J. Korean Math. Soc. **49** (2012), No. 5, pp. 907–925
http://dx.doi.org/10.4134/JKMS.2012.49.5.907

SOME MULTI-SUBLINEAR OPERATORS ON GENERALIZED MORREY SPACES WITH NON-DOUBLING MEASURES

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Abstract. In this paper the boundedness for a large class of multi-sublinear operators is established on product generalized Morrey spaces with non-doubling measures. As special cases, the corresponding results for multilinear Calderón-Zygmund operators, multilinear fractional integrals and multi-sublinear maximal operators will be obtained.

1. Introduction

A Radon measure μ on $\mathbb{R}^d$ is called a doubling measure if it satisfies the doubling condition, i.e., there is a constant $C > 0$ such that $\mu(2Q) \leq C\mu(Q)$ for any cube $Q \subset \mathbb{R}^d$. This doubling condition on μ seems to be a key assumption in classical Fourier analysis, however, it has been shown recently that many results also hold without the doubling assumption, see [4, 8, 9, 14, 15] among other literatures. In this paper we will investigate a large class of multi-sublinear operators, including multilinear Calderón-Zygmund operators, multilinear fractional integrals and multi-sublinear maximal operators, on product generalized Morrey spaces with non-doubling measures.

In fact, for $m \in \mathbb{N}$ and m-tuple $(f_1, f_2, \ldots, f_m)$, we consider the multi-sublinear operators $\mathcal{T}$ satisfying the following size condition,

(1.1)
$$|\mathcal{T}(f_1, \ldots, f_m)(x)| \leq C \int_{(\mathbb{R}^d)^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(|x - y_1| + \cdots + |x - y_m|)^{mn-\alpha}} \prod_{i=1}^{m} d\mu(y_i)$$

Received September 9, 2010; Revised December 26, 2011.

2010 *Mathematics Subject Classification.* 42B20, 42B25, 42B35.

Key words and phrases. multilinear Calderón-Zygmund operator, multilinear fractional integral, multi-sublinear maximal function, generalized Morrey spaces, non-doubling measure.

The research of the first author was supported by the Foundation of Zhejiang Pharmaceutical College under grant #ZPCSR2010013.

The research of the second author was partially supported by the National Nature Science Foundation of China under grant #11171306 and #11071065, and sponsored by the Scientific Project of Zhejiang Provincial Science Technology Department under grant #2011C33012 and the Scientific Research Fund of Zhejiang Provincial Education Department under grant #Z201017584.

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with $0 \le \alpha < mn$, where the non-doubling measure μ satisfies the growth condition

$$\mu(Q) \le C_0[\ell(Q)]^n, \quad 0 < n \le d, \tag{1.2}$$

for any cube $Q \subset \mathbb{R}^d$ with its side length $\ell(Q) > 0$.

Recall the multilinear fractional integral, for $0 < \alpha < mn$,

$$I_{\alpha,m}(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}} \prod_{i=1}^m d\mu(y_i),$$

and the m-linear Calderón-Zygmund operator defined by

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^d)^m} \mathcal{K}(x, y_1, \dots, y_m) f_1(y_1)\cdots f_m(y_m) \prod_{i=1}^m d\mu(y_i)$$

for functions f_i with compact support and $x \notin \bigcap_{i=1}^m \operatorname{supp} f_i$, where $\mathcal{K}$ is an m-Calderón-Zygmund kernel defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^d)^{m+1}$ satisfying the size condition

$$|\mathcal{K}(x, y_1, y_2, \dots, y_m)| \le \frac{C}{(\sum_{l=0}^m |x-y_l|)^{mn}} \tag{1.3}$$

and some smoothness condition, see [1, 15] for details.

When μ is a doubling measure, i.e., $n = d$, Grafakos and Torres showed in [1] that the m-linear Calderón-Zygmund operator T is bounded from $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ into $L^p(\mu)$ with $0 < 1/p = 1/p_1 + \cdots + 1/p_m < 1$; Kenig and Stein showed in [2] that $I_{\alpha,m}$ is a bounded operator from $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ into $L^p(\mu)$ with $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$ and $p_1, \dots, p_m > 1$. These results had been extended to Herz spaces, Morrey spaces and generalized Morrey spaces in [5, 6, 10, 11, 13], and also extended in [4] and [15] to Lebesgue spaces with non-doubling measures.

On the other hand, Sawano and Tanaka [9] had studied the Calderón-Zygmund operator and the fractional integral on Morrey spaces with non-doubling measure. One purpose of this paper is to establish the boundedness of the m-linear Calderón-Zygmund operator and the multilinear fractional integral on Morrey spaces with non-doubling measure.

Let us give some notations. The capital letter Q always denotes a cube with sides parallel to the coordinate axes and $\ell(Q)$ stands for the side length of Q. Besides, for $c > 0$, cQ will mean the cube with the same center as Q and with $\ell(cQ) = c\ell(Q)$, $Q(x, r)$ will be the cube centered at x with side length r and we denote by $\mathscr{Q}(\mu)$ the set of all doubling cubes with positive μ-measure.

Definition 1.1 ([9])**.** For $k > 1$ and $1 \le p \le q < \infty$, the Morrey space $\mathscr{M}_p^q(k, \mu)$ is defined as

$$\mathscr{M}_p^q(k, \mu) := \left\{ f \in L^p_{loc}(\mu) : \|f\|_{\mathscr{M}_p^q(k,\mu)} < \infty \right\},$$

where the Morrey norm $\| f \|_{\mathscr{M}_p^q(k,\mu)}$ is given by

$$\|f\|_{\mathscr{M}_p^q(k,\mu)} = \sup_{Q \in \mathscr{Q}(\mu)} \mu(kQ)^{\frac{1}{q}-\frac{1}{p}} \left(\int_Q | f |^p \, d\mu \right)^{\frac{1}{p}}.$$

It's worthy to point out that the parameter $k > 1$ does not affect the set of $\mathscr{M}_p^q(k,\mu)$. More precisely, for $k_1 > k_2 > 1$, $\mathscr{M}_p^q(k_1,\mu)$ and $\mathscr{M}_p^q(k_2,\mu)$ coincide as a set and their norms are mutually equivalent. See [9] for details. We will denote $\mathscr{M}_p^q(2,\mu)$ by $\mathscr{M}_p^q(\mu)$.

Definition 1.2 ([8])**.** For $k > 1$ and $1 \leq p < \infty$, and let ϕ be an increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, then the generalized Morrey space, $\mathcal{L}^{p,\phi}(k,\mu)$, is defined as

$$\mathcal{L}^{p,\phi}(k,\mu) := \left\{ f \in L^p_{loc}(\mu) : \|f\|_{\mathcal{L}^{p,\phi}(k,\mu)} < \infty \right\}$$

with the norm $\| f \|_{\mathcal{L}^{p,\phi}(k,\mu)}$ that is given by

$$\|f\|_{\mathcal{L}^{p,\phi}(k,\mu)} = \sup_{Q \in \mathscr{Q}(\mu)} \left(\frac{1}{\phi(\mu(kQ))} \int_Q |f|^p d\mu \right)^{1/p}.$$

We remark that $\mathcal{L}^{p,\phi}(k,\mu) = \mathscr{M}_p^q(k,\mu)$ for $\phi(t) = t^{1-p/q}$ and $1 \leq p \leq q < \infty$. Thus, the generalized Morrey spaces is a generalization of Morrey spaces. Similarly, it's proved in [8] that, for $k_1 > k_2 > 1$, $\mathcal{L}^{p,\phi}(k_1,\mu)$ and $\mathcal{L}^{p,\phi}(k_2,\mu)$ coincide as a set and their norms are mutually equivalent. Thus we can write $\mathcal{L}^{p,\phi}(\mu)$ instead of $\mathcal{L}^{p,\phi}(2,\mu)$ for simplicity. In 2008 Sawano [8] had shown the boundedness of the Calderón-Zygmund operator and the fractional integral on $\mathcal{L}^{p,\phi}(\mu)$.

