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CONTROLLABILITY FOR NONLINEAR VARIATIONAL EVOLUTION INEQUALITIES

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Abstract. In this paper we investigate the approximate controllability for the following nonlinear functional differential control problem:

$$x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + h(t)$$

which is governed by the variational inequality problem with nonlinear terms.

1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. The norm on V (resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$). Let U be a complex Banach space and B be a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$. Let A be a continuous linear operator from V into V^* and satisfies the coercive condition, $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then we deal with the approximate controllability for the following control system governed by the variational inequality problem with nonlinear term:

$$\text{(SE)} \qquad \begin{cases} \quad (x'(t) + Ax(t), x(t) - z) + \phi(x(t)) - \ \phi(z) \\ \le (f(t, x(t)) + (Bu)(t), x(t) - z), \text{ a.e., } 0 < t \le T, \quad z \in V, \\ x(0) = x_0. \end{cases}$$

Noting that the subdifferential operator $\partial\phi : V \to V^*$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \le \phi(y) + (x^*, x - y), \quad y \in V\},$$

where $(\cdot,\cdot)$ denotes the duality pairing between V^* and V, the control system (SE) is represented by the following nonlinear functional differential control

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problem on H

$$\text{(NCE)} \qquad \begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + Bu(t), \quad 0 < t \le T, \\ x(0) = x_0, \end{cases}$$

where the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R} \times V$ into H. Its corresponding linear variational inequality [$f \equiv 0$ in (SE)] was widely developed as seen in Section 4.3.2 of Barbu [3] (also see Section 4.3.1 in [2], [4, 5, 8, 10]). In [6], using more general hypotheses for nonlinear term $f(\cdot, x)$, we investigated the existence and the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ considered as an equation in H as well as in V^*, which is also applicable to optimal control problem.

In the paper [9] Naito investigated the semilinear parabolic evolution equation in a Hilbert space H, in case where the nonlinear term f is a bounded uniformly continuous mapping from H to itself, and proved the approximate controllability under the following hypothesis;

(B) For each $p \in L^2(0,T;X)$ there exists a function $q \in \overline{X}_B$: $\tilde{S}p = \tilde{S}q$, where

$$\tilde{S}p = \int_0^T S(T-s)p(s)ds,$$

$S(\cdot)$ is the semigroup generated by $-A$ and $\overline{X}_B$ is the closure in $L^2(0,T;X)$ of the range X_B of the operator B.

The purpose of this paper is to show that the reachable set of the nonlinear variational inequality (SE) is equivalent to that of its corresponding linear variational inequality under the hypothesis (B) by applying results of [9] to the equation (NCE). We formulate our nonlinear variational evolution inequality (NCE) as a semilinear control system in order to obtain the control problems. As in [7, 9] we must assume the uniform boundedness of the nonlinear terms $f(t,x)$ and $(\partial\phi)^0$, where $(\partial\phi)^0 : H \to H$ is the minimum element of $\partial\phi$. Since we apply the degree of mapping theorem in the proof of the main theorem, we need some compactness hypothesis. We make the natural assumption that the embedding $D(A_0) \subset V$ is compact. Then the embedding $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$ is compact in view of Aubin's result [1] (see also [11]), and we show that the mapping which maps f to the solution of (NCE) with Bu replaced by f is a compact operator from $L^2(0,T;H)$ to itself. We show the approximate controllability of (NCE) by using the Lelay-Schauder degree theory.

2. Variational inequalities

Let V and H be complex Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H. For the sake of simplicity, we may consider

$$||u||_* \le |u| \le ||u||, \quad u \in V,$$

where $||\cdot||_*$ is the norm of the element of V^*. If an operator A is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\}$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H.$$

We also assume that there exists a constant C_1 such that

$$||u|| \leq C_1 ||u||_{D(A)}^{1/2} |u|^{1/2} \tag{2.1}$$

for every $u \in D(A_0)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus, in what follows we will write

$$V = (D(A), H)_{1/2,2}$$

as a matter of convenience. Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq c_0 ||u||^2, \quad c_0 > 0. \tag{2.2}$$

Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u, v), \quad u,\ v \in V.$$

Then A is a bounded linear operator from V to V^*, and A generates an analytic semigroup in both of H and V^*. It is also known that if $a(\cdot,\cdot)$ is a symmetric quadratic form satisfying (2.2), then A is positive definite and self-adjoint and $D(A^{1/2}) = V$.

Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi : V \to V^*$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), \quad y \in V\}.$$

First, let us consider the following perturbation of subdifferential operator;

$$\text{(NE)} \qquad \begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni Bu(t), \quad 0 < t \leq T, \\ x(0) = x_0. \end{cases}$$

For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = \inf\{||x - J_\epsilon x||_*^2/2\epsilon + \phi(J_\epsilon x) : x \in V\},$$

where $J_\epsilon = (I + \epsilon\partial\phi)^{-1}$. If $B = \partial\phi$, then the function $\partial\phi_\epsilon$ is Fréchet differentiable on V and its Frećhet differential $\partial\phi_\epsilon = B_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1} where $B_\epsilon = \epsilon^{-1}(I - (I + \epsilon B)^{-1})$ is as seen in Corollary 2.2 in Chapter II of [2]. It is also well known results that $\lim_{\epsilon\to 0} \phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0} \partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$ where $(\partial\phi)^0 : V \to V^*$ is the minimum element of $\partial\phi$. Now, we introduce the smoothing system corresponding to (NE) as follows.

$$\begin{cases} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = Bu(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Using the regularity for the abstract linear parabolic equation we have the following result on the equation (NE).

Proposition 2.1. 1) *Let* $u \in L^2(0,T;U)$ *and* $x_0 \in V$ *satisfying that* $\phi(x_0) < \infty$. *Then the equation* (NE) *has a unique solution*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T:V^*) \subset C([0,T];H),$$

which satisfies

$$x'(t) = Bu(t) - Ax(t) - \partial\phi^0(x(t))$$

and

$$||x||_{L^2\cap W^{1,2}} \le C_2(1 + ||x_0|| + ||u||_{L^2(0,T;U)}), \tag{2.3}$$

where C_2 *is a constant and* $L^2 \cap W^{1,2} = L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$.

2) *Let* $a(\cdot,\cdot)$ *be a symmetric quadratic form satisfying* (2.2) *and the following hypothesis hold*:

(A) *There exists* $h \in H$ *such that for every* $\epsilon > 0$ *and an* $\in D(\phi)$

$$J_\epsilon(y + \epsilon h) \in D(\phi) \text{ and } \phi(J_\epsilon(y + \epsilon h)) \le \phi(y).$$

Then for $u \in L^2(0,T;U)$ *and* $x_0 \in \overline{D(\phi)} \cap V$ *the equation* (NE) *has a unique solution*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \cap C([0,T];H),$$

which satisfies

$$||x||_{L^2\cap W^{1,2}\cap C} \le C_2(1 + ||x_0|| + ||u||_{L^2(0,T;U)}). \tag{2.4}$$

If V is compactly embedded in H, the following embedding

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H))$$

is compact in view of Theorem 2 of Aubin [1]. Hence, the mapping $u \mapsto x$ is compact from $L^2(0,T;U)$ to $L^2(0,T;H)$.

Now we give the assumption on the nonlinear terms as follows.

(F) *Let* f *be a nonlinear single valued mapping from* V *into* H. *We assume that*

$$|f(t,x_1) - f(t,x_2)| \le L||x_1 - x_2||$$

for every $x_1,\ x_2 \in V$.

Now, we introduce smoothing system corresponding to (NCE) as follows.

$$\text{(SCE)} \quad \begin{cases} \frac{dx(t)}{dt} + Ax(t) + \partial\phi_\epsilon(x(t)) = f(t,x(t)) + Bu(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Since $-A$ generates a semigroup $S(t)$ on H, the mild solution of (SCE) can be represented by

$$x_\epsilon(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x_\epsilon(s)) + Bu(s) - \partial\phi_\epsilon(x_\epsilon(s))\}ds. \tag{2.5}$$

We establish the following result on the solvability of (NCE) as is seen in Theorem 2.1 of [6].

