K_1 is some positive constant.

For sufficiently small $\varepsilon\left(0 < \varepsilon < \frac{1}{2C_1'}\right)$, using Proposition 3.1, we get

$$E(t) \leq \ell^{-1}e(t) \leq Ke^{-\nu t},$$

where $K = 2\ell^{-1}(K_1 + e_\varepsilon(0))$ and $\nu = C_\varepsilon$.

The proof of Theorem 3.2 is completed. □

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