It's interesting for us to ask whether the m-linear Calderón-Zygmund operator T and the multilinear fractional integral $I_{\alpha,m}$ are bounded on product generalized Morrey spaces with non-doubling measure.

In this paper, we will discuss more general multi-sublinear operators $\mathcal{T}$ satisfying the condition (1.1), and prove the boundedness of $\mathcal{T}$ on product Morrey spaces and generalized Morrey spaces with non-doubling measure, see Theorems 2.3, 2.6, 3.1 and 3.2 below.

Particularly, we will give the following estimates for the m-linear Calderón-Zygmund operator T, see Theorems 2.4 and 3.3,

$$\|T(f_1, \ldots, f_m)\|_{\mathscr{M}_p^q(\mu)} \leq C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)}$$

and

$$\|T(f_1, \ldots, f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \leq C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}$$

for $1/p = 1/p_1 + \cdots + 1/p_m$, $1/q = 1/q_1 + \cdots + 1/q_m$ and $\phi^{1/p} = \prod_{i=1}^m \phi_i^{1/p_i}$ under the assumptions

$$\sup_{0<r<\infty} \frac{r}{\phi_i(r)} \int_r^\infty \frac{\phi_i(t)}{t} \frac{dt}{t} < \infty \tag{1.4}$$

and

$$\frac{\phi_i(t)}{t} \le C \frac{\phi_i(s)}{s} \qquad \text{for} \quad t \ge s \tag{1.5}$$

for each $i = 1, 2, \ldots, m$.

Simultaneously, we will show the following estimates for the multilinear fractional integral $I_{\alpha,m}$, see Theorems 2.7, 3.4 and 3.6,

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{\mathscr{M}_p^q(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)} \tag{1.6}$$

and

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}$$

for $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n$ and $\phi^{1/p} = \prod_{i=1}^m \phi_i^{1/p_i}$, where ϕ_i satisfies (1.5) and

$$\sup_{0<r<\infty} \frac{r^{1-\alpha p_i/mn}}{\phi_i(r)} \int_r^\infty \frac{\phi_i(t)}{t^{1-\alpha p_i/mn}} \frac{dt}{t} < \infty, \quad i = 1, 2, \ldots, m. \tag{1.7}$$

In the paper we will also consider the generalization of Hardy-Littlewood maximal operator, which is an important example of multi-sublinear operators. In 1994 Nakai [7] introduced the similar assumptions on ϕ to show the boundedness of maximal operator in generalized Morrey space $L^{p,\phi}(\mathbb{R}^n)$. In 2008 Sawano [8] studied modified maximal operator under similar assumptions on ϕ in the case of non-doubling measure. Inspired by these works, we will also consider the modified multi-sublinear maximal operator $\mathcal{M}_\kappa$, which defined by

$$\mathcal{M}_\kappa(f_1, \ldots, f_m)(x) = \sup_{x \in Q \in \mathscr{Q}(\mu)} \prod_{i=1}^m \frac{1}{\mu(\kappa Q)} \int_Q |f_i(y_i)| d\mu(y_i), \quad \kappa > 1.$$

This operator was defined in [3] when μ is a Lebesgue measure. It is easy to see that $\mathcal{M}_\kappa$ is a multi-version of the maximal operator M_κ introduced by Tolsa [14], and it's worthy to notice that $\mathcal{M}_\kappa$ is strictly smaller than the m-fold produce of M_κ. We will establish the boundedness of $\mathcal{M}_\kappa$ on product generalized Morrey spaces, i.e., Theorems 2.9 and 3.5 below, which implies that

$$\|\mathcal{M}_\kappa(f_1, \ldots, f_m)\|_{\mathscr{M}_p^q(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)}$$

and

$$\|\mathcal{M}_\kappa(f_1, \ldots, f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}$$

for $1/p = 1/p_1 + \cdots + 1/p_m$, $1/q = 1/q_1 + \cdots + 1/q_m$ and $\phi^{1/p} = \prod_{i=1}^m \phi_i^{1/p_i}$, where each ϕ_i satisfies (1.5). Here we point out that we do not postulate anything on measure μ when we study the operator $\mathcal{M}_\kappa$, that is, here μ is just a Radon measure.

The paper is organized as follows. The Section 2 focuses on the boundedness on generalized Morrey spaces $L^{p,\phi}(\mu)$. In Section 3, we devote to the boundedness on Morrey spaces $\mathscr{M}_p^q(\mu)$. Throughout this paper, the letter C always remains to denote a positive constant that may varies at each occurrence but is independent of the essential variable.

2. Boundedness on generalized Morrey spaces $L^{p,\phi}(\mu)$

The present section consists of three parts which are about the bounded estimates on generalized Morrey spaces for the multilinear Calderón-Zygmund operator, the multilinear fractional integral and the multi-sublinear maximal operator, respectively.

2.1. Boundedness of multilinear Calderón-Zygmund operators

Let us begin with some requisite lemmas. Using integration by part, see Lemma 2 in [7] or Lemma 5.3 in [11], we have that:

Lemma 2.1 ([7, 11])**.** *Suppose that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function satisfying*

$$\int_r^\infty \psi(s)\frac{ds}{s} \le C'\psi(r) \quad \text{for all } r > 0$$

with the positive constant C'. Then we can take the constant $C = C'/(1-C'\varepsilon)$ for small real $\varepsilon > 0$ such that

$$\int_r^\infty \psi(s)s^\varepsilon\frac{ds}{s} \le C\psi(r)r^\varepsilon \quad \text{for all } r > 0.$$

Lemma 2.2. *Suppose that ψ is the same as in Lemma 2.1 and $0 < \eta \le 1$. Then there is a constant $C > 0$ so that*

$$\int_r^\infty \psi(s)^\eta\frac{ds}{s} \le C\psi(r)^\eta \quad \text{for all } r > 0.$$

Proof. For any $R > 0$ and $0 < \eta \le 1$, the Hölder inequality and Lemma 2.1 imply that

$$\begin{aligned}\int_r^R \psi(s)^\eta\frac{ds}{s} &= \int_r^R [\psi(s)^\eta s]\frac{1}{s}\frac{ds}{s}\\ &\le C\left(\int_r^R \psi(s)s^{\frac{1}{\eta}}\frac{ds}{s}\right)^\eta \left(\int_r^R \frac{1}{s^{\frac{1}{1-\eta}+1}}ds\right)^{1-\eta}\\ &\le C\psi(r)^\eta r\left(\int_r^R s^{-\frac{1}{1-\eta}-1}ds\right)^{1-\eta}\end{aligned}$$

$$\leq C\psi(r)^\eta r\left(-\frac{1}{1-\eta}R^{-\frac{1}{1-\eta}}+\frac{1}{1-\eta}r^{-\frac{1}{1-\eta}}\right)^{1-\eta}.$$

Then, let $R\to\infty$, we get the conclusion of the lemma. □

Let us state our main result of this subsection.