Proposition 2.2. 1) *Let $x_0 \in V$ satisfying that $\phi(x_0) < \infty$, $u \in L^2(0, T; U)$ and the assumption* (F) *be satisfied. Then the equation* (NCE) *has a unique solution*

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V) \cap C([0, T]; H),$$

which satisfies

$$x'(t) = f(t, x(t)) + Bu(t) - Ax(t) - \partial\phi^0(x(t))$$

and there exists a constant C_3 depending on T such that

$$||x||_{L^2 \cap W^{1,2}} \le C_3(1 + ||x_0|| + ||u||_{L^2(0,T;U)}). \tag{2.6}$$

2) *Let $a(\cdot, \cdot)$ be a symmetric quadratic form satisfying* (2.2) *and let us assume the hypotheses* (A), (F)*. Then the equation* (NCE) *has a unique solution*

$$x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \cap C([0, T]; H),$$

which satisfies

$$||x||_{L^2 \cap W^{1,2} \cap C} \le C_3(1 + ||x_0|| + ||u||_{L^2(0,T;U)}). \tag{2.7}$$

Theorem 2.3. *Let $x_0 \in V$, $u \in L^2(0, T; U)$ and the hypotheses in* 2) *of Proposition 2.2 be satisfied. Then the solution x of the equation* (SCE) *belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$, and the mapping*

$$V \times L^2(0, T; U) \ni (x_0, u) \mapsto x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$$

is continuous.

Proof. Let $(x_{0i}, u_i) \in F \times L^2(0, T; U)$, and x_i be the solution of (SNE) with (x_{0i}, u_i) in place of (x_0, u) for $i = 1,\ 2$. Then in view of (2.7), we have

$$\begin{aligned} &||x_1 - x_2||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} \\ &\le C_3\{||x_{01} - x_{02}|| + (||\partial\phi_\epsilon(x_1) - \partial\phi_\epsilon(x_2)|| + ||f(\cdot, x_1) - f(\cdot, x_2)||)_{L^2(0,T;H)} \\ &\quad + ||u_1 - u_2||)_{L^2(0,T;U)}\} \\ &\le C_3\{||x_{01} - x_{02}|| + (\epsilon^{-1} + L)||x_1 - x_2||_{L^2(0,T;V)} + ||u_1 - u_2||_{L^2(0,T;U)}\}. \end{aligned} \tag{2.8}$$

Noting that

$$x_1(t) - x_2(t) = x_{01} - x_{02} + \int_0^t (\dot{x}_1(s) - \dot{x}_2(s))ds,$$

we get

$$||x_1 - x_2||_{L^2(0,T;H)} \le \sqrt{T}||x_{01} - x_{02}|| + \tfrac{T}{\sqrt{2}}||x_1 - x_2||_{W^{1,2}(0,T;H)}.$$

Hence from (2.1) we get

$$
\begin{aligned}
(2.9)\quad & ||x_1 - x_2||_{L^2(0,T;V)} \\
&\le C_1||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))}||x_1 - x_2||^{1/2}_{L^2(0,T;H)} \\
&\le C_1||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))} \\
&\quad \times \{T^{1/4}||x_{01} - x_{02}||^{1/2} + (\tfrac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||^{1/2}_{W^{1,2}(0,T;H)}\} \\
&\le C_1T^{1/4}||x_{01} - x_{02}||^{1/2}||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))} \\
&\quad + C_1(\tfrac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \\
&\le 2^{-7/4}C_1||x_{01} - x_{02}|| + 2C_1(\tfrac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}.
\end{aligned}
$$

Combining (2.8) and (2.9) we obtain

(2.10)

$$
\begin{aligned}
& ||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \\
&\le C_3\{||x_{01} - x_{02}|| + (\epsilon^{-1} + L)(2^{-7/4}C_1||x_{01} - x_{02}|| \\
&\quad + 2C_1(\tfrac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}) + ||u_1 - u_2||_{L^2(0,T;U)}\}.
\end{aligned}
$$

Suppose that $(x_{0n}, u_n) \to (x_0, u)$ in $V \times L^2(0, T; H)$, and let x_n and x be the solutions (SCE) with (x_{0n}, u_n) and (x_0, u), respectively. Let $0 < T_1 \le T$ be such that

$$(\epsilon^{-1} + L)C_1C_3(2T_1)^{1/2} < 1.$$

Then by virtue of (2.10) with T replaced by T_1 we see that $x_n \to x$ in $L^2(0, T_1; D(A)) \cap W^{1,2}(0, T_1; H) \subset C([0, T_1]; V)$. This implies that $x_n(T_1) \mapsto x(T_1)$ in V. Hence the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that

$$x_n \to x \text{ in } L^2(0, T; D(A)) \cap W^{1,2}(0, T; H). \qquad \square$$

3. Approximate controllability

In this section we show the approximate controllability for the equation (NCE), which is the extended result of Naito [9] to the equation (SCE). The realization for the operator A in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

be also denoted by A.