Theorem 2.3. *Assume that $\mathcal{T}$ is a multi-sublinear operator satisfying* (1.1) *with $\alpha=0$. Let $1<p_i<\infty$ satisfy $1/p=1/p_1+\cdots+1/p_m<1$,*

$$\phi^{1/p}=\phi_1^{1/p_1}\phi_2^{1/p_2}\cdots\phi_m^{1/p_m},$$

where each function $\phi_i:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the assumptions (1.4) *and* (1.5). *If $\mathcal{T}$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^p(\mu)$, then there is a constant $C>0$ independent of f_i such that*

$$\|\mathcal{T}(f_1,\dots,f_m)\|_{\mathcal{L}^{p,\phi}(\mu)}\leq C\prod_{i=1}^m\|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.$$

Proof. Fix a cube $Q\in\mathscr{Q}(\mu)$ and write $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{2Q}$ and $f_i^\infty=f_i-f_i^0$ for $i=1,\dots,m$. Then along this decomposition, we get

$$\begin{aligned}|\mathcal{T}(f_1,\dots,f_m)(x)|&\leq\sum_{\tau_1,\dots,\tau_m\in\{0,\infty\}}|\mathcal{T}(f_1^{\tau_1},\dots,f_m^{\tau_m})(x)|\\&=|\mathcal{T}\left(f_1^0,\dots,f_m^0\right)(x)|+\sum{}'|\mathcal{T}(f_1^{\tau_1},\dots,f_m^{\tau_m})(x)|,\end{aligned}$$

where each term in $\sum'$ contains at least one $\tau_i\neq 0$. This implies

$$\left(\frac{1}{\phi(\mu(4Q))}\int_Q|\mathcal{T}(f_1,\dots,f_m)(x)|^p d\mu(x)\right)^{1/p}\leq C(H_1+H_2),$$

where

$$H_1:=\left(\frac{1}{\phi(\mu(4Q))}\int_Q|\mathcal{T}\left(f_1^0,\dots,f_m^0\right)(x)|^p d\mu(x)\right)^{1/p}$$

and

$$H_2:=\sum{}'\left(\frac{1}{\phi(\mu(4Q))}\int_Q|\mathcal{T}(f_1^{\tau_1},\dots,f_m^{\tau_m})(x)|^p d\mu(x)\right)^{1/p}.$$

Next we analyze each term separately. For H_1, by the L^p boundedness of $\mathcal{T}$, it is easy to see that

$$H_1\leq C\prod_{i=1}^m\frac{\|f_i\chi_{2Q}\|_{L^{p_i}}}{\phi_i^{1/p_i}(\mu(4Q))}\leq C\prod_{i=1}^m\|f_i\|_{L^{p_i,\phi_i}(\mu)}.$$

For H_2, let us begin with the case $\tau_1=\cdots=\tau_m=\infty$. Denoting by c_Q the center of Q, we have $|y_i-x|\sim|y_i-c_Q|$ for $x\in Q$ and $y_i\in(2Q)^c$. Hence, by the condition (1.1), we obtain the following point-wise estimate for $x\in Q$,

$$|\mathcal{T}(f_1^\infty,\dots,f_m^\infty)(x)|$$

$$\leq C \int\limits_{(\mathbb{R}^d \setminus B(c_Q,\ell(Q)))^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{(|y_1-c_Q|+\cdots+|y_m-c_Q|)^{mn}} d\mu(y_1)\cdots d\mu(y_m).$$

Then, the following equality

$$mn\int_{|y_1-c_Q|+\cdots+|y_m-c_Q|}^{\infty} \frac{dl}{l^{mn+1}} = \frac{1}{(|y_1-c_Q|+\cdots+|y_m-c_Q|)^{mn}}$$

and Fubini theorem yield

$$\begin{aligned}
&|\mathcal{T}(f_1^\infty,\ldots,f_m^\infty)(x)| \\
&\leq C\int_{\ell(Q)}^{\infty}\frac{1}{l^{mn+1}}\int_{\Sigma_{i=1}^m|y_i-c_Q|<l}|f_1(y_1)\cdots f_m(y_m)|d\mu(y_1)\cdots d\mu(y_m)dl \\
&\leq C\int_{\ell(Q)}^{\infty}\frac{1}{l^{mn+1}}\left(\int_{B(c_Q,l)}\cdots\int_{B(c_Q,l)}|f_1(y_1)\cdots f_m(y_m)|d\mu(y_1)\cdots d\mu(y_m)\right)dl.
\end{aligned}$$

By using the Hölder inequality and the growth condition (1.2) of μ, we get

$$\begin{aligned}
&|\mathcal{T}(f_1^\infty,\ldots,f_m^\infty)(x)| \\
&\leq C\int_{\ell(Q)}^{\infty}\frac{1}{l^{mn+1}}\prod_{i=1}^m\left(\mu(B(c_Q,l))^{1-1/p_i}\left[\int_{B(c_Q,l)}|f_i(y_i)|^{p_i}d\mu(y_i)\right]^{1/p_i}\right)dl \\
&\leq C\int_{\ell(Q)}^{\infty}\frac{1}{l^{n/p+1}}\left(\prod_{i=1}^m\phi_i^{1/p_i}(\mu(B(c_Q,2l)))\|f_i\|_{L^{p_i,\phi_i}(\mu)}\right)dl \\
&\leq C\left(\int_{\ell(Q)}^{\infty}\frac{\phi_1(C_0 2^n l^n)^{1/p_1}\cdots\phi_m(C_0 2^n l^n)^{1/p_m}}{l^{n/p}}\frac{dl}{l}\right)\prod_{i=1}^m\|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&= C\left(\int_{\ell(Q)}^{\infty}\frac{\phi_1(C_0 2^n l^n)^{1/p_1}}{l^{n/p_1}}\times\cdots\times\frac{\phi_m(C_0 2^n l^n)^{1/p_m}}{l^{n/p_m}}\frac{dl}{l}\right)\prod_{i=1}^m\|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&\leq C\prod_{i=1}^m\left[\int_{\ell(Q)}^{\infty}\left(\frac{\phi_i(C_0 2^n l^n)}{l^n}\right)^{1/p}\frac{dl}{l}\right]^{p/p_i}\|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&\leq C\prod_{i=1}^m\left(\frac{\phi_i(C_0 2^n\ell(Q)^n)}{\ell(Q)^n}\right)^{1/p_i}\|f_i\|_{L^{p_i,\phi_i}(\mu)}.
\end{aligned}$$

Here, we used Lemma 2.2 by taking $\psi(l) = \frac{\phi_i(C_0 2^n l^n)}{C_0 2^n l^n}$ and $\eta = 1/p < 1$ in the last relation.

Hence, applying the assumption (1.5), we obtain

$$|\mathcal{T}(f_1^\infty,\ldots,f_m^\infty)(x)| \leq C\prod_{i=1}^m\left(\frac{\phi_i(C_0 2^n d^n\ell(Q)^n)}{\ell(Q)^n}\right)^{1/p_i}\|f_i\|_{L^{p_i,\phi_i}(\mu)}$$

$$\leq C \prod_{i=1}^{m} \left(\frac{\phi_i(\mu(2dQ))}{\mu(2dQ)} \right)^{1/p_i} \|f_i\|_{L^{p_i,\phi_i}(\mu)}.$$

Integrating this over Q, by the assumption (1.5), we obtain

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(4Q))} \int_Q |T\left(f_1^\infty, \ldots, f_m^\infty\right)(x)|^p d\mu(x) \right)^{1/p} \\
&\leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\phi_i}(\mu)} \left(\frac{\mu(Q)}{\phi(\mu(4Q))} \times \frac{\phi_i(\mu(2dQ))}{\mu(2dQ))} \right)^{1/p_i} \\
&\leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\phi_i}(\mu)}.
\end{aligned}$$

What remains to be considered are the terms in H_2 such that $\tau_{i_1} = \cdots = \tau_{i_\ell} = 0$ for some $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, m\}$, where $1 \leq \ell < m$. Applying partly the technique used in the estimates for $|\mathcal{T}\left(f_1^\infty, \ldots, f_m^\infty\right)(x)|$ and the Hölder inequality, we get

$$\begin{aligned}
&|\mathcal{T}\left(f_1^{\tau_1}, \ldots, f_m^{\tau_1}\right)(x)| \\
&\leq C \int\limits_{(2Q)^\ell} \prod_{i\in\{i_1,\ldots,i_\ell\}} |f_i(y_i)| d\mu(y_i) \times \int\limits_{(\mathbb{R}^d\backslash 2Q)^{m-\ell}} \frac{\prod_{i\notin\{i_1,\ldots,i_\ell\}} |f_i(y_i))| d\mu(y_i)}{(\sum_{i\notin\{i_1,\ldots,i_\ell\}} |y_i - c_Q|)^{mn}} \\
&\leq C \prod_{i\in\{i_1,\ldots,i_\ell\}} \left(\frac{\phi_i(\mu(4Q))}{\mu(4Q)} \right)^{1/p_i} \mu(4Q) \|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&\quad \times \int_{\ell(Q)}^{\infty} \frac{1}{l^{\sum_{i\notin\{i_1,\ldots,i_\ell\}} n/p_i + \ell n + 1}} \left(\prod_{i\notin\{i_1,\ldots,i_\ell\}} \phi_i^{1/p_i}(\mu(B(c_Q, 2l)) \|f_i\|_{L^{p_i,\phi_i}(\mu)} \right) dl \\
&\leq C \prod_{i\in\{i_1,\ldots,i_\ell\}} \left(\frac{\phi_i(\mu(4Q))}{\mu(4Q)} \right)^{1/p_i} \|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&\quad \times \int_{\ell(Q)}^{\infty} \frac{1}{l^{\sum_{i\notin\{i_1,\ldots,i_\ell\}} n/p_i + 1}} \left(\prod_{i\notin\{i_1,\ldots,i_\ell\}} \phi_i^{1/p_i}(\mu(B(c_Q, 2l)) \|f_i\|_{L^{p_i,\phi_i}(\mu)} \right) dl \\
&\leq C \prod_{i\in\{i_1,\ldots,i_l\}} \frac{\phi_i(\mu(4Q))^{1/p_i}}{\mu(4Q)^{1/p_i}} \|f_i\|_{L^{p_i,\phi_i}(\mu)} \times \prod_{i\notin\{i_1,\ldots,i_\ell\}} \frac{\phi_i(\mu(2dQ))^{1/p_i}}{\mu(2dQ)^{1/p_i}} \|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&\leq C \prod_{i=1}^{m} \frac{\phi_i(\mu(4Q))^{1/p_i}}{\mu(4Q)^{1/p_i}} \|f_i\|_{L^{p_i,\phi_i}(\mu)} \\
&= C \frac{\phi(\mu(4Q))^{1/p}}{\mu(4Q)^{1/p}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\phi_i}(\mu)}.
\end{aligned}$$

Then we obtain that

$$\left(\frac{1}{\phi(\mu(4Q))}\int_Q |\mathcal{T}\left(f_1^{\tau_1},\ldots,f_m^{\tau_1}\right)(x)|^p d\mu(x)\right)^{1/p} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(\mu)},$$

and so we have arrived at the expression considered in the previous case. Combining the arguments above, we complete the proof of the theorem. □

It is easy to see from (1.3) that the m-linear Calderón-Zygmund operator T satisfy (1.1). Thus, by its L^p boundedness, we get the following theorem immediately.

Theorem 2.4. *Let ϕ,ϕ_i,p,p_i be the same as in Theorem 2.3 for each $i=1,2,\ldots,m$. Then there is a constant $C>0$ independent of f_i such that*

$$\|T(f_1,\ldots,f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C\prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.$$

2.2. Boundedness of multilinear fractional integrals operators

To prove our theorem we need a covering lemma.

Lemma 2.5 ([8]). *Let $b>a>0$ be fixed positive numbers and $\rho>1$. Suppose that μ is a Randon measure and $\mathcal{Q}_{a,b}=\{Q\in\mathcal{Q}(\mu): a\le\mu(\rho^2Q)\le b\}\neq\emptyset$. Then there exist N subfamilies $\mathcal{Q}(\mu)_{a,b,1},\ldots,\mathcal{Q}(\mu)_{a,b,N}$ such that*

$$\{\rho Q : Q\in\mathcal{Q}(\mu)_{a,b,j}\} \quad \text{is disjoint for all } j=1,\ldots,N$$

and for all $Q\in\mathcal{Q}(\mu)_{a,b}$ we can find $Q'\in\cup_{j=1}^N\mathcal{Q}(\mu)_{a,b,j}$ such that $Q\subset\rho Q'$. Here N does not depend on a nor b.

Let us begin with our main conclusion in this part.

Theorem 2.6. *Assume that $0<\alpha<mn$ and $\mathcal{T}$ is a multi-sublinear operator satisfying* (1.1). *Let $1<p_i<mn/\alpha$ satisfy $1/p=1/p_1+\cdots+1/p_m-\alpha/n$,*

$$\phi^{1/p}=\phi_1^{1/p_1}\phi_2^{1/p_2}\cdots\phi_m^{1/p_m},$$

where each function $\phi_i:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the conditions (1.5) *and* (1.7). *If $\mathcal{T}$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^p(\mu)$, then there is a constant $C>0$ independent of f_i such that*

$$\|\mathcal{T}(f_1,\ldots,f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C\prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.$$

Proof. Fix a cube $Q\in\mathcal{Q}(\mu)$ as before and $\rho>1$, decompose f_i according to $2KQ$. Namely, we split $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{2KQ}$ and $f_i^\infty=f_i-f_i^0$ for $i=1,\ldots,m$, where K is large enough according to the value of ρ. Along this decomposition, we get

$$|\mathcal{T}(f_1,\ldots,f_m)(x)|\le|\mathcal{T}\left(f_1^0,\ldots,f_m^0\right)(x)|+{\sum}'|\mathcal{T}\left(f_1^{\tau_1},\ldots,f_m^{\tau_m}\right)(x)|,$$

where each term in $\sum'$ contains at least one $\tau_i\neq 0$.

Then we need show

$$\left(\frac{1}{\phi(\mu(4KQ))}\int_Q |\mathcal{T}(f_1^{\tau_1},\dots,f_m^{\tau_m})(x)|^p d\mu(x)\right)^{1/p} \le C\prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}$$

for $\{\tau_1,\dots,\tau_m\}\in\{0,\infty\}$.