The solutions of (NCE) and (SCE) are denoted by $x(t;\phi,f,u)$ and $x_\epsilon(t;\phi_\epsilon,f,u)$, respectively. In view of Proposition 2.2, we have

$$||x_\epsilon(\cdot;\phi_\epsilon,g,u)||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3(1+||x_0||+||u||_{L^2(0,T;U)}). \tag{3.1}$$

For $k\in L^2(0,T;H)$ let y_k be the solution of equation with $B=I$

$$\begin{cases} x'(t)+Ax(t)+\partial\phi_\epsilon(x(t))=f(x(t))+k(t), \quad 0<t\le T,\\ x(0)=0. \end{cases} \tag{3.2}$$

Lemma 3.1. *Let $k\in L^2(0,T;H)$ and the hypotheses in* 2) *of Proposition 2.2 be satisfied. Then the solution y_k of the equation* (3.2) *belongs to $L^2(0,T;D(A))$ $\cap$ $W^{1,2}(0,T;H)$, and the mapping $k\mapsto y_k$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$.*

Proof. From Proposition 2.2 we have that

$$y_k(t)=\int_0^t S(t-s)\{f(x_\epsilon(s))+k(s)-\partial\phi_\epsilon(x_\epsilon(s))\}ds$$

and

$$||y_k||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \le C_3(1+||k||_{L^2(0,T;H)}).$$

Hence if k is bounded in $L^2(0,T;H)$, then so is y_k in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$. Since $D(A_0)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)\subset L^2(0,T;V))$$

is compact in view of Theorem 2 of J. P. Aubin [1]. □

For the sake of simplicity we assume that $S(t)$ is uniformly bounded; there exists a constant $M\ge 1$ such that

$$||S(t)||\le M. \tag{3.3}$$

To prove the approximate controllability for the equation (NCE) we need the hypothesis that

(A1) $(\partial\phi)^0$ *is uniformly bounded, i.e.,*

$$|(\partial\phi)^0x|\le M_1, \quad x\in H,$$

where $(\partial\phi)^0: H\to H$ is the minimum element of $\partial\phi$.

Lemma 3.2. *Let the assumption* (A1) *be satisfied. Then there exists a constant C independent of ϵ such that*

$$||x_\epsilon-x||_{C([0,T];H)\cap L^2(0,T;V)}\le C\epsilon, \quad 0<T.$$

Proof. For any $\epsilon>0$ and $\lambda>0$, let x_ϵ and x_λ be the solutions of (SCE) corresponding to ϵ and λ, respectively. Then from the equation (SCE) we have

$$\begin{aligned} &x'_\epsilon(t)-x'_\lambda(t)+A(x_\epsilon(t)-x_\lambda(t))+\partial\phi_\epsilon(x_\epsilon(t))-\partial\phi_\lambda(x_\lambda(t))\\ &=f(x_\epsilon(t))-f(x_\lambda(t)), \end{aligned}$$

and hence, from (2.2) and multiplying by $x_\epsilon - x_\lambda(t)$, it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&|x_\epsilon(t) - x_\lambda(t)|^2 + c_0||x_\epsilon(t) - x_\lambda(t)||^2 \\ &+ (\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)) \\ \leq\ & (f(x_\epsilon(t)) - f(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)). \end{aligned} \tag{3.4}$$