The estimate for the case $\tau_1=\cdots=\tau_m=0$ is simple. By the L^p boundedness of $\mathcal{T}$, we have

$$\begin{aligned}&\left(\frac{1}{\phi(\mu(4KQ))}\int_Q |\mathcal{T}(f_1^{0},\dots,f_m^{0})(x)|^p d\mu(x)\right)^{1/p}\\ &\le C\prod_{i=1}^m\left(\frac{1}{\phi_i(\mu(4KQ))}\int_{2KQ}|f_i(x)|^{p_i}d\mu(x)\right)^{1/p_i}\\ &\le C\prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.\end{aligned}$$

We now turn our attention to the estimate for $\tau_1=\cdots=\tau_m=\infty$. For $x\in Q$ and $y_i\in(\mathbb{R}^d\setminus 2KQ)$, assume that $x,y_i\in R\in\mathscr{Q}(\mu)$, then we can find a constant $L>K$ which is large enough such that $\rho R\subset LQ$ for $\rho>1$, these facts imply $L\ell(Q)\ge\rho\ell(R)>\ell(R)\ge|x-y_i|\ge(K-1/2)\ell(Q)$. Then, using the growth condition (1.2) of measure μ, we have a pointwise estimate

$$\begin{aligned}&|\mathcal{T}(f_1^\infty,\dots,f_m^\infty)(x)|\\ &\le C\int\limits_{(\mathbb{R}^d\setminus 2KQ)^m}\left(\sup_{x,y_i\in R\in\mathscr{Q}(\mu)}\prod_{i=1}^m\frac{1}{\mu(\rho R)^{1-\alpha/mn}}\right)\prod_{i=1}^m|f_i(y_i)|d\mu(y_i)\\ &\le C\prod_{i=1}^m\int\limits_{\mathbb{R}^d\setminus 2KQ}\sup_{x,y_i\in R\in\mathscr{Q}(\mu)}\frac{1}{\mu(\rho R)^{1-\alpha/mn}}|f_i(y_i)|d\mu(y_i).\end{aligned}$$

A geometric observation again shows that $\sqrt{\rho}R$ engulfs $2Q$, the center of Q in its interior, provided K is large enough and R intersects both Q and $\mathbb{R}^d\setminus 2KQ$. Therefore, we get

$$|\mathcal{T}(f_1^\infty,\dots,f_m^\infty)(x)|\le C\prod_{i=1}^m\int\limits_{\mathbb{R}^d\setminus 2KQ}\sup_{\substack{R\in\mathscr{Q}(\mu)\\ \{y_i\}\bigcup 2Q\subset R}}\frac{1}{\mu(\sqrt{\rho}R)^{1-\alpha/mn}}|f_i(y_i)|d\mu(y_i).$$

Let us set, for each $i=1,2,\dots,m$ and $j\ge1$,

$$\mathcal{D}_{ij}:=\left\{y_i\in\mathbb{R}^d\setminus 2KQ: 2^{j-1}\mu(2Q)\le\inf_{\substack{R\in\mathscr{Q}(\mu)\\ \{y_i\}\bigcup 2Q\subset \mathrm{Int}(R)}}\mu(\sqrt{\rho}R)\le 2^j\mu(2Q)\right\},$$

where $\mathrm{Int}(R)$ denotes the interior of a set R. Applying Lemma 2.5, we can find $N\in\mathbb{N}$ depending only on ρ and d, and a collection of cubes $Q_1^{ij},Q_2^{ij},\dots,Q_N^{ij}$ which contain x such that $\mathcal{D}_{ij}\subset\sqrt{\rho}Q_1^{ij}\cup\sqrt{\rho}Q_2^{ij}\cup\cdots\cup\sqrt{\rho}Q_N^{ij}$ and $\mu(\sqrt{\rho}Q_l^{ij})\le$

$2^j\mu(2Q)$ for all $1 \le l \le N$, $j \in \mathbb{N}$ and $i \in \{1,\dots,m\}$. Thus, by the Hölder inequality, we obtain

$$\begin{aligned}
&|\mathcal{T}(f_1^\infty,\dots,f_m^\infty)(x)| \\
&\le C\prod_{i=1}^m\left(\sum_{j=1}^\infty \frac{1}{(2^j\mu(2Q))^{1-\alpha/mn}}\int_{\mathcal{D}_{ij}}|f_i(y_i)|d\mu(y_i)\right) \\
&\le C\prod_{i=1}^m\left(\sum_{j=1}^\infty\sum_{l=1}^N \frac{1}{(2^j\mu(2Q))^{1-\alpha/mn}}\int_{\sqrt{\rho}Q_l^{ij}}|f_i(y_i)|d\mu(y_i)\right) \\
&\le C\prod_{i=1}^m\left(\sum_{j=1}^\infty\sum_{l=1}^N \frac{1}{(2^j\mu(2Q))^{1/p_i-\alpha/mn}}\left(\int_{\sqrt{\rho}Q_l^{ij}}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{1/p_i}\right) \\
&\le C\prod_{i=1}^m\left(\sum_{j=1}^\infty\sum_{l=1}^N \frac{\phi_i(\mu(2\sqrt{\rho}Q_l^{ij}))^{1/p_i}}{(2^j\mu(2Q))^{1/p_i-\alpha/mn}}\|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}\right) \\
&\le C\prod_{i=1}^m\left(\sum_{j=1}^\infty\sum_{l=1}^N \frac{\phi_i(2^j\mu(2Q))^{1/p_i}}{(2^j\mu(2Q))^{1/p_i-\alpha/mn}}\|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}\right) \\
&\le C\prod_{i=1}^m\|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}\sum_{j=1}^\infty \frac{\phi_i(2^j\mu(2Q))^{1/p_i}}{(2^j\mu(2Q))^{1/p_i-\alpha/mn}}
\end{aligned}$$

for all $x \in Q$.

By the assumption (1.7) and Lemma 2.2, we have

$$\sum_{j=1}^\infty \frac{\phi_i(2^j\mu(2Q))^{1/p_i}}{(2^j\mu(2Q))^{1/p_i-\alpha/mn}} \le C\left(\frac{\phi_i(\mu(2Q))}{(\mu(2Q))^{1-\alpha p_i/mn}}\right)^{1/p_i}.$$

Thus, we obtain

$$\left(\frac{1}{\phi(\mu(4KQ))}\int_Q|\mathcal{T}(f_1^\infty,\dots,f_m^\infty)(x)|^p d\mu(x)\right)^{1/p} \le C\prod_{i=1}^m\|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.$$

At last, we have to consider the case $\tau_{i_1} = \cdots = \tau_{i_\ell} = 0$ for some $\{i_1,\dots,i_\ell\} \subset \{1,\dots,m\}$, where $1 \le \ell < m$. Applying partly the technique used in the estimate of $|\mathcal{T}(f_1^\infty,\dots,f_m^\infty)(x)|$ and the Hölder inequality, we get

$$\begin{aligned}
&|\mathcal{T}(f_1^{\tau_1},\dots,f_m^{\tau_1})(x)| \\
&\le C\int\limits_{(2KQ)^\ell}\prod_{i\in\{i_1,\dots,i_\ell\}}|f_i(y_i)|d\mu(y_i)\times\int\limits_{(\mathbb{R}^d\backslash 2KQ)^{m-\ell}}\frac{\prod_{i\notin\{i_1,\dots,i_\ell\}}|f_i(y_i))|d\mu(y_i)}{(\sum_{i\notin\{i_1,\dots,i_\ell\}}|x-y_i|)^{mn-\alpha}} \\
&\le C\prod_{i\in\{i_1,\dots,i_\ell\}}\mu(2KQ)^{1-1/p_i}\left(\int\limits_{2KQ}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{1/p_i}
\end{aligned}$$

$$
\begin{aligned}
&\times \int\limits_{(\mathbb{R}^d\setminus 2KQ)^{m-\ell}} \frac{\prod_{i\notin\{i_1,\ldots,i_\ell\}}|f_i(y_i))|d\mu(y_i)}{\prod_{i\notin\{i_1,\ldots,i_\ell\}}|x-y_i|^{n-\alpha/(m-\ell)+n\ell/(m-\ell)}}\\
&\le C \prod_{i\in\{i_1,\ldots,i_\ell\}} \frac{\phi_i(\mu(4KQ))^{1/p_i}}{\mu(2Q)^{1/p_i-1}}\|f_i\|_{L^{p_i,\phi_i}(\mu)}\\
&\times \mu(2Q)^{\ell} \prod_{i\notin\{i_1,\ldots,i_\ell\}} \int\limits_{\mathbb{R}^d\setminus 2KQ} \left(\sup_{x,y_i\in R\in\mathscr{Q}(\mu)} \frac{1}{\mu(\rho R)^{1-\alpha/(m-\ell)n}}\right)|f_i(y_i))|d\mu(y_i)\\
&\le C \prod_{i\in\{i_1,\ldots,i_\ell\}} \frac{\phi_i(\mu(4KQ))^{1/p_i}}{\mu(2Q)^{1/p_i}}\|f_i\|_{L^{p_i,\phi_i}(\mu)}\\
&\times \prod_{i\notin\{i_1,\ldots,i_\ell\}} \frac{\phi_i(\mu(2Q))^{1/p_i}}{\mu(2Q)^{1/p_i-\alpha/(m-\ell)n}}\|f_i\|_{L^{p_i,\phi_i}(\mu)}\\
&\le C \prod_{i=1}^{m} \frac{\phi_i(\mu(4KQ))^{1/p_i}}{\mu(2Q)^{1/p_i-\alpha/mn}}\|f_i\|_{L^{p_i,\phi_i}(\mu)}\\
&= C\left(\frac{\phi(\mu(4KQ))}{\mu(2Q)}\right)^{1/p} \prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(\mu)}.
\end{aligned}
$$

Hence, we get

$$
\left(\frac{1}{\phi(\mu(4KQ))}\int_Q |\mathcal{T}\left(f_1^{\tau_1},\ldots,f_m^{\tau_1}\right)(x)|^p d\mu(x)\right)^{1/p} \le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(\mu)}.
$$

If we combine this with the estimate for $\{\tau_1,\ldots,\tau_m\}\in\{0,\infty\}$, the proof of this theorem is completed. □

As a corollary, we obtain the following theorem for the multilinear fractional integral operator $I_{\alpha,m}$, $0<\alpha<mn$.

Theorem 2.7. *Let ϕ,ϕ_i,α,p,p_i be the same as in Theorem 2.6 for each $i=1,2,\ldots,m$. Then there is a constant $C>0$ independent of f_i such that*

$$
\|I_{\alpha,m}(f_1,\ldots,f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C\prod_{i=1}^{m}\|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.
$$

2.3. Boundedness of multi-sublinear maximal operator $\mathcal{M}_\kappa$

In this part we investigate the maximal operator $\mathcal{M}_\kappa$ defined in Section 1. Since $\mathcal{M}_\kappa$ is strictly smaller than the m-fold produce of the operator M_κ, then the Hölder inequality yields that:

Lemma 2.8. *Let $\kappa, p, p_i > 1$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Then there is a constant $C > 0$ independent of f_i such that*

$$\|\mathcal{M}_\kappa(f_1, \ldots, f_m)\|_{L^p(\mu)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}.$$

Now we prove the boundedness of $\mathcal{M}_\kappa$ on product generalized Morrey spaces.

Theorem 2.9. *Let ϕ, p, p_i be the same as in Theorem 2.3, and let ϕ_i satisfy the condition* (1.5) *for each $i = 1, 2, \ldots, m$. Then there is a constant $C > 0$ independent of f_i such that*

$$\|\mathcal{M}_\kappa(f_1, \ldots, f_m)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\mu)}.$$

Proof. Fix $Q \in \mathscr{Q}(\mu)$ be a cube with side length $\ell(Q)$, let Q^* be the cube with the same center of Q and side length $\frac{\kappa+7}{\kappa-1}$ times the side-length of Q. Obviously, in order to prove the theorem, it suffices to establish the following inequality

$$\left(\frac{1}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\int_Q |\mathcal{M}_\kappa(f_1, \ldots, f_m)(x)|^p d\mu(x)\right)^{1/p} \le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\frac{2\kappa}{\kappa+1},\mu)}.$$

Write f_i as $f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{Q^*}$ and $f_i^\infty = f_i - f_i^0$ for each $i = 1, \ldots, m$. Then, it is easy to see that

$$\mathcal{M}_\kappa(f_1, \ldots, f_m)(x) \le \mathcal{M}_\kappa\left(f_1^0, \ldots, f_m^0\right)(x) + \sum{}' \mathcal{M}_\kappa\left(f_1^{\tau_1}, \ldots, f_m^{\tau_m}\right)(x),$$

where each term in $\sum'$ contains at least one $\tau_i \neq 0$.

For the case $\tau_1 = \cdots = \tau_k = 0$, using the boundedness of $\mathcal{M}_\kappa$ from $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ into $L^p(\mu)$, i.e., Theorem 2.8, we have

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\int_Q |\mathcal{M}_\kappa\left(f_1^0, \ldots, f_m^0\right)(x)|^p d\mu(x)\right)^{1/p} \\
&\le C \prod_{i=1}^m \left(\frac{1}{\phi_i(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\int_{\frac{\kappa+7}{\kappa-1}Q} |f_i(x)|^{p_i} d\mu(x)\right)^{1/p_i} \\
&\le C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\frac{2\kappa}{\kappa+1},\mu)}.
\end{aligned}$$

Now, consider the case $\tau_i = \infty$ for each $i = 1, \ldots, m$. For every $x \in Q$, in order to let $R \bigcap \frac{\kappa+7}{\kappa-1}Q \neq \emptyset$ for cube R, suppose that $y_i \in R \in \mathscr{Q}(\mu)$ and $\ell(R) \ge \frac{8}{\kappa-1}\frac{\ell(Q)}{2}$. A geometric observation shows

$$\mathcal{M}_\kappa\left(f_1^\infty, \ldots, f_m^\infty\right)(x) = \sup_{x \in R \in \mathscr{Q}(\mu)} \prod_{i=1}^m \frac{1}{\mu(\kappa R)}\int_R |f_i^\infty(y_i)| d\mu(y_i)$$

$$\leq \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i=1}^{m} \frac{1}{\mu(\frac{2\kappa}{\kappa+1}R)} \int_R |f_i(y_i)| d\mu(y_i).$$