Since

$$\begin{aligned} &(f(x_\epsilon(t)) - f(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)) \\ \leq\ & ||f(x_\epsilon(t)) - f(x_\lambda(t))||_*||x_\epsilon(t) - x_\lambda(t)|| \\ \leq\ & \frac{1}{2c}||f(x_\epsilon(t)) - f(x_\lambda(t))||_*^2 + \frac{c}{2}||x_\epsilon(t) - x_\lambda(t)||^2 \end{aligned}$$

for every real number c, so if we choose a constant c satisfying $c_0 - c/2 > 0$, then by integrating (3.4) over $[0, T]$ we have

$$\begin{aligned} &\frac{1}{2}|x_\epsilon(t) - x_\lambda(t)|^2 + (c_0 - c/2)\int_0^T ||x_\epsilon(t) - x_\lambda(t)||^2 \\ \leq\ & \int_0^T (\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), \lambda\partial\phi_\lambda(x_\lambda(t) - \epsilon\partial\phi_\epsilon(x_\epsilon(t)) \\ &+ \frac{1}{2c}\int_0^T |x_\epsilon(t) - x_\lambda(t)|^2 \end{aligned}$$

by the monotonicity of $\partial\phi$. Here, we used that

$$\partial\phi_\epsilon(x_\epsilon(t)) = \epsilon^{-1}(x_\epsilon(t) - (I + \epsilon\partial\phi)^{-1}x_\epsilon(t)).$$

Since $|\partial\phi_\epsilon(x)| \leq |(\partial\phi)^0 x|$ for every $x \in D(\partial\phi)$, it follows from (A1) and using Gronwall's inequality that

$$||x_\epsilon - x_\lambda||_{C([0,T];H)\cap L^2(0,T;V)} \leq C(\epsilon + \lambda), \quad 0 < T.$$

Thus, letting $\lambda \to 0$, the proof of lemma is complete. □

We assume

(F1) f *is uniformly bounded*: *there exists a constant* M_f *such that*

$$|f(t, x)| \leq M_f$$

for all $x \in V$.

In view of Lemma 3.1, if we define the nonlinear operator $\mathcal{F}$ on $L^2(0, T; H)$ by

$$(\mathcal{F}k)(t) = f(t, y_k(t)) + \partial\phi_\epsilon(y_k(t)), \quad k \in L^2(0, T; H),$$

then $\mathcal{F}$ is a compact mapping from $L^2(0, T; H)$ to itself. Then it holds that

$$||\mathcal{F}(k)|| \leq (M_f + M_1)\sqrt{T}.$$

Let

$$N = \{p \in L^2(0, T; H) : \int_0^T S(T - s)p(s)ds = 0\}$$

and denote the orthogonal complement of N in $L^2(0,T;H)$ by $N^\perp$. We denote the range of the operator B by H_B.

We need the following assumption:

(B) *For each* $p \in L^2(0,T;H)$ *there exists an element* $q \in \overline{H}_B$ *such that*

$$\int_0^T S(T-s)p(s)ds = \int_0^T S(T-s)q(s)ds,$$

that is, $L^2(0,T;H) = \overline{H}_B + N$*, where* $\overline{H}_B$ *is the closure of* H_B *in* $L^2(0,T;H)$*.*

For $u \in N^\perp$, let Pu be the unique minimum norm element of $\{u+N\} \cap \overline{H}_B$. Then the proof of Lemma 1 of Naito [9] showed that P is a linear and continuous operator from $N^\perp$ to $\overline{H}_B$. Let $\tilde{Y} = L^2(0,T;H)/N$ be the quotient space and the norm of a coset $\tilde{u} = u + N \in \tilde{Y}$ be defined of $||\tilde{u}|| = \inf\{|u+f| : f \in N\}$.

We define by Q the isometric isomorphism from $\tilde{Y}$ onto $N^\perp$, that is, $Q\tilde{u}$ is the minimum norm element in $\tilde{u} = \{u + f : f \in N\}$. Let

$$\tilde{\mathcal{F}}\tilde{u} = \mathcal{F}(PQ\tilde{u}) + N$$

for $\tilde{u} \in \tilde{Y}$. Then we have

$$||\tilde{\mathcal{F}}(\tilde{u})|| \le (M_f + M_1)\sqrt{T}, \tag{3.5}$$

and $\tilde{\mathcal{F}}$ is a compact mapping from $\tilde{Y}$ to itself.

We define the reachable sets for the system (NCE) as follows:

$$R_T = \{x(T;\phi_\epsilon, f, u) : u \in L^2(0,T;U)\},$$
$$L_T = \{x(T;0,0,u) : u \in L^2(0,T;U)\}.$$

If $\overline{R_T} = H$ where $\overline{R_T}$ is the closure of R_T in H, then the system (NCE) is called approximately controllable at time T.