Then, the fact $Q \subset R \subset \frac{2\kappa}{\kappa+1}R$ and the condition (1.5) yield that

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\int_Q |\mathcal{M}_\kappa\left(f_1^\infty,\dots,f_m^\infty\right)(x)|^p d\mu(x)\right)^{1/p}\\
&\leq \left(\frac{\mu(Q)}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\right)^{1/p} \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i=1}^{m} \frac{1}{\mu(\frac{2\kappa}{\kappa+1}R)} \int_R |f_i(y_i)| d\mu(y_i)\\
&\leq \left(\frac{\mu(Q)}{\phi(\mu(Q))}\right)^{1/p}\\
&\quad\times \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i=1}^{m} \left(\frac{\phi_i(\mu(\frac{2\kappa}{\kappa+1}R))}{\mu(\frac{2\kappa}{\kappa+1}R)}\right)^{1/p_i} \left(\frac{1}{\phi_i(\mu(\frac{2\kappa}{\kappa+1}R))}\int_R |f_i(y_i)|^{p_i} d\mu(y_i)\right)^{1/p_i}\\
&\leq \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i=1}^{m} \left(\frac{\mu(Q)}{\phi_i(\mu(Q))} \times \frac{\phi_i(\mu(\frac{2\kappa}{\kappa+1}R))}{\mu(\frac{2\kappa}{\kappa+1}R)}\right)^{1/p_i}\\
&\quad\times \left(\frac{1}{\phi_i(\mu(\frac{2\kappa}{\kappa+1}R))}\int_R |f_i(y_i)|^{p_i} d\mu(y_i)\right)^{1/p_i}\\
&\leq C\prod_{i=1}^{m} \left(\frac{1}{\phi_i(\mu(\frac{2\kappa}{\kappa+1}R))}\int_R |f_i(y_i)|^{p_i} d\mu(y_i)\right)^{1/p_i}\\
&\leq C\prod_{i=1}^{m} \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\frac{2\kappa}{\kappa+1},\mu)}.
\end{aligned}$$

We are left now to consider the case $\tau_{i_1} = \cdots = \tau_{i_l} = 0$ for some $\{i_1,\dots,i_l\} \subset \{1,\dots,m\}$, where $1 \leq l < m$. It is to see that

$$\begin{aligned}
\mathcal{M}_\kappa\left(f_1^{\tau_1},\dots,f_m^{\tau_m}\right)(x) &= \sup_{x\in R\in\mathscr{Q}(\mu)} \prod_{i=1}^{m} \frac{1}{\mu(\kappa R)} \int_R |f_i^{\tau_i}(y_i)| d\mu(y_i)\\
&\leq \sup_{x\in R\in\mathscr{Q}(\mu)} \prod_{i\in\{i_1,\dots,i_l\}} \frac{1}{\mu(\kappa R)} \int_R |f_i^{0}(y_i)| d\mu(y_i)\\
&\quad\times \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i\notin\{i_1,\dots,i_l\}} \frac{1}{\mu(\frac{2\kappa}{\kappa+1}R)} \int_R |f_i^{\infty}(y_i)| d\mu(y_i)\\
&\leq \mathcal{M}_\kappa\left(f_{i_1}^0,\dots,f_{i_l}^0\right)(x)
\end{aligned}$$

$$\times \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i\notin\{i_1,\dots,i_l\}} \frac{1}{\mu(\frac{2\kappa}{\kappa+1}R)} \int_R |f_i(y_i)|d\mu(y_i).$$

Let $1/h_1 = \sum_{i\in\{i_1,\dots,i_l\}} 1/p_i$ and $1/h_2 = \sum_{i\notin\{i_1,\dots,i_l\}} 1/p_i$. Then $1/p = 1/h_1 + 1/h_2$. Applying the Hölder inequality and (1.5), we get

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\int_Q |\mathcal{M}_\kappa(f_1^{\tau_1},\dots,f_m^{\tau_m})(x)|^p d\mu(x)\right)^{1/p}\\
&\leq \left(\frac{1}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\int_Q |\mathcal{M}_\kappa(f_{i_1}^0,\dots,f_{i_l}^0)(x)|^{h_1} d\mu(x)\right)^{1/h_1}\\
&\quad\times\left(\frac{\mu(Q)}{\phi(\mu(\frac{2\kappa(\kappa+7)}{\kappa^2-1}Q))}\right)^{1/h_2} \sup_{\substack{x\in R\in\mathscr{Q}(\mu)\\ Q\subset R}} \prod_{i\notin\{i_1,\dots,i_l\}} \frac{1}{\mu(\frac{2\kappa}{\kappa+1}R)} \int_R |f_i(y_i)|d\mu(y_i)\\
&\leq C\prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,\phi_i}(\frac{2\kappa}{\kappa+1},\mu)}.
\end{aligned}$$

Finally, a combination of the estimates above finishes the proof of the theorem. □

3. Boundedness on Morrey spaces $\mathscr{M}_p^q(\mu)$

It is easy to see that the function $\phi_i(t) = t^{1-p_i/q_i}$, $1 < p_i \leq q_i < \infty$, satisfies the assumptions (1.4), (1.5) and (1.7) for each $i = 1, 2, \dots, m$. Then we can obtain the following three theorems from the theorems above.

Theorem 3.1. *Assume that $\mathcal{T}$ is a multi-sublinear operator satisfying* (1.1) *with $\alpha = 0$. Let $1 < p_i \leq q_i < \infty$, $1 < p \leq q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/q_1 + \cdots + 1/q_m$. If $\mathcal{T}$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^p(\mu)$, then there is a constant $C > 0$ independent of f_i such that*

$$\|\mathcal{T}(f_1,\dots,f_m)\|_{\mathscr{M}_p^q(\mu)} \leq C\prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)}.$$

Theorem 3.2. *Assume that $0 < \alpha < mn$ and $\mathcal{T}$ is a multi-sublinear operator satisfying* (1.1). *Let $0 < \alpha < mn$, $1 < p_i \leq q_i < mn/\alpha$, $1 < p \leq q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n$ and $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n$. If $\mathcal{T}$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^p(\mu)$, then there is a constant $C > 0$ independent of f_i such that*

$$\|\mathcal{T}(f_1,\dots,f_m)\|_{\mathscr{M}_p^q(\mu)} \leq C\prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)}.$$

Theorem 3.3. *Assume that T is a m-linear Calderón-Zygmund operator, $1 < p_i \le q_i < \infty$, $1 < p \le q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/q_1 + \cdots + 1/q_m$. Then there is a constant $C > 0$ independent of f_i such that*

$$\|T(f_1, \dots, f_m)\|_{\mathscr{M}_p^q(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)} .$$

Theorem 3.4. *Assume that $I_{\alpha,m}$ is a multilinear fractional integral operator. Let $0 < \alpha < mn$, $1 < p_i \le q_i < mn/\alpha$, $1 < p \le q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n$ and $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n$. Then there is a constant $C > 0$ independent of f_i such that*

$$\|I_{\alpha,m}(f_1, \dots, f_m)\|_{\mathscr{M}_p^q(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)} .$$

Theorem 3.5. *Let $\kappa > 1$, $1 < p_i \le q_i < \infty$, $1 < p \le q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/q_1 + \cdots + 1/q_m$. Then there is a constant $C > 0$ independent of f_i such that*

$$\|\mathcal{M}_\kappa(f_1, \dots, f_m)\|_{\mathscr{M}_p^q(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)} .$$

Here we point out that, in Theorem 3.4, we get the boundedness for $I_{\alpha,m}$ under the assumption $1 < p_i < mn/\alpha$ for each $i = 1, 2, \dots, m$. But this assumption is unnatural, we want to obtain the estimates for the operator $I_{\alpha,m}$ in the natural condition $1 < p_i < \infty$. The following theorem will arrive at our desire.

Theorem 3.6. *Suppose that $0 < \alpha < mn$, $1 < p_i \le q_i < \infty$, $1 < p \le q < \infty$. Let $1/l = 1/p_1 + \cdots + 1/p_m$, $1/h = 1/q_1 + \cdots + 1/q_m$ and $p/q = l/h$, $1/q = 1/h - \alpha/n > 0$. Then there is a constant $C > 0$ independent of f_i such that*

$$\|I_{\alpha,m}(f_1, \dots, f_m)\|_{\mathscr{M}_p^q(\mu)} \le C \prod_{i=1}^m \|f_i\|_{\mathscr{M}_{p_i}^{q_i}(\mu)} .$$

Proof. Obviously, for all $x, y_i \in \mathbb{R}^d$, $x \ne y_i$, $i = 1, 2, \dots, m$, there is

$$\begin{aligned} &|I_{\alpha,m}(f_1, \dots, f_m)(x)| \\ &\le \int_{(\mathbb{R}^d)^m} \frac{|f_1(y_1)| \cdots |f_m(y_m)|}{(|x - y_1| + \cdots + |x - y_m|)^{mn-\alpha}} d\mu(y_1) \cdots d\mu(y_m). \end{aligned}$$

Since $0 < \alpha < mn$, we can get

$$\frac{1}{(|x - y_1| + \cdots + |x - y_m|)^{mn-\alpha}} = C \int_{|x-y_1|+\cdots+|x-y_m|}^{\infty} \sigma^{\alpha-mn-1} d\sigma.$$

Thus, applying Fubini theorem, we have, for any $\delta > 0$, that

$$|I_{\alpha,m}(f_1, \dots, f_m)(x)(x)|$$

$$\leq C\int_0^\infty \int_{\{|x-y_1|+\cdots+|x-y_m|<\sigma\}} \sigma^{\alpha-mn-1}\prod_{i=1}^m |f_i(y_i)|d\mu(y_i)d\sigma$$
$$\leq C\int_0^\infty \left(\prod_{i=1}^m \frac{1}{\sigma^n}\int_{B(x,\sigma)}|f_i(y_i)|d\mu(y_i)\right)\sigma^{\alpha-1}d\sigma$$
$$\leq C\left\{\int_0^\delta + \int_\delta^\infty\right\}\left(\prod_{i=1}^m F_i(\sigma,x)\right)\sigma^{\alpha-1}d\sigma =: C(I_1+I_2),$$

where

$$F_i(\sigma,x) := \frac{1}{\sigma^n}\int_{B(x,\sigma)}|f_i(y_i)|d\mu(y_i) \quad \text{for } i=1,2,\ldots,m.$$

For I_1, by the growth condition of μ, it is easy to see that

$$I_1 \leq C\int_0^\delta \mathcal{M}_\kappa(f_1,\ldots,f_m)(x)\sigma^{\alpha-1}d\sigma \leq C\delta^\alpha \mathcal{M}_\kappa(f_1,\ldots,f_m)(x).$$

Meanwhile, for I_2, let $Q(x,\sigma)$ be the cube whose center is x and side length is σ, by the Hölder's inequality and the growth condition, we obtain

$$F_i(\sigma,x) \leq \frac{\mu(B(x,\sigma))^{1-\frac{1}{p_i}}}{\sigma^n}\left(\int_{B(x,\sigma)}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{\frac{1}{p_i}}$$
$$\leq C\sigma^{-\frac{n}{q_i}}\sigma^{\frac{n}{q_i}-\frac{n}{p_i}}\left(\int_{B(x,\sigma)}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{\frac{1}{p_i}}$$
$$\leq C\sigma^{-\frac{n}{q_i}}\mu(Q(x,4\sigma))^{\frac{1}{q_i}-\frac{1}{p_i}}\left(\int_{Q(x,2\sigma)}|f_i(y_i)|^{p_i}d\mu(y_i)\right)^{\frac{1}{p_i}}$$
$$\leq C\sigma^{-\frac{n}{q_i}}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}.$$

Thus, by $1/q = 1/q_1+\cdots+1/q_m-\alpha/n$, it follows that

$$I_2 \leq C\int_\delta^\infty\left(\prod_{i=1}^m \sigma^{-\frac{n}{q_i}}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}\right)\sigma^{\alpha-1}d\sigma$$
$$\leq C\prod_{i=1}^m\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}\int_\delta^\infty \sigma^{-\frac{n}{q}-1}d\sigma$$
$$\leq C\delta^{-\frac{n}{q}}\prod_{i=1}^m\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}.$$

Summing the estimates for I_1 and I_2, we get

$$|I_{\alpha,m}(f_1,\ldots,f_m)(x)| \leq C\left[\delta^\alpha\mathcal{M}_\kappa(f_1,\ldots,f_m)(x)+\delta^{-\frac{n}{q}}\prod_{i=1}^m\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}\right].$$

Now we take δ such that

$$\delta^{\alpha}\mathcal{M}_{\kappa}(f_1,\dots,f_m)(x)=\delta^{-\frac{n}{q}}\prod_{i=1}^{m}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}.$$

This implies

(3.1)

$$|I_{\alpha,m}(f_1,\dots,f_m)(x)|\le C\left(\prod_{i=1}^{m}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}\right)^{1-h/q}\mathcal{M}_{\kappa}(f_1,\dots,f_m)(x)^{h/q}$$

with a constant $C>0$ independent of f_i.

Since $1<p_i\le q_i<\infty$, $1<p\le q<\infty$, $1/l=1/p_1+\cdots+1/p_m$, $1/h=1/q_1+\cdots+1/q_m$ and $p/q=l/h$. For any $Q\in\mathscr{Q}(\mu)$, using the Hölder inequality and Theorem 3.5, we obtain

$$\begin{aligned}&\mu(2Q)^{\frac{p}{q}-1}\int_Q\mathcal{M}_{\kappa}(f_1,\dots,f_m)(x)^{\frac{ph}{q}}d\mu(x)\\ &=\left\{\mu(2Q)^{\frac{1}{h}-\frac{1}{l}}\left[\int_Q\mathcal{M}_{\kappa}(f_1,\dots,f_m)(x)^l d\mu(x)\right]^{\frac{1}{l}}\right\}^l\\ &\le C\left(\prod_{i=1}^{m}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}\right)^l.\end{aligned}$$

Then, combine the above estimates with the inequality (3.1), we will immediately get the desired inequality, i.e.,

$$\begin{aligned}\|I_{\alpha,m}(f_1,\dots,f_m)\|_{\mathscr{M}^{q}_{p}(\mu)}&\le C\left(\prod_{i=1}^{m}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}\right)^{1-\frac{h}{q}+\frac{l}{p}}\\ &=C\prod_{i=1}^{m}\|f_i\|_{\mathscr{M}^{q_i}_{p_i}(\mu)}.\end{aligned}$$

The proof of Theorem 3.6 is completed. □

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