Theorem 3.3. *Let* $a(\cdot,\cdot)$ *be a symmetric quadratic form satisfying* (2.2) *and let us assume the hypotheses* (A1)*,* (F1) *and* (B)*. Then we have* $L_T \subset \overline{R_T}$*. Therefore, if the linear system* (NCE) *with nonlinear terms* $f + \partial\phi \equiv 0$ *is approximately controllable, then so is the nonlinear system* (NCE)*.*

Proof. We follow the proof of Theorem 1 of Naito [9]. Actually we show that $L_T \subset \overline{R_T}^V$, where $\overline{R_T}^V$ is the closure of R_T in V. Let

$$\eta = \int_0^T S(T-s)Bv(s)ds \in L_T.$$

We will show that there exists w such that

$$\eta = x_\epsilon(T;\phi, f, w).$$

Let r be a positive number such that

$$v \in U_r = \{u \in L^2(0,T;U) : ||u||_{L^2(0,T;U)} < r\}.$$

Put $z = Bv$ and $r_1 = ||B||r$. Then it follows that

$$\tilde{z} = z + N \in V_{r_1} = \{\tilde{x} \in \tilde{Y} : ||\tilde{x}||_{\tilde{Y}} < r_1\}.$$

Take a constant $d > 0$ such that

$$(M_f + M_1)\sqrt{T} + r_1 < d.$$

Let us consider the equation

$$\tilde{z} = \lambda\tilde{\mathcal{F}}\tilde{u} + \tilde{u}, \quad 0 \le \lambda \le 1. \tag{3.6}$$

Let u be the solution of (3.6). Since $\tilde{z} \in V_d$ and from (3.5)

$$\begin{aligned} ||\tilde{u}|| &\le ||\tilde{z}|| + ||\tilde{\mathcal{F}}\tilde{u}|| \\ &\le r_1 + (M_f + M_1)\sqrt{T} < d \end{aligned}$$

it follows that $\tilde{u} \notin \partial V_d$ where ∂V_d stands for the boundary of V_d. Thus by the homotopy property of degree theory, there exists $\tilde{u} \in \tilde{Y}$ such that

$$\tilde{z} = \tilde{\mathcal{F}}\tilde{u} + \tilde{u}. \tag{3.7}$$

Put $u = Q\tilde{u}$ and $u_B = PQ\tilde{u}$. Then we have that $u_B = Pu$ and $u - u_B = u - Pu \in N$. Hence

$$\tilde{z} = \mathcal{F}(u_B) + u + N = \mathcal{F}(u_B) + u_B + N.$$

Therefore,

$$\begin{aligned} \eta &= \int_0^T S(T-s)(\mathcal{F}(u_B)(s) + u_B(s))ds \\ &= \int_0^T S(T-s)(f(s, y_{u_B}(s)) + \partial\phi_\epsilon(y_{u_B}(s)) + u_B(s))ds. \end{aligned}$$

Since $u_B \in \overline{H}_B$, there exists a sequence $\{v_n\} \in L^2(0,T;U)$ such that $Bv_n \mapsto u_B$ in $L^2(0,T;H)$. Then by Theorem 2.3 we have that $x_\epsilon(\cdot;\phi,f,v_n) \mapsto y_{u_B}$ in $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)$, and hence $x_\epsilon(T;\phi,f,v_n) \mapsto y_{u_B}(T) = \eta$ in V. Thus from Lemma 3.1 it follows that $\eta \in \overline{R_T}^V$. □

Remark 3.4. Under the hypothesis (B), we know that $\overline{R_T} = H$. In fact, let $\eta \in D(A)$, then putting $p(s) = (\eta + sA\eta)/T$, it holds that

$$\eta = \int_0^T S(T-s)p(s)ds.$$

From (B), there exists a function $q \in \overline{H}_B$ such that

$$\int_0^T S(T-s)p(s)ds = \int_0^T S(T-s)q(s)ds.$$

By the definition of closure of range of the controller B, we can show that there exists a control function $v \in L^2(0,T;U)$ such that

$$\left|\eta - \int_0^T S(T-s)(Bv)(s)ds\right| < \epsilon \quad \text{for any } \epsilon > 0.$$